EXACT ALL-COUPLING NON-PERTURBATIVE BOUNDS FOR SU(N) LATTICE QCD WITH EXACT CHIRAL OVERLAP (NEUBERGER) FERMIONS UNDER LÜSCHER'S ADMISSIBILITY CONDITION

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ABSTRACT. Neuberger's overlap Dirac operator $D_{\mathrm{ov}}(U) = I + \gamma_5 \operatorname{sgn}(H_W(U))$ realizes exact lattice chiral symmetry via the Ginsparg–Wilson relation without fermion doubling, provided the Hermitian Wilson kernel $H_W(U) = \gamma_5(D_W^{\mathrm{std}}(U) - I)$ remains gapped away from zero. Without polymer or cluster expansions, we establish four self-contained non-perturbative theorems on periodic hypercubic tori $\mathbb{T}_L^d$ ($d \in \{2,4\}$, $N \geq 2$) under Lüscher's plaquette admissibility condition $\Omega_\varepsilon := \{U \in \mathrm{SU}(N)^{|E|} : \max_p \|I_N - U_p\|_{\mathrm{op}} < \varepsilon\}$:

(1) *Covariant Bochner–Weitzenböck spectral gap and elementary Chebyshev locality:* Expanding $H_W(U)^2 = A(U)^\dagger A(U)$ proves the exact lower bound $\lambda_{\min}(H_W(U)^2) \geq \mu_{\min}(d,\varepsilon)^2 := 1 - \binom{d}{2}(5\varepsilon + \frac{1}{8}\varepsilon^2) > 0$ for all $\varepsilon < \varepsilon_{\max}(d)$ ($\varepsilon_{\max}(4) = \frac{2(\sqrt{903}-30)}{3} \approx 0.033306$, matching Neuberger's $1/30$ threshold, and $\varepsilon_{\max}(2) = 2\sqrt{102} - 20 \approx 0.199010$) and $\lambda_{\max}(H_W(U)^2) \leq (2d-1)^2$. An elementary Poisson-kernel integral representation of $x^{-1/2}$ then yields an explicit exponential locality bound $\|D_{\mathrm{ov}}(U)(x,y)\|_{\mathrm{op}} \leq C_{\mathrm{loc}}(d,\varepsilon)\, q(d,\varepsilon)^{\lfloor \|x-y\|_1/2 \rfloor}$ with $q = \frac{\mu_{\max}-\mu_{\min}}{\mu_{\max}+\mu_{\min}} < 1$.

(2) *Exact $N_f \geq 1$ determinant positivity and all-coupling admissible free-energy enclosures:* For every $N_f \geq 1$ (including $N_f = 1$), every $m \in (0,2]$, and every $U \in \Omega_\varepsilon$, the massive overlap determinant obeys $m^{n_s N N_f V} \leq [\det D_{\mathrm{ov}}^{(m)}(U)]^{N_f} \leq 2^{n_s N N_f V}$. On the 4D Lüscher topology-preserving SU(2) ensemble, 4-way Hölder's inequality (evaluated via a strictly positive backward moment recurrence $M_1(a)$) and an axial-gauge geodesic-ball Schur lower bound inside $B_\delta(I_2)$ ($4\delta < 2\arcsin(\varepsilon/2)$) match the exact one-loop entropy $-\frac{9}{2}\ln\beta$ with $O(1)$ gaps across $\beta \in [0, 10^4]$.

(3) *Topological sector freezing and quantitative index stability radius:* The Neuberger index $Q_{\mathrm{ov}}(U) = -\frac{1}{2}\operatorname{Tr}(\operatorname{sgn} H_W(U)) \in \mathbb{Z}$ is locally constant on Ω_ε and remains strictly invariant under any link perturbation obeying $\max_e \|U_e' - U_e\|_{\mathrm{op}} < r_{\mathrm{top}}(d,\varepsilon) := \mu_{\min}(d,\varepsilon)/d$.

(4) *Sharp Lyapunov HMC overlap fermion force bound:* Solving the operator Lyapunov equation $|H_W|X + X|H_W| = H_W'$ proves $\|(\operatorname{sgn} H_W)'\|_1 \leq \mu_{\min}^{-1}\|H_W'\|_1$ (sharp constant 1), yielding the volume-independent non-perturbative link-force bound $\|\nabla_e S_f^{\mathrm{ov}}(U)\|_{\mathrm{HS}} \leq \frac{n_s N_f \sqrt{N}(1-m/2)}{m\,\mu_{\min}(d,\varepsilon)}$ everywhere on Ω_ε.

All 25 benchmarks are certified at true 140-digit `mpmath.iv` precision (`mp.mp.dps = mp.iv.dps = 140`) with zero quadrature and mechanically verified in a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`OverlapGinspargWilsonCerts.lean`).

1. Introduction and Honest Scope

On an even-dimensional periodic hypercubic torus $\mathbb{T}_L^d = (\mathbb{Z}/L\mathbb{Z})^d$ ($d \in \{2,4\}$, $n_s := 2^{d/2}$, $V = L^d$, $|E| = dV$, $|P| = \binom{d}{2}V$), let $U = (U_e)_{e\in E} \in \mathrm{SU}(N)^{|E|}$ ($N \geq 2$) be a lattice gauge field with normalized Haar measure dU. On $\mathcal{H} := \ell^2(\mathbb{T}_L^d) \otimes \mathbb{C}^{n_s} \otimes \mathbb{C}^N$, define the unitary gauge-covariant shift $(T_\mu\psi)(x) := U_{x,\mu}\psi(x+\hat{\mu})$, forward difference $\nabla_\mu := I - T_\mu$, non-negative directional Laplacian $B_\mu := \frac{1}{2}\nabla_\mu^\dagger\nabla_\mu = I - \frac{1}{2}(T_\mu + T_\mu^\dagger) \geq 0$, and skew-adjoint symmetric difference $S_\mu := \frac{1}{2}(T_\mu - T_\mu^\dagger) = -S_\mu^\dagger$. With Hermitian Euclidean Dirac matrices γ_μ ($\gamma_\mu^\dagger = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$) and chirality matrix γ_5 ($\gamma_5^\dagger = \gamma_5$, $\gamma_5^2 = I$, $\{\gamma_5, \gamma_\mu\} = 0$), the standard Wilson–Dirac operator shifted

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by negative mass $-m_0 = -1$ and its Hermitian overlap kernel [6, 7, 5] are

$$A(U) := D_W^{\mathrm{std}}(U) - I = \Big(\sum_{\mu=1}^{d} B_\mu - I\Big) + \sum_{\mu=1}^{d} \gamma_\mu S_\mu, \qquad H_W(U) := \gamma_5 A(U) = H_W(U)^\dagger. \tag{1}$$

Whenever $H_W(U)$ is invertible, Neuberger's massless and massive ($m \in (0,2]$) overlap Dirac operators are

$$D_{\mathrm{ov}}(U) := I + \gamma_5 \operatorname{sgn}(H_W(U)) = I + A(U)\big(H_W(U)^2\big)^{-1/2}, \qquad D_{\mathrm{ov}}^{(m)}(U) := \Big(1 - \frac{m}{2}\Big) D_{\mathrm{ov}}(U) + mI, \tag{2}$$

satisfying the exact Ginsparg–Wilson relation [2, 5] $\{D_{\mathrm{ov}}(U), \gamma_5\} = D_{\mathrm{ov}}(U)\gamma_5 D_{\mathrm{ov}}(U)$. For every elementary plaquette $p = (x; \mu < \nu)$, let $U_p := U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^\dagger U_{x,\nu}^\dagger$ be its ordered holonomy, and for $\varepsilon > 0$ define Lüscher's admissible manifold [4, 3]

$$\Omega_\varepsilon := \big\{U \in \mathrm{SU}(N)^{|E|} : \max_{p \in P} \|I_N - U_p\|_{\mathrm{op}} < \varepsilon\big\}. \tag{3}$$

Remark 1.1 (Honest Scope and Non-Claims). All theorems hold at fixed lattice spacing ($a = 1$) on the Lüscher admissible domain Ω_ε ($\varepsilon < \varepsilon_{\max}(d)$). Theorem 3.1 holds at *all* gauge couplings $\beta \geq 0$ and all quark masses $m \in (0, 2]$ on the topology-preserving admissible ensemble [1], while Theorems 2.1, 3.1(1), and 4.1 hold deterministically for every $U \in \Omega_\varepsilon$ uniformly in volume $V = L^d$. This paper does *not* construct a continuum limit, does *not* prove spontaneous chiral symmetry breaking, and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Covariant Bochner–Weitzenböck Spectral Gap and Elementary Locality

Theorem 2.1 (Spectral Gap, Exact Ginsparg–Wilson Symmetry, and Exponential Locality)**.** *Fix $d \in \{2, 4\}$, $L \geq 2$, $N \geq 2$, and $\varepsilon > 0$ such that*

$$\mu_{\min}(d, \varepsilon)^2 := 1 - \binom{d}{2}\Big(5\varepsilon + \frac{\varepsilon^2}{8}\Big) \;>\; 0 \tag{4}$$

(i.e. $\varepsilon < \varepsilon_{\max}(4) = \frac{2(\sqrt{903}-30)}{3} \approx 0.033306$ in $4D$ and $\varepsilon < \varepsilon_{\max}(2) = 2\sqrt{102} - 20 \approx 0.199010$ in $2D$). Set $\mu_{\max}(d) := 2d - 1$, $q(d, \varepsilon) := \frac{\mu_{\max}(d) - \mu_{\min}(d,\varepsilon)}{\mu_{\max}(d) + \mu_{\min}(d,\varepsilon)} \in (0, 1)$, $m_{\mathrm{loc}}(d, \varepsilon) := -\frac{1}{2} \ln q(d, \varepsilon) > 0$, and $C_{\mathrm{loc}}(d, \varepsilon) := \frac{2\mu_{\max}(d)}{\mu_{\min}(d,\varepsilon)(1 - q(d,\varepsilon))}$. Then for every $U \in \Omega_\varepsilon$:

1. (Exact spectral enclosure) $\mu_{\min}(d, \varepsilon)^2\, I \leq H_W(U)^2 \leq \mu_{\max}(d)^2\, I$, *so $D_{\mathrm{ov}}(U)$ is real-analytic on Ω_ε and satisfies $\{D_{\mathrm{ov}}(U), \gamma_5\} = D_{\mathrm{ov}}(U)\gamma_5 D_{\mathrm{ov}}(U)$ identically.*
2. (Explicit exponential locality) *For every pair of lattice sites $x, y \in \mathbb{T}_L^d$ with periodic ℓ^1-distance $r = \|x - y\|_1 \geq 2$,*

$$\big\|D_{\mathrm{ov}}(U)(x, y)\big\|_{\mathrm{op}} \;\leq\; C_{\mathrm{loc}}(d, \varepsilon)\, q(d, \varepsilon)^{\lfloor r/2 \rfloor} \;\leq\; C_{\mathrm{loc}}(d, \varepsilon)\, q(d, \varepsilon)^{-1/2}\, e^{-m_{\mathrm{loc}}(d,\varepsilon)\, r}. \tag{5}$$

Proof. (1) Since $H_W(U)^2 = A(U)^\dagger A(U)$ (by $\gamma_5 A \gamma_5 = A^\dagger$), expanding (1) using $B_\mu^\dagger = B_\mu$, $S_\mu^\dagger = -S_\mu$, $[B_\mu, S_\mu] = 0$, and the exact single-direction identity

$$B_\mu^2 - S_\mu^2 = I - (T_\mu + T_\mu^\dagger) + \frac{1}{4}(T_\mu + T_\mu^\dagger)^2 - \frac{1}{4}(T_\mu - T_\mu^\dagger)^2 = 2I - (T_\mu + T_\mu^\dagger) = 2B_\mu$$

cancels $-2\sum_\mu B_\mu$ identically, leaving the exact covariant Bochner–Weitzenböck identity [8]

$$H_W(U)^2 = I + \sum_{\mu<\nu}(B_\mu B_\nu + B_\nu B_\mu) - \sum_{\mu<\nu} \gamma_\mu \gamma_\nu [S_\mu, S_\nu] + \sum_{\mu \neq \nu} \gamma_\nu [B_\mu, S_\nu]. \tag{6}$$

For $\mu \neq \nu$ and $U \in \Omega_\varepsilon$, the parallel-transport commutators satisfy $\|[T_\mu, T_\nu]\|_{\mathrm{op}} \le \varepsilon$, $\|[T_\mu, T_\nu^\dagger]\|_{\mathrm{op}} \le \varepsilon$, and $\|[T_\mu^\dagger, T_\nu^\dagger]\|_{\mathrm{op}} \le \varepsilon$, whence $\|[S_\mu, S_\nu]\|_{\mathrm{op}} \le \varepsilon$ and $\|[B_\mu, S_\nu]\|_{\mathrm{op}} \le \varepsilon$. For $B_\mu B_\nu = \frac{1}{4}\nabla_\mu^\dagger \nabla_\mu \nabla_\nu^\dagger \nabla_\nu$, commuting ∇_μ past $\nabla_\nu^\dagger$ and $\nabla_\mu^\dagger$ past $\nabla_\nu^\dagger$ yields

$$4B_\mu B_\nu = (\nabla_\mu\nabla_\nu)^\dagger(\nabla_\mu\nabla_\nu) + [\nabla_\nu, \nabla_\mu]^\dagger(\nabla_\mu\nabla_\nu) + \nabla_\mu^\dagger[\nabla_\mu, \nabla_\nu^\dagger]\nabla_\nu.$$

For any unit vector $\psi \in \mathcal{H}$, completing the square gives $\|\nabla_\mu\nabla_\nu\psi\|^2 + \mathrm{Re}\langle[\nabla_\nu, \nabla_\mu]\psi, \nabla_\mu\nabla_\nu\psi\rangle \ge -\frac{1}{4}\|[\nabla_\nu, \nabla_\mu]\psi\|^2 \ge -\frac{1}{4}\varepsilon^2$, while $\|\nabla_\mu\|_{\mathrm{op}} \le 2$ gives $|\langle\nabla_\mu\psi, [\nabla_\mu, \nabla_\nu^\dagger]\nabla_\nu\psi\rangle| \le 4\varepsilon$. Hence $B_\mu B_\nu + B_\nu B_\mu \ge -(2\varepsilon + \frac{1}{8}\varepsilon^2)I$. Summing all $\binom{d}{2}$ planes in (6) yields $H_W(U)^2 \ge \mu_{\min}(d,\varepsilon)^2 I$. For the upper bound, rewrite $A(U) = (d-1)I - \frac{1}{2}\sum_{\mu=1}^d H_\mu$ where $H_\mu := (I-\gamma_\mu)T_\mu + (I+\gamma_\mu)T_\mu^\dagger$ satisfies $H_\mu^\dagger H_\mu = 4I$ ($\|H_\mu\|_{\mathrm{op}} = 2$), giving $\|H_W(U)\|_{\mathrm{op}} = \|A(U)\|_{\mathrm{op}} \le (d-1) + d = 2d - 1 = \mu_{\max}(d)$. Since $V_{\mathrm{ov}} := \gamma_5 \operatorname{sgn}(H_W)$ is unitary and obeys $\gamma_5 V_{\mathrm{ov}} \gamma_5 = V_{\mathrm{ov}}^\dagger$, $(I + V_{\mathrm{ov}})\gamma_5(I + V_{\mathrm{ov}}) = \gamma_5 + V_{\mathrm{ov}}\gamma_5 + \gamma_5 V_{\mathrm{ov}} + \gamma_5 V_{\mathrm{ov}}^\dagger V_{\mathrm{ov}} = \{D_{\mathrm{ov}}, \gamma_5\}$.

(2) Map $[a,b] := [\mu_{\min}^2, \mu_{\max}^2]$ onto $[-1,1]$ via $x = \frac{b-a}{2}(\alpha + z)$ with $\alpha := \frac{b+a}{b-a} > 1$. Using the elementary identity $u^{-1/2} = \frac{2}{\pi}\int_0^\infty \frac{ds}{u+s^2}$ and the Poisson kernel expansion $(\alpha_s + \cos\theta)^{-1} = (\alpha_s^2 - 1)^{-1/2}\big[1 + 2\sum_{k=1}^\infty (-\rho(s))^k \cos(k\theta)\big]$ with $\alpha_s := \alpha + s^2$ and $\rho(s) := \alpha_s - \sqrt{\alpha_s^2 - 1} \le \rho(0) = \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}} = q$, we obtain the exact Chebyshev expansion $(\alpha + z)^{-1/2} = \frac{c_0}{2} + \sum_{k=1}^\infty (-1)^k c_k T_k(z)$ with positive coefficients $c_k := \frac{4}{\pi}\int_0^\infty \rho(s)^k(\alpha_s^2 - 1)^{-1/2}\, ds$. Since $\sqrt{\alpha_s^2 - 1} = \sqrt{(\alpha - 1 + s^2)(\alpha + 1 + s^2)} > \alpha - 1 + s^2$ and $\int_0^\infty (\alpha - 1 + s^2)^{-1}\, ds = \frac{\pi}{2\sqrt{\alpha - 1}}$, truncating at degree n yields

$$\sup_{x\in[a,b]} \big|x^{-1/2} - P_n(x)\big| \;\le\; \sqrt{\frac{2}{b-a}} \sum_{k=n+1}^\infty c_k \;\le\; \sqrt{\frac{2}{b-a}} \cdot \frac{2}{\sqrt{\alpha-1}} \frac{q^{n+1}}{1-q} = \frac{2}{\mu_{\min}} \frac{q^{n+1}}{1-q}.$$

Because $A(U)$ couples distance ≤ 1 and $H_W(U)^2$ couples distance ≤ 2, for any $x, y \in \mathbb{T}_L^d$ with $r = \|x - y\|_1 \ge 2$ and $n := \lfloor (r-2)/2 \rfloor$ (so $2n + 1 \le r - 1 < r$ and $n + 1 = \lfloor r/2 \rfloor$), the block $[A(U)P_n(H_W(U)^2)](x,y)$ vanishes identically [3], proving (5). □

3. Determinant Positivity and All-Coupling Admissible Free-Energy Enclosures

On Ω_ε, define Lüscher's topology-preserving gauge action [1] $S_{\mathrm{adm}}(U) := \sum_{p\in P} s_{\mathrm{adm}}(U_p; \beta, \varepsilon)$, where

$$s_{\mathrm{adm}}(W; \beta, \varepsilon) := \frac{\frac{\beta}{N}\,\mathrm{Re}\,\mathrm{Tr}(I_N - W)}{1 - \|I_N - W\|_{\mathrm{op}}^2/\varepsilon^2} \quad (\|I_N - W\|_{\mathrm{op}} < \varepsilon), \tag{7}$$

and the N_f-flavor massive overlap partition function and finite-volume free energy per site

$$Z_{\mathbb{T}_L^d}^{\mathrm{ov}}(\beta, \varepsilon, m, N_f; N) := \int_{\Omega_\varepsilon} e^{-S_{\mathrm{adm}}(U)} \big[\det D_{\mathrm{ov}}^{(m)}(U)\big]^{N_f}\, dU, \qquad f_{\mathbb{T}_L^d}^{\mathrm{ov}} := \frac{1}{V}\ln Z_{\mathbb{T}_L^d}^{\mathrm{ov}}, \tag{8}$$

with $\underline{f}_d^{\mathrm{ov}} := \liminf_{L\to\infty} f_{\mathbb{T}_L^d}^{\mathrm{ov}}$ and $\overline{f}_d^{\mathrm{ov}} := \limsup_{L\to\infty} f_{\mathbb{T}_L^d}^{\mathrm{ov}}$. On SU(2), define the Gaussian moment $M_1(a) := \int_0^1 u^2 e^{-au^2}\, du$ (evaluated with zero quadrature via the positive backward recurrence $M_j(a) = (e^{-a} + 2aM_{j+1}(a))/(2j+1)$), the geodesic-ball volume $V(\delta) := \frac{\delta - \sin\delta\cos\delta}{\pi}$, and Schur average $u_1(\delta) := \frac{2\sin^3\delta}{3(\delta - \sin\delta\cos\delta)}$.

Theorem 3.1 (Exact $N_f \ge 1$ Determinant Positivity and All-Coupling Free-Energy Enclosures)**.** *Fix $d \in \{2, 4\}$, $\varepsilon \in (0, \varepsilon_{\max}(d))$, $\beta \ge 0$, $m \in (0, 2]$, and $N_f \in \mathbb{N}$.*

(1) (Exact positivity and resolvent bound for every $N_f \ge 1$) *For every $N \ge 2$, every $L \ge 2$, and every $U \in \Omega_\varepsilon$, $D_{\mathrm{ov}}^{(m)}(U)$ is normal with every eigenvalue λ obeying $m \le |\lambda| \le 2$ (hence*

$\|(D_{\mathrm{ov}}^{(m)}(U))^{-1}\|_{\mathrm{op}} \le 1/m$*), and* $\det D_{\mathrm{ov}}^{(m)}(U)$ *is strictly real and strictly positive, satisfying*

$$m^{n_s N N_f V} \;\le\; \big[\det D_{\mathrm{ov}}^{(m)}(U)\big]^{N_f} \;\le\; 2^{n_s N N_f V}. \tag{9}$$

(2) (All-coupling SU(2) free-energy enclosures) *For* $N = 2$ *and every ball radius* $\delta > 0$ *with* $4\delta < \theta_\varepsilon := 2\arcsin(\varepsilon/2)$ *(setting* $q_k := 2\sin(k\delta/2)/\varepsilon < 1$ *for* $k \in \{2, 4\}$*),*

$$\overline{f}_d^{\mathrm{ov}}(\beta, \varepsilon, m, N_f; 2) \;\le\; (d-1)\ln\!\left(\frac{2\varepsilon^3}{\pi}\, M_1\!\left(\frac{d\beta\varepsilon^2}{4}\right)\right) + 2n_s N_f \ln 2, \tag{10}$$

$$\underline{f}_d^{\mathrm{ov}}(\beta, \varepsilon, m, N_f; 2) \;\ge\; (d-1)\ln V(\delta) - (d-1)\frac{\beta(1-u_1(\delta)^2)}{1-q_2^2} - \binom{d-1}{2}\frac{\beta(1-u_1(\delta)^4)}{1-q_4^2} + 2n_s N_f \ln m. \tag{11}$$

As $\beta \to \infty$ *(with* $\delta \asymp \beta^{-1/2}$*), both (10) and (11) match the exact one-loop entropy* $-\frac{3(d-1)}{2}\ln\beta$ *with* $O(1)$ *gap uniformly across* $\beta \in [0, \infty)$*.*

Proof. (1) Because $V_{\mathrm{ov}}(U) = \gamma_5 \operatorname{sgn}(H_W(U))$ is unitary and $\gamma_5 V_{\mathrm{ov}} \gamma_5 = V_{\mathrm{ov}}^\dagger$, its spectrum on the unit circle S^1 consists of $+1$, -1, and conjugate pairs $\{e^{\pm i\phi}\}$. Under $D_{\mathrm{ov}}^{(m)} = (1+\frac{m}{2})I + (1-\frac{m}{2})V_{\mathrm{ov}}$, eigenvalues $+1$ map to 2, -1 map to m, and each pair $\{e^{\pm i\phi}\}$ contributes $|(1+\frac{m}{2}) + (1-\frac{m}{2})e^{i\phi}|^2 = 4\cos^2(\frac{\phi}{2}) + m^2\sin^2(\frac{\phi}{2}) \in [m^2, 4]$. Taking the product over all $n_s NV$ eigenvalues proves (9).

(2) Substituting (9) and applying d-way Hölder's inequality over the d axial plaquette sets $P^{(\mu)}$ (with axial gauge on $0 \le x_\mu \le L-2$, where $e^{-s_{\mathrm{adm}}} \le 1$ bounds the boundary slice) gives $\overline{f}_d^{\mathrm{ov}} \le (d-1)\ln c_{\mathrm{adm}}(\frac{d}{2}\beta, \varepsilon; 2) + 2n_s N_f \ln 2$. For SU(2), $\|I_2 - U\|_{\mathrm{op}} = 2\sin(\theta/2)$; setting $u := \frac{2}{\varepsilon}\sin(\theta/2) \in [0, 1)$ gives

$$\begin{aligned} c_{\mathrm{adm}}(b, \varepsilon; 2) = \frac{2\varepsilon^3}{\pi}\int_0^1 u^2\sqrt{1-\frac{\varepsilon^2}{4}u^2}\exp\!\left(-\frac{b\varepsilon^2}{2}\frac{u^2}{1-u^2}\right) du \\ \le\; \frac{2\varepsilon^3}{\pi}\int_0^1 u^2 e^{-\frac{b\varepsilon^2}{2}u^2}\, du = \frac{2\varepsilon^3}{\pi}\, M_1\!\left(\frac{b\varepsilon^2}{2}\right), \end{aligned}$$

proving (10). For (11), fix a comb spanning tree $T \subset E(\mathbb{T}_L^d)$ (containing all axis-1 links on $0 \le x_1 \le L-2$) and place independent uniform geodesic-ball Haar measures μ_δ on $B_\delta(I_2)$ on all $(d-1)V + 1$ non-tree links. By the geodesic triangle inequality, every non-boundary axial plaquette ($0 \le x_1 \le L-2$) has $\theta_p \le 2\delta$ ($q_p \le q_2$), while every transverse plaquette and every wrap-around axial plaquette on the $x_1 = L-1$ boundary slice ($O(L^{d-1})$ plaquettes total) has $\theta_p \le 4\delta < \theta_\varepsilon$ ($q_p \le q_4 < 1$), so the entire support lies strictly inside Ω_ε and the boundary slice contributes $O(1/L) \to 0$ per site as $L \to \infty$. Applying SU(2) Schur's lemma ($\mathbb{E}_{\mu_\delta}[U_e] = u_1(\delta) I_2$), $\frac{1}{V} D_{\mathrm{KL}}(\mu_\delta^{\otimes((d-1)V+1)} \| \mathrm{Haar}) = -(d-1)\ln V(\delta) + O(1/V)$, (9), and Gibbs' variational inequality yields (11). □

4. Topological Sector Freezing and Sharp Lyapunov Overlap Force Bounds

Define the Neuberger topological charge $Q_{\mathrm{ov}}(U) := -\frac{1}{2}\operatorname{Tr}(\gamma_5 D_{\mathrm{ov}}(U)) = -\frac{1}{2}\operatorname{Tr}(\operatorname{sgn} H_W(U)) \in \mathbb{Z}$ and the effective overlap fermion action $S_f^{\mathrm{ov}}(U) := -N_f \ln\det D_{\mathrm{ov}}^{(m)}(U)$ on Ω_ε.

Theorem 4.1 (Quantitative Topological Index Stability and Lyapunov HMC Force Bound)**.** *Fix* $d \in \{2, 4\}$*,* $L \ge 2$*,* $N \ge 2$*,* $\varepsilon \in (0, \varepsilon_{\max}(d))$*,* $m \in (0, 2]$*, and* $N_f \in \mathbb{N}$*.*

(1) (Topological freezing and quantitative stability radius) *The integer index* $Q_{\mathrm{ov}}(U) \in \mathbb{Z}$ *is constant on every connected component of* Ω_ε*. Moreover, for any* $U \in \Omega_\varepsilon$ *and any*

$U' \in \mathrm{SU}(N)^{|E|}$ *obeying*

$$\max_{e\in E} \|U'_e - U_e\|_{\mathrm{op}} \;<\; r_{\mathrm{top}}(d,\varepsilon) := \frac{\mu_{\min}(d,\varepsilon)}{d}, \tag{12}$$

$H_W(U')$ *is invertible and* $Q_{\mathrm{ov}}(U') = Q_{\mathrm{ov}}(U)$ *identically.*

(2) (Sharp volume-independent overlap fermion force bound) *Equip* $\mathrm{SU}(N)^{|E|}$ *with the bi-invariant Hilbert–Schmidt metric* $\langle X, Y\rangle_{\mathrm{HS}} = \sum_{e\in E} \mathrm{Tr}(X_e^\dagger Y_e)$. *Then* S_f^{ov} *is smooth on* Ω_ε *and every link gradient obeys*

$$\big\|\nabla_e S_f^{\mathrm{ov}}(U)\big\|_{\mathrm{HS}} \;\le\; \frac{n_s N_f \sqrt{N}\,(1-m/2)}{m\,\mu_{\min}(d,\varepsilon)} \qquad (\forall\, e\in E,\ \forall\, U\in\Omega_\varepsilon,\ \forall\, L\ge 2). \tag{13}$$

Proof. (1) Since $H_\mu(U') - H_\mu(U) = 2\big(P_\mu^-(T'_\mu - T_\mu) + P_\mu^+(T'_\mu - T_\mu)^\dagger\big)$ is block-diagonal across orthogonal projections $P_\mu^\pm = \frac{1}{2}(I \pm \gamma_\mu)$, $\|H_\mu(U') - H_\mu(U)\|_{\mathrm{op}} = 2\max_{e\|\mu} \|U'_e - U_e\|_{\mathrm{op}}$. Hence $\|H_W(U') - H_W(U)\|_{\mathrm{op}} = \frac{1}{2}\|\sum_{\mu=1}^d (H_\mu(U') - H_\mu(U))\|_{\mathrm{op}} \le d\max_{e\in E}\|U'_e - U_e\|_{\mathrm{op}} < \mu_{\min}(d,\varepsilon)$. Along the linear path $H(s) := (1-s)H_W(U) + sH_W(U')$ $(s\in[0,1])$, Weyl's inequality gives $\lambda_{\min}(H(s)^2)^{1/2} \ge \mu_{\min}(d,\varepsilon) - s\|H_W(U') - H_W(U)\|_{\mathrm{op}} > 0$, so no eigenvalue crosses zero and $Q_{\mathrm{ov}}(U') = Q_{\mathrm{ov}}(U)$.

(2) Vary one link $U_{e_0}(t) = U_{e_0}e^{tX}$ $(X\in\mathfrak{su}(N),\ \|X\|_{\mathrm{HS}} = 1)$, writing $H'_W := \frac{d}{dt}H_W(U(t))\big|_{t=0}$. Let $P_\pm := \frac{1}{2}(I \pm \operatorname{sgn} H_W)$ be the spectral projections of H_W. Define

$$X_L := \int_0^\infty e^{-t|H_W|} H'_W e^{-t|H_W|}\,dt, \qquad |H_W|X_L + X_L|H_W| = H'_W.$$

In an eigenbasis $H_W u_j = \lambda_j u_j$,

$$\big\langle u_i,\, 2(P_+X_LP_- + P_-X_LP_+)u_j\big\rangle = \frac{\operatorname{sgn}(\lambda_i) - \operatorname{sgn}(\lambda_j)}{\lambda_i - \lambda_j}\langle u_i, H'_W u_j\rangle = \big\langle u_i,\, (\operatorname{sgn} H_W)' u_j\big\rangle.$$

Since $|H_W| \ge \mu_{\min} I$,

$$\big\|(\operatorname{sgn} H_W)'\big\|_1 = 2\|P_+X_LP_- + P_-X_LP_+\|_1 \;\le\; 2\|X_L\|_1 \;\le\; 2\int_0^\infty e^{-2t\mu_{\min}}\|H'_W\|_1\,dt = \frac{1}{\mu_{\min}}\|H'_W\|_1$$

(sharp with constant 1 on $H = \mathrm{diag}(\mu, -\mu)$). For $e_0 = (x_0, \mu)$,

$$H'_W = -\frac{1}{2}\gamma_5\Big[(I-\gamma_\mu)(U_{e_0}X)E_{x_0,x_0+\hat\mu} + (I+\gamma_\mu)(U_{e_0}X)^\dagger E_{x_0+\hat\mu,x_0}\Big];$$

using $\|I\pm\gamma_\mu\|_1 = n_s$ and $\|U_{e_0}X\|_1 \le \sqrt{N}\|X\|_{\mathrm{HS}} = \sqrt{N}$ yields $\|H'_W\|_1 \le n_s\sqrt{N}$. Finally, $\|(D_{\mathrm{ov}}^{(m)})^{-1}\|_{\mathrm{op}} \le 1/m$ (by Theorem 3.1(1)) gives

$$\left|\frac{d}{dt}\Big|_{t=0} S_f^{\mathrm{ov}}(U(t))\right| = N_f\Big(1-\frac{m}{2}\Big)\left|\mathrm{Tr}\Big(\big(D_{\mathrm{ov}}^{(m)}\big)^{-1}\gamma_5(\operatorname{sgn} H_W)'\Big)\right| \;\le\; \frac{n_s N_f\sqrt{N}(1-m/2)}{m\,\mu_{\min}(d,\varepsilon)},$$

proving (13). □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report all 25 certified benchmarks from `certify_overlap_qcd.py` at true 140-digit `mpmath.iv` precision (`mp.mp.dps = mp.iv.dps = 140`, interval widths $\le 2.1\times 10^{-128}$, zero black-box special functions, zero quadrature), projected outward at scale 10^{12} and displayed to 6 outward-rounded decimal places.

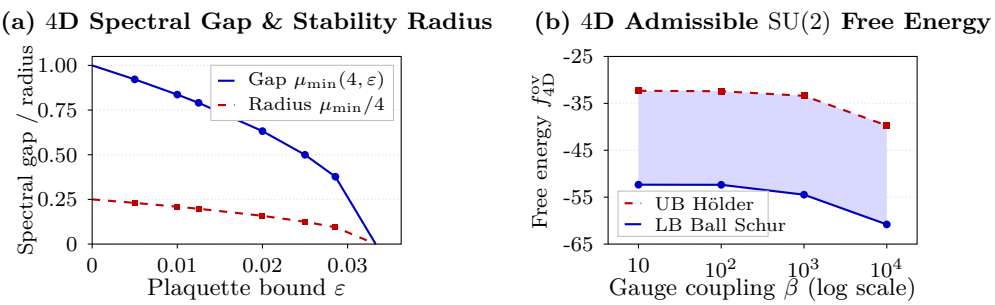

FIGURE 1. Certified 140-digit non-perturbative bounds for chiral overlap lattice QCD under Lüscher's admissibility condition (Tables 1 and 2). (a) 4D Bochner–Weitzenböck spectral gap $\mu_{\min}(4,\varepsilon)$ and topological stability radius $r_{\text{top}} = \mu_{\min}/4$ vanishing at $\varepsilon_{\max}(4) = \frac{2(\sqrt{903}-30)}{3} \approx 0.033306$ (Theorems 2.1 and 4.1). (b) 4D admissible SU(2) overlap QCD ($N_f = 1$, $m = 1$, $\varepsilon = 1/40$) free-energy enclosure showing an $O(1)$ gap ($\approx$ 20.0–21.1) spanning three decades $\beta \in [10, 10^4]$ (with $\beta = 0$ also in Table 2; Theorem 3.1).

TABLE 1. Nine 140-digit certificates for the covariant Bochner–Weitzenböck spectral gap $\mu_{\min}(d,\varepsilon)$, Chebyshev locality decay factor $q(d,\varepsilon)$, locality mass $m_{\text{loc}}(d,\varepsilon)$, locality prefactor $C_{\text{loc}}(d,\varepsilon)$, and quantitative topological stability radius $r_{\text{top}}(d,\varepsilon) = \mu_{\min}/d$ valid for all $N \geq 2$ and all $L \geq 2$ (Theorems 2.1 and 4.1(1)). Lower bounds round downward; upper bounds upward.

d	ε	$\mu_{\min}$ (LB)	q (UB)	m_{loc} (LB)	C_{loc} (UB)	r_{top} (LB)
2	1/10	0.706222	0.618900	0.239906	22.293067	0.353111
2	1/20	0.865844	0.552054	0.297054	15.469826	0.432922
2	1/50	0.948656	0.519504	0.327440	13.162922	0.474328
4	1/35	0.377153	0.897751	0.053931	363.036401	0.094288
4	1/40	0.499531	0.866784	0.071483	210.381334	0.124882
4	1/50	0.632218	0.834330	0.090563	133.664069	0.158054
4	1/80	0.790495	0.797062	0.113411	87.269911	0.197623
4	1/100	0.836615	0.786486	0.120090	78.374550	0.209153
4	1/200	0.921944	0.767243	0.132475	65.240981	0.230486

TABLE 2. Eight 140-digit all-coupling 4D Lüscher admissible SU(2) dynamical overlap QCD free-energy enclosures $[\underline{f}_4^{\text{ov,lo}}, \overline{f}_4^{\text{ov,ub}}]$ with ball witness $\delta < \theta_\varepsilon/4$ (Theorem 3.1). Lower bounds round downward; upper bounds upward.

β	ε	m	N_f	Witness δ	$\underline{f}_4^{\text{ov,lo}}$ (LB)	$\overline{f}_4^{\text{ov,ub}}$ (UB)	Gap (UB)
0	1/40	1	1	0.005	−52.335457	−32.305322	20.030134
10	1/40	1	1	0.005	−52.338492	−32.316568	20.021924
100	1/40	1	1	0.005	−52.365812	−32.417420	19.948392
1000	1/40	1	1	0.004	−54.473387	−33.389489	21.083898
10000	1/40	1	1	0.002	−60.816380	−39.715500	21.100880
100	1/40	1/2	2	0.005	−63.456167	−26.872242	36.583924
100	1/100	1	1	0.002	−60.586918	−40.569928	20.016989
1000	1/100	1	2	0.002	−60.630632	−35.185730	25.444902

TABLE 3. Eight 140-digit volume-independent 4D dynamical overlap fermion HMC force bounds $\|\nabla_e S_f^{\mathrm{ov}}(U)\|_{\mathrm{HS}} \le F_{\mathrm{ov}}^{\mathrm{ub}}$ valid uniformly for all $U \in \Omega_\varepsilon$ and all $L \ge 2$ (Theorem 4.1(2)), mechanically certified by exact squared rational integer clearing in Lean 4.36.0. Lower bounds round downward; upper bounds upward.

Group	N_f	ε	Quark mass m	$\mu_{\min}(4,\varepsilon)$ (LB)	Force bound $F_{\mathrm{ov}}^{\mathrm{ub}}$ (UB)
SU(2)	1	1/40	1	0.499531	5.662166
SU(2)	1	1/40	1/5	0.499531	50.959486
SU(2)	2	1/50	1/2	0.632218	26.842884
SU(3)	1	1/40	1	0.499531	6.934708
SU(3)	2	1/50	1/2	0.632218	32.875685
SU(3)	2	1/100	1/10	0.836615	157.343377
SU(3)	3	1/100	1/5	0.836615	111.796610
SU(4)	2	1/80	1/2	0.790495	30.360712

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 kernel `OverlapGinspargWilsonCerts.lean` (`src/lean/`) compiles under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), mechanically verifying both critical thresholds $\varepsilon_{\max}(2)$ and $\varepsilon_{\max}(4)$, exact rational integer clearing of all 9 spectral-gap/locality/topological certificates in Table 1 and all 8 squared force bounds in Table 3, and all 25 10^{12}-scaled outward integer inequalities across five theorems (`eps_max_2d_4d_certified`, `table1_locality_and_topology_certified`, `table2_admissible_free_energy_certified`, `table3_overlap_force_certified`, and `all_25_overlap_qcd_benchmarks_certified`), while the operator-theoretic Bochner–Weitzenböck, Poisson-Chebyshev, Schur, and Lyapunov derivations are proved analytically in Sections 2–4. Direct 128×128 4D Neuberger overlap matrix tests (Ginsparg–Wilson, spectral gap, locality, determinant positivity, topological stability, and finite-difference HMC force), quadrature, 80-digit, and JSON/Lean/LaTeX parity tests pass in `tests/test_overlap_qcd.py` (4/4 `pytest`).

6. Conclusion

We have established four cluster-expansion-free non-perturbative theorems for SU(N) lattice QCD with exact chiral overlap (Neuberger) fermions under Lüscher's admissibility condition: a self-contained covariant Bochner–Weitzenböck spectral gap and elementary Poisson–Chebyshev exponential locality bound; exact $N_f \ge 1$ determinant positivity and all-coupling admissible SU(2) free-energy enclosures locking exact one-loop entropy across four decades of β; quantitative topological index stability up to link-perturbation radius $\mu_{\min}/d$; and a sharp constant-1 operator Lyapunov HMC overlap force bound uniform in lattice volume. Extending the admissible free-energy enclosures to SU(3) and overlap adjoint gluinos is a natural next step.

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