EXACT ALL-COUPLING NON-PERTURBATIVE BOUNDS FOR LÜSCHER'S LATTICE YANG–MILLS GRADIENT FLOW (WILSON FLOW) ON $\mathrm{SU}(N)^{|E|}$

LLUÍS ERIKSSON

Abstract. We establish four cluster-expansion-free, non-perturbative theorems for Lüscher's lattice Yang–Mills gradient flow (the Wilson flow) $\dot{V}_e(\tau) = -N(\nabla_e S_W(V(\tau)))V_e(\tau)$ on $\mathrm{SU}(N)^{|E|}$ for all dimensions $d \geq 2$, all colors $N \geq 2$, and all flow times $\tau \geq 0$. First, using the exact quadratic Casimir of the bi-invariant Hilbert–Schmidt metric on $\mathrm{SU}(N)$, we prove the exact Laplace–Beltrami Liouville identity $N\Delta_{\mathrm{LB}} S_W(V) = 4(N^2-1)(|P| - S_W(V))$ on every finite lattice Λ, yielding exact global Jacobian contraction bounds and an exact N-independent Riemannian Hessian norm bound $\|N \operatorname{Hess} S_W(V)\|_{\mathrm{op}} \leq 8(d-1)$ that implies volume- and N-uniform two-sided diffeomorphism Lipschitz enclosures $e^{-8(d-1)\tau} \leq \|d\Phi_\tau(V)[X]\|_{\mathrm{HS}}/\|X\|_{\mathrm{HS}} \leq e^{8(d-1)\tau}$. Second, for a single SU(2) plaquette initialized at Haar measure (and lifted via any maximal spanning tree to an exact volume-extensive trivializing diffeomorphism $\widetilde{\Psi}_\tau$ on every open 2D lattice Λ), we derive exact closed-form formulas for the flowed energy density $E^{\mathrm{2D}}_{\mathrm{flow}}(0,\tau) = 2x^2(3+x)/(1+x)^3$ ($x = e^{-4\tau}$) via four-harmonic Poisson-kernel orthogonality and for the Kullback–Leibler divergence $D_{\mathrm{KL}}(\mu^{(1)}_\tau \| \mu^{(1)}_{\mathrm{Haar}}) = 12\tau - 6\ln(2/(1+x)) + \frac{3}{2}((1-x)/(1+x))^2$ directly from the Liouville identity, giving rigorous all-coupling Gibbs–Donsker–Varadhan free-energy lower bounds that match the exact 2D free energy to 0.15% at $\beta = 500$. Third, combining the N-uniform Lipschitz enclosure with the Bakry–Émery curvature floor $\kappa_0(d,\gamma) = \frac{1}{2} - 8(d-1)\gamma > 0$ of the planar Wilson measure ($\beta = \gamma N^2$), we prove that the exact pushed-forward flowed gauge measure $\mu_\tau = (\Phi_\tau)_* \mu_\beta$ satisfies a volume-uniform logarithmic Sobolev inequality with constant $\rho_{\mathrm{LS}}(\tau)/N \geq \kappa_0(d,\gamma) e^{-16(d-1)\tau}$ for all $N \geq 2$ and $\tau \geq 0$, implying $O(1/N^2)$ planar variance bounds $N^2 \operatorname{Var}_{\mu_\tau}(\tau^2 E_p) \leq 4\tau^4 e^{16(d-1)\tau}/\kappa_0(d,\gamma)$ and sub-Gaussian Herbst concentration for Lüscher's scale-setting observable $\tau^2 E_p(\tau)$. All 25 quantitative benchmark certificates are computed in 140-digit outward-rounded interval arithmetic without black-box special functions or quadrature, cross-checked by direct RK4 Wilson-flow and finite-difference Laplace–Beltrami simulations on 2^3 SU(2) and SU(3) lattices, and their integer-scaled (10^{12}) benchmark inequalities and partial-fraction combinatorics are mechanically verified in a standalone zero-import, zero-axiom Lean 4.36.0 kernel.

1. Introduction

Lüscher's Yang–Mills gradient flow—the Wilson flow on the lattice—has become a foundational tool across lattice gauge theory, providing high-precision scale setting via the flowed action density $\tau^2\langle E(\tau)\rangle$, automatic ultraviolet renormalization of composite operators at positive flow time $\tau > 0$, topological charge smoothing, and the theoretical backbone of trivializing maps and flow-based sampling algorithms [9, 10, 11, 5, 1, 2]. Despite its central role in modern lattice QCD, rigorous non-perturbative control of the lattice gradient flow has largely remained restricted to small-τ perturbative expansions [11] or heuristic estimates of Jacobian determinants and sampling variances in neural trivializing flows [1, 2].

In this paper we establish exact, completely non-perturbative, cluster-expansion-free bounds for the lattice Wilson flow on $\mathrm{SU}(N)^{|E|}$ valid for every dimension $d \geq 2$, every number of colors $N \geq 2$, every finite periodic torus $\mathbb{T}^d_L$ ($L \geq 2$) or open hypercubic box Λ ($|P| \geq 1$), and every flow time $\tau \geq 0$ (with Theorems 2.2, 2.3, and 3.1 holding at all couplings/configurations and Theorem 4.1 holding in the Bakry–Émery strong-coupling window $\gamma < \frac{1}{16(d-1)}$). Our analysis rests exclusively on bi-invariant Riemannian geometry on $\mathrm{SU}(N)^{|E|}$, elementary Fourier orthogonality, Liouville's

Date: November 29, 2026.

Jacobian identity, and Bakry–Émery functional inequalities [3, 8, 4]. Specifically, we prove four main results:

(1) **Exact Liouville Laplacian identity and Jacobian bounds (Theorem 2.2):** With the bi-invariant Hilbert–Schmidt metric $\langle X, Y\rangle_{\mathrm{HS}} = \sum_{e\in E} \mathrm{Tr}(X_e^\dagger Y_e)$ and planar-scaled Wilson flow $\dot{V}_e = -N(\nabla_e S_W(V))V_e$, the Laplace–Beltrami operator acts on the Wilson action $S_W(V) = \sum_{p\in P}(1 - \frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\,U_p)$ by the exact linear eigenfunction relation

$$N\Delta_{\mathrm{LB}} S_W(V) = 4(N^2-1)\big(|P| - S_W(V)\big),$$

valid identically for all $V \in \mathrm{SU}(N)^{|E|}$, all $N \geq 2$, and all $d \geq 2$. Along every flow trajectory $V(\tau) = \Phi_\tau(V_0)$, the log-Jacobian satisfies the exact Liouville integral identity and the universal all-configuration bounds $-4(N^2-1)|P|\tau \leq \ln|\det d\Phi_\tau(V_0)| \leq 4(N^2-1)|P|\tau$ (with the sharp one-sided bound $\ln|\det d\Phi_\tau(V_0)| \leq 0$ whenever $S_W(V(\tau)) \leq |P|$ along the trajectory, and in particular globally whenever $S_W(V_0) \leq |P|$ by monotonic action dissipation).

(2) **N-independent Riemannian Hessian bound and two-sided Lipschitz envelopes (Theorem 2.3):** Using Cauchy–Schwarz in the Hilbert–Schmidt norm on each plaquette, we prove that the Riemannian Hessian of NS_W satisfies the exact, volume- and N-independent operator bound

$$\|N\,\mathrm{Hess}\, S_W(V)\|_{\mathrm{op}} \leq 8(d-1) \qquad \forall V \in \mathrm{SU}(N)^{|E|},\ \forall N \geq 2.$$

Grönwall's inequality along the variational equation immediately yields the sharp two-sided diffeomorphism distortion enclosure

$$e^{-8(d-1)\tau}\|X\|_{\mathrm{HS}} \leq \|d\Phi_\tau(V)[X]\|_{\mathrm{HS}} \leq e^{8(d-1)\tau}\|X\|_{\mathrm{HS}}$$

uniformly in lattice volume and $N \geq 2$.

(3) **Exact closed-form 2D SU(2) Haar-initialized flow and Lüscher trivializing-map bounds (Theorem 3.1):** On a single SU(2) plaquette, the exact link-variable Wilson flow reduces to $\dot{\theta} = -4\sin\theta$, yielding $\tan(\theta(\tau)/2) = x\tan(\theta_0/2)$ with $x = e^{-4\tau}$. Four-harmonic Poisson-kernel orthogonality gives the exact flowed energy density

$$E^{\mathrm{2D}}_{\mathrm{flow}}(0,\tau) = \frac{2x^2(3+x)}{(1+x)^3},$$

and integrating the exact Liouville identity $\frac{d}{d\tau}D_{\mathrm{KL}}(\mu_\tau^{(1)}\|\mu_{\mathrm{Haar}}^{(1)}) = 12(1 - E^{\mathrm{2D}}_{\mathrm{flow}}(0,\tau))$ gives the exact closed-form relative entropy

$$D_{\mathrm{KL}}(\mu_\tau^{(1)}\|\mu_{\mathrm{Haar}}^{(1)}) = 12\tau - 6\ln\left(\frac{2}{1+x}\right) + \frac{3}{2}\left(\frac{1-x}{1+x}\right)^2.$$

Lifting this single-plaquette flow via any maximal spanning tree $T \subset E$ produces an exact volume-extensive trivializing diffeomorphism $\widetilde{\Psi}_\tau$ on every open 2D lattice Λ whose Gibbs–Donsker–Varadhan variational bound certifies the exact 2D SU(2) thermodynamic free energy $f^W_{\mathrm{2D}}(\beta) = \ln(2I_1(\beta)/\beta)$ from below across all $\beta \geq 0$ (reaching 0.15% relative accuracy at $\beta = 500$).

(4) **Volume-uniform pushforward LSI, Herbst concentration, and $O(1/N^2)$ variance bounds (Theorem 4.1):** For the planar Wilson gauge measure $d\mu_\beta \propto e^{-\gamma N^2 S_W}dV$ in the Bakry–Émery strong-coupling window $\gamma < \frac{1}{16(d-1)}$ (where $\kappa_0(d,\gamma) := \frac{1}{2} - 8(d-1)\gamma > 0$), pushing the Bakry–Émery LSI forward through the exact multi-plaquette Wilson flow Φ_τ yields a volume-uniform log-Sobolev inequality for $\mu_\tau = (\Phi_\tau)_*\mu_\beta$:

$$\rho_{\mathrm{LS}}(\tau) \geq N\,\kappa_0(d,\gamma)\,e^{-16(d-1)\tau} \qquad \forall N \geq 2,\ \forall \tau \geq 0.$$

Combining this with the exact plaquette gradient bound $\|\nabla E_p\|_{\mathrm{HS}}^2 \le 4/N$ yields a non-perturbative $O(1/N^2)$ variance bound for Lüscher's flowed action density $\tau^2 E_p(\tau)$,

$$N^2 \operatorname{Var}_{\mu_\tau}(\tau^2 E_p) \le \frac{4\tau^4 e^{16(d-1)\tau}}{\kappa_0(d,\gamma)},$$

together with volume-uniform sub-Gaussian Herbst concentration $\mathbb{P}_{\mu_\tau}(|E_p - \mathbb{E}_{\mu_\tau}[E_p]| \ge \delta) \le 2\exp(-c_H N^2 \delta^2)$ with $c_H = \kappa_0(d,\gamma) e^{-16(d-1)\tau}/8$.

Every quantitative benchmark across Tables 1–3 (25 rows total) is computed via explicit 140-digit outward-rounded interval arithmetic (`mpmath.iv`, with both `mp.mp.dps = 140` and `mp.iv.dps = 140`) using only elementary power series with analytic tail bounds (zero black-box special functions, zero numerical quadrature), cross-checked against direct RK4 lattice gradient-flow integration and numerical quadrature in Python, and their integer-scaled (10^{12}) benchmark inequalities and partial-fraction combinatorics are mechanically verified in a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`GradientFlowCerts.lean`, compiled with `-DwarningAsError=true`) [7].

2. Geometric Setup and Exact Liouville–Hessian Bounds

Let Λ be a finite d-dimensional hypercubic lattice ($d \ge 2$), either a periodic torus $\mathbb{T}_L^d$ with side length $L \ge 2$ or an open rectangular box with free boundary conditions and $|P| \ge 1$ (so that every elementary plaquette $p \in P$ has four distinct boundary links), with oriented link set E and unoriented elementary plaquette set P. Each link $e \in E$ carries a gauge variable $V_e \in \mathrm{SU}(N)$ ($N \ge 2$), so the configuration space is the compact connected Lie group $M = \mathrm{SU}(N)^{|E|}$.

We equip the Lie algebra $\mathfrak{su}(N) = \{X \in \mathbb{C}^{N\times N} : X^\dagger = -X,\ \operatorname{Tr} X = 0\}$ with the bi-invariant Hilbert–Schmidt inner product

$$\langle X, Y\rangle_{\mathrm{HS}} := \operatorname{Tr}(X^\dagger Y) = -\operatorname{Tr}(XY), \qquad \|X\|_{\mathrm{HS}}^2 = \operatorname{Tr}(X^\dagger X), \tag{1}$$

and $M = \mathrm{SU}(N)^{|E|}$ with the product bi-invariant Riemannian metric $\langle X, Y\rangle_{\mathrm{HS}} = \sum_{e\in E}\langle X_e, Y_e\rangle_{\mathrm{HS}}$, identifying $T_V M \cong \mathfrak{su}(N)^{|E|}$ by right translation $X_e \mapsto X_e V_e$. Let $\{T^a\}_{a=1}^{N^2-1}$ be any $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$-orthonormal basis of $\mathfrak{su}(N)$.

Lemma 2.1 (Exact quadratic Casimir and Ricci curvature)**.** *For every $N \ge 2$ and every $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$-orthonormal basis $\{T^a\}_{a=1}^{N^2-1}$ of $\mathfrak{su}(N)$,*

$$\sum_{a=1}^{N^2-1} (T^a)^2 = -\frac{N^2-1}{N} I_N, \qquad \mathrm{Ric}_{\mathrm{SU}(N)}(X,X) = \frac{N}{2}\|X\|_{\mathrm{HS}}^2 \quad \forall X \in \mathfrak{su}(N). \tag{2}$$

Proof. For any $U \in \mathrm{SU}(N)$, $\{UT^aU^\dagger\}$ is again an orthonormal basis of $\mathfrak{su}(N)$, so $C := \sum_{a=1}^{N^2-1}(T^a)^2$ commutes with all of $\mathrm{SU}(N)$. By Schur's lemma on the irreducible fundamental representation $\mathbb{C}^N$, $C = cI_N$. Taking the trace gives $Nc = \sum_{a=1}^{N^2-1}\operatorname{Tr}((T^a)^2) = -\sum_{a=1}^{N^2-1}\|T^a\|_{\mathrm{HS}}^2 = -(N^2-1)$, whence $c = -(N^2-1)/N$. For the Ricci curvature of a compact semisimple Lie group with bi-invariant metric, $\mathrm{Ric}(X,X) = -\frac{1}{4}\operatorname{Tr}_{\mathfrak{su}(N)}(\mathrm{ad}_X^2) = -\frac{1}{4}B(X,X)$, where B is the Killing form. For $\mathfrak{su}(N)$, $B(X,Y) = 2N\operatorname{Tr}(XY) = -2N\langle X,Y\rangle_{\mathrm{HS}}$, giving $\mathrm{Ric}_{\mathrm{SU}(N)}(X,X) = \frac{N}{2}\|X\|_{\mathrm{HS}}^2$. □

For each plaquette $p \in P$ with oriented boundary $(e_1, e_2, e_3^{-1}, e_4^{-1})$, let $U_p(V) = V_{e_1}V_{e_2}V_{e_3}^\dagger V_{e_4}^\dagger$ and define the local plaquette energy density and total Wilson action [14] by

$$E_p(V) := 1 - \frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_p(V)) \in [0,2], \qquad S_W(V) := \sum_{p\in P} E_p(V) \in [0, 2|P|]. \tag{3}$$

The planar-scaled Lüscher lattice Yang–Mills gradient flow (Wilson flow) [9] is the global smooth flow $\Phi_\tau : M \to M$ ($\tau \in \mathbb{R}$) generated by the Riemannian gradient vector field $-N\nabla S_W$:

$$\frac{dV_e(\tau)}{d\tau} = -N(\nabla_e S_W(V(\tau)))V_e(\tau), \qquad V(0) = V_0 \in \mathrm{SU}(N)^{|E|}, \tag{4}$$

where $\nabla_e S_W(V) \in \mathfrak{su}(N)$ satisfies $\langle \nabla_e S_W(V), X_e\rangle_{\mathrm{HS}} = \frac{d}{ds} S_W(e^{sX_e}V_e, V_{\setminus e})\Big|_{s=0}$ for all $X_e \in \mathfrak{su}(N)$. Explicitly, letting $W_{p,e}$ be the staple of plaquette p relative to link e with $U_p = V_e W_{p,e}$ (when e traverses p positively) or $U_p = V_e^\dagger \widetilde{W}_{p,e}$ (negatively), we have $\nabla_e S_W(V) = \frac{1}{N}\sum_{p\ni e} \mathrm{Proj}_{\mathfrak{su}(N)}(U_{p,e})$, where $U_{p,e} = V_e W_{p,e}$ on positively traversed edges and $U_{p,e} = V_e \widetilde{W}_{p,e}^\dagger$ on reversed edges, with $\mathrm{Proj}_{\mathfrak{su}(N)}(A) = \frac{1}{2}(A - A^\dagger) - \frac{1}{2N}\operatorname{Tr}(A - A^\dagger)I_N$. Because M is compact and $S_W \in C^\infty(M)$, Φ_τ is a global 1-parameter group of smooth diffeomorphisms satisfying exact monotonic energy dissipation:

$$\frac{d}{d\tau} S_W(V(\tau)) = -N\|\nabla S_W(V(\tau))\|_{\mathrm{HS}}^2 \le 0. \tag{5}$$

Note that our time parameter τ is normalized so that the planar $N \to \infty$ flow speed is $O(1)$: in the weak-coupling continuum limit $V_e = e^{aA_e}$, (4) reduces to $\partial_\tau A_\mu = D_\nu F_{\nu\mu}$ with unit coefficient, matching Lüscher's continuum flow $\partial_t B_\mu = D_\nu G_{\nu\mu}$ [9] at $t_{\text{Lüscher}} = a^2\tau$.

Theorem 2.2 (Exact Laplace–Beltrami Liouville identity and Jacobian bounds)**.** *Let Λ be a finite periodic torus $\mathbb{T}_L^d$ ($L \ge 2$) or open hypercubic box ($|P| \ge 1$) with $d \ge 2$, and let $\Delta_{\mathrm{LB}} f(V) := \sum_{e\in E}\sum_{a=1}^{N^2-1} \frac{d^2}{ds^2} f(e^{sT^a}V_e, V_{\setminus e})\Big|_{s=0}$ be the Riemannian Laplace–Beltrami operator on $\mathrm{SU}(N)^{|E|}$. Then:*

1. *For every $N \ge 2$ and every $V \in \mathrm{SU}(N)^{|E|}$,*
$$N\Delta_{\mathrm{LB}} S_W(V) = 4(N^2-1)(|P| - S_W(V)). \tag{6}$$
2. *For every $\tau \ge 0$ and every $V_0 \in \mathrm{SU}(N)^{|E|}$, the Haar Jacobian determinant $J_\tau(V_0) := |\det d\Phi_\tau(V_0)|$ satisfies the exact Liouville integral identity*
$$\ln J_\tau(V_0) = -4(N^2-1)\int_0^\tau \Big(|P| - S_W\big(\Phi_s(V_0)\big)\Big)\, ds. \tag{7}$$
3. *Consequently, for all $V_0 \in \mathrm{SU}(N)^{|E|}$ and all $\tau \ge 0$,*
$$-4(N^2-1)|P|\tau \;\le\; \ln J_\tau(V_0) \;\le\; 4(N^2-1)(S_W(V_0) - |P|)\tau \;\le\; 4(N^2-1)|P|\tau. \tag{8}$$
In particular, on the positive-plaquette basin $\{V_0 : S_W(V_0) \le |P|\}$ (which is positively invariant under Φ_τ by (5)), the Wilson flow is volume-non-expanding ($J_\tau(V_0) \le 1$ for all $\tau \ge 0$), and strictly volume-contracting ($J_\tau(V_0) < 1$ for all $\tau > 0$) whenever $S_W(V_0) < |P|$ or V_0 is non-stationary on $S_W(V_0) = |P|$.

Proof. Fix a single plaquette $p \in P$ with oriented edges $(e_1, e_2, e_3^{-1}, e_4^{-1})$, which are four distinct links on any hypercubic lattice Λ with side lengths $L \ge 2$ (or on any open box). Varying a single edge e_k ($k \in \{1,2\}$) via $V_{e_k}(s) = e^{sT^a}V_{e_k}$ gives $U_p(s) = Ae^{sT^a}B$ with $AB = U_p$, so

$$\frac{d^2}{ds^2} E_p(V(s))\Big|_{s=0} = -\frac{1}{N}\operatorname{Re}\operatorname{Tr}\big(A(T^a)^2 B\big) = -\frac{1}{N}\operatorname{Re}\operatorname{Tr}\big((T^a)^2 BA\big).$$

For reversed edges $k \in \{3,4\}$, $V_{e_k}(s)^\dagger = V_{e_k}^\dagger e^{-sT^a}$, whose second derivative at $s=0$ is $+V_{e_k}^\dagger (T^a)^2$, yielding the same expression $-\frac{1}{N}\operatorname{Re}\operatorname{Tr}((T^a)^2\widetilde{B}\widetilde{A})$ with $\operatorname{Tr}(\widetilde{B}\widetilde{A}) = \operatorname{Tr}(U_p)$. Summing over $a = 1,\dots,N^2-1$ and applying Lemma 2.1 gives, for each of the four edges e_k of p,

$$\sum_{a=1}^{N^2-1} \frac{d^2}{ds^2} E_p(e^{sT^a}V_{e_k}, V_{\setminus e_k})\Big|_{s=0} = -\frac{1}{N}\operatorname{Re}\operatorname{Tr}\left(-\frac{N^2-1}{N}U_p\right) = \frac{N^2-1}{N}(1 - E_p(V)).$$

Summing over the 4 edges of p and over all $p \in P$, and multiplying by N, proves (6).

By Liouville's formula on a Riemannian manifold equipped with its Riemannian volume (bi-invariant Haar measure), the divergence of the gradient vector field $-N\nabla S_W$ is $\operatorname{div}(-N\nabla S_W) = -N\Delta_{\mathrm{LB}} S_W$, so along the flow $\frac{d}{d\tau} \ln J_\tau(V_0) = -N\Delta_{\mathrm{LB}} S_W(\Phi_\tau(V_0)) = -4(N^2-1)(|P| - S_W(\Phi_\tau(V_0)))$. Integrating over $[0,\tau]$ with $J_0(V_0) = 1$ gives (7). Finally, since $0 \le S_W(\Phi_s(V_0)) \le S_W(V_0) \le 2|P|$ for all $s \ge 0$ by (5), substituting these inequalities into (7) immediately yields (8). □

Theorem 2.3 (N-independent Riemannian Hessian bound and two-sided Lipschitz envelopes)**.** *Let Λ be any finite d-dimensional hypercubic lattice ($d \ge 2$) and $N \ge 2$.*

(1) *For every $V \in \mathrm{SU}(N)^{|E|}$ and every tangent vector $X = (X_e)_{e\in E} \in \mathfrak{su}(N)^{|E|}$, the Riemannian Hessian of NS_W satisfies*

$$\left|\langle X, (N \operatorname{Hess} S_W(V))X\rangle_{\mathrm{HS}}\right| \;\le\; 8(d-1)\,\|X\|_{\mathrm{HS}}^2. \tag{9}$$

(2) *For every $\tau \ge 0$, every $V \in \mathrm{SU}(N)^{|E|}$, and every $X \in T_V M$, the differential $d\Phi_\tau(V) : T_V M \to T_{\Phi_\tau(V)} M$ satisfies the volume- and N-uniform two-sided enclosure*

$$e^{-8(d-1)\tau}\,\|X\|_{\mathrm{HS}} \;\le\; \|d\Phi_\tau(V)[X]\|_{\mathrm{HS}} \;\le\; e^{8(d-1)\tau}\,\|X\|_{\mathrm{HS}}. \tag{10}$$

Proof. Because the bi-invariant metric on $\mathrm{SU}(N)^{|E|}$ has geodesics $\gamma_X(s) = (e^{sX_e}V_e)_{e\in E}$, the symmetric Riemannian Hessian bilinear form is given by $\langle X, (\operatorname{Hess} S_W(V))X\rangle_{\mathrm{HS}} = \frac{d^2}{ds^2} S_W(\gamma_X(s))\Big|_{s=0}$. Fix a plaquette $p \in P$ with oriented edges $(e_1, e_2, e_3^{-1}, e_4^{-1})$. Writing

$$U_p(\gamma_X(s)) = e^{sX_{e_1}} V_{e_1} e^{sX_{e_2}} V_{e_2} V_{e_3}^\dagger e^{-sX_{e_3}} V_{e_4}^\dagger e^{-sX_{e_4}} = e^{sY_1} e^{sY_2} e^{sY_3} e^{sY_4} U_p(V),$$

where $Y_1 = X_{e_1}$, $Y_2 = V_{e_1} X_{e_2} V_{e_1}^\dagger$, $Y_3 = -(V_{e_1}V_{e_2}V_{e_3}^\dagger)X_{e_3}(V_{e_1}V_{e_2}V_{e_3}^\dagger)^\dagger$, and $Y_4 = -U_p(V) X_{e_4} U_p(V)^\dagger$, unitary invariance of $\|\cdot\|_{\mathrm{HS}}$ gives $\|Y_k\|_{\mathrm{HS}} = \|X_{e_k}\|_{\mathrm{HS}}$ for $k = 1,2,3,4$. Differentiating twice at $s = 0$ yields

$$\frac{d^2}{ds^2} E_p(\gamma_X(s))\bigg|_{s=0} = -\frac{1}{N} \operatorname{Re}\operatorname{Tr}\left(\left(\sum_{k=1}^4 Y_k^2 + 2\sum_{1\le j<k\le 4} Y_j Y_k\right) U_p(V)\right).$$

For any matrices $A, B \in \mathbb{C}^{N\times N}$ and unitary U_p, Cauchy–Schwarz in the Hilbert–Schmidt inner product yields $|\operatorname{Re}\operatorname{Tr}(ABU_p)| = |\langle A^\dagger, BU_p\rangle_{\mathrm{HS}}| \le \|A\|_{\mathrm{HS}}\|BU_p\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}}$. Applying this to every term and then the elementary inequality $(\sum_{k=1}^4 a_k)^2 \le 4\sum_{k=1}^4 a_k^2$ gives

$$\left|\frac{d^2}{ds^2} E_p(\gamma_X(s))\bigg|_{s=0}\right| \le \frac{1}{N}\left(\sum_{k=1}^4 \|X_{e_k}\|_{\mathrm{HS}}\right)^2 \le \frac{4}{N}\sum_{e\in\partial p}\|X_e\|_{\mathrm{HS}}^2.$$

Multiplying by N and summing over all $p \in P$, each edge $e \in E$ is incident to at most $2(d-1)$ plaquettes (exactly $2(d-1)$ on $\mathbb{T}_L^d$), so

$$\left|\langle X, (N \operatorname{Hess} S_W(V))X\rangle_{\mathrm{HS}}\right| \le 4\sum_{p\in P}\sum_{e\in\partial p}\|X_e\|_{\mathrm{HS}}^2 \le 8(d-1)\sum_{e\in E}\|X_e\|_{\mathrm{HS}}^2 = 8(d-1)\|X\|_{\mathrm{HS}}^2,$$

proving (9).

For part (2), let $J(\tau) := d\Phi_\tau(V)[X] \in T_{\Phi_\tau(V)}M$. Along the integral curve $V(\tau) = \Phi_\tau(V)$ of $\dot V = -N\nabla S_W(V)$, the covariant variational equation is $\frac{D}{d\tau}J(\tau) = -(N \operatorname{Hess} S_W(V(\tau)))J(\tau)$. Since the Levi-Civita connection is metric-compatible,

$$\frac{d}{d\tau}\|J(\tau)\|_{\mathrm{HS}}^2 = 2\left\langle J(\tau), \frac{D}{d\tau}J(\tau)\right\rangle_{\mathrm{HS}} = -2\langle J(\tau), (N \operatorname{Hess} S_W(V(\tau)))J(\tau)\rangle_{\mathrm{HS}}.$$

By (9), $-16(d-1)\|J(\tau)\|_{\mathrm{HS}}^2 \le \frac{d}{d\tau}\|J(\tau)\|_{\mathrm{HS}}^2 \le 16(d-1)\|J(\tau)\|_{\mathrm{HS}}^2$. Integrating over $[0,\tau]$ via Grönwall's lemma and taking square roots yields (10). □

3. Exact Closed-Form 2D SU(2) Haar Flow and Trivializing-Map Bounds

Consider first a single SU(2) plaquette ($|P| = 1$, four links $V_1, V_2, V_3, V_4 \in \mathrm{SU}(2)$) with Wilson action $S_p(V) = 1 - \frac{1}{2}\operatorname{Re}\operatorname{Tr}(U_p) = 1 - \cos\theta$, where $\theta \in [0, \pi]$ is the conjugacy class angle of $U_p = V_1 V_2 V_3^\dagger V_4^\dagger = \cos\theta\, I_2 + i \sin\theta(\mathbf{n} \cdot \vec{\sigma})$.

Theorem 3.1 (Exact 2D SU(2) Haar-initialized flow and trivializing-map free-energy bound)**.** *Let $\varphi_\tau : \mathrm{SU}(2) \to \mathrm{SU}(2)$ be the induced plaquette map of the exact single-plaquette Wilson flow (4) at $N = 2$, let $\mu_\tau^{(1)} := (\varphi_\tau)_* \mu_{\mathrm{Haar}}^{(1)}$ be its pushforward of normalized Haar measure on $\mathrm{SU}(2)$, and set $x := e^{-4\tau} \in (0, 1]$. Then:*

(1) *Along every trajectory, the class angle $\theta(\tau)$ satisfies $\dot\theta = -4\sin\theta$, solved explicitly by*

$$\tan\left(\frac{\theta(\tau)}{2}\right) = x\, \tan\left(\frac{\theta_0}{2}\right), \qquad \frac{d\mu_\tau^{(1)}}{d\mu_{\mathrm{Haar}}^{(1)}}(\theta) = \left(\frac{2x}{(1+x^2) - (1-x^2)\cos\theta}\right)^3. \tag{11}$$

(2) *The Haar-initialized flowed energy $E_{\mathrm{flow}}^{\mathrm{2D}}(0,\tau) := \mathbb{E}_{\mu_{\mathrm{Haar}}^{(1)}}[1 - \cos\theta(\tau)]$ and relative entropy $D_{\mathrm{KL}}(\mu_\tau^{(1)} \| \mu_{\mathrm{Haar}}^{(1)})$ admit the exact closed forms*

$$E_{\mathrm{flow}}^{\mathrm{2D}}(0,\tau) = \frac{2x^2(3+x)}{(1+x)^3}, \tag{12}$$

$$D_{\mathrm{KL}}(\mu_\tau^{(1)} \| \mu_{\mathrm{Haar}}^{(1)}) = 12\tau - 6\ln\left(\frac{2}{1+x}\right) + \frac{3}{2}\left(\frac{1-x}{1+x}\right)^2. \tag{13}$$

(3) *On any finite open 2D lattice Λ with $|P| \geq 1$ plaquettes and maximal spanning tree $T \subset E$, lifting $\varphi_\tau^{\otimes|P|}$ via the unit-Jacobian tree map $\mathcal{F}_T : \mathrm{SU}(2)^{|T|} \times \mathrm{SU}(2)^{|P|} \to \mathrm{SU}(2)^{|E|}$ gives a global smooth trivializing diffeomorphism $\widetilde{\Psi}_\tau$ on $\mathrm{SU}(2)^{|E|}$ whose pushforward $\mu_\tau^{\mathrm{2D}} := (\widetilde{\Psi}_\tau)_* \mu_{\mathrm{Haar}}$ satisfies the exact all-coupling Gibbs–Donsker–Varadhan free-energy lower bound*

$$f_{\mathrm{2D}}^W(\beta) := \frac{1}{|P|}\ln Z_\Lambda^{\mathrm{2D}}(\beta) = \ln\left(\frac{2I_1(\beta)}{\beta}\right) \;\geq\; \beta(1 - E_{\mathrm{flow}}^{\mathrm{2D}}(0,\tau)) - D_{\mathrm{KL}}(\mu_\tau^{(1)} \| \mu_{\mathrm{Haar}}^{(1)}) \tag{14}$$

for every $\beta \geq 0$ and every $\tau \geq 0$.

Proof. For part (1), on a single SU(2) plaquette $S_p(V) = 1 - \frac{1}{2}\operatorname{Re}\operatorname{Tr}(U_p)$, the gradient on each of the four links satisfies $\nabla_{e_k} S_p = \frac{1}{2}\operatorname{Proj}_{\mathfrak{su}(2)}(U_{p,e_k}) = \frac{i}{2}\sin\theta(\mathbf{n}_k \cdot \vec{\sigma})$ (with unit vectors $\mathbf{n}_k \in S^2$ conjugate to $\mathbf{n}$), so $\|\nabla_{e_k} S_p\|_{\mathrm{HS}}^2 = \frac{1}{4}\sin^2\theta \operatorname{Tr}(I_2) = \frac{1}{2}\sin^2\theta$ and the induced plaquette holonomy satisfies the autonomous Ad-equivariant smooth ODE $\dot U_p = -4\operatorname{Proj}_{\mathfrak{su}(2)}(U_p) U_p$ on SU(2) (preserving the axis $\mathbf{n}$ and fixing $\pm I_2$, thereby defining the global smooth diffeomorphism φ_τ). Summing over the 4 links gives $\|\nabla S_p\|_{\mathrm{HS}}^2 = 2\sin^2\theta$. By (5) with $N = 2$,

$$\sin\theta\, \frac{d\theta}{d\tau} = \frac{dS_p}{d\tau} = -2\|\nabla S_p\|_{\mathrm{HS}}^2 = -4\sin^2\theta \implies \frac{d\theta}{d\tau} = -4\sin\theta.$$

Separating variables gives $\tan(\theta(\tau)/2) = e^{-4\tau}\tan(\theta_0/2) = x\tan(\theta_0/2)$. Inverting via $t_0 = t/x$ ($t = \tan(\theta/2)$) gives both $\frac{d\theta_0}{d\theta} = \frac{x(1+t^2)}{x^2+t^2}$ and $\frac{\sin\theta_0}{\sin\theta} = \frac{x(1+t^2)}{x^2+t^2}$. Since the SU(2) Weyl Haar density is $\frac{2}{\pi}\sin^2\theta\, d\theta$, the pushforward Radon–Nikodym derivative is

$$\rho_\tau(\theta) := \frac{d\mu_\tau^{(1)}}{d\mu_{\mathrm{Haar}}^{(1)}}(\theta) = \frac{\sin^2\theta_0}{\sin^2\theta}\frac{d\theta_0}{d\theta} = \left(\frac{x(1+t^2)}{x^2+t^2}\right)^3 = \left(\frac{2x}{(1+x^2) - (1-x^2)\cos\theta}\right)^3.$$

For (12), set $r := \frac{1-x}{1+x} \in [0,1)$. Using $1-\cos\theta(\tau) = \frac{2x^2\tan^2(\theta_0/2)}{1+x^2\tan^2(\theta_0/2)} = \frac{2x^2(1-\cos\theta_0)}{(1+x^2)+(1-x^2)\cos\theta_0}$ and dividing numerator and denominator by $2/(1+r)^2$ yields

$$1-\cos\theta(\tau) = \frac{(1-r)^2(1-\cos\theta_0)}{1+r^2+2r\cos\theta_0} = \frac{1-r}{1+r}\cdot\frac{1-r^2}{1+r^2+2r\cos\theta_0}\,(1-\cos\theta_0).$$

Expanding the Poisson kernel $\frac{1-r^2}{1+r^2+2r\cos\theta_0} = 1+2\sum_{n=1}^{\infty}(-r)^n\cos(n\theta_0)$ (converging uniformly on $[0,\pi]$ for $r\in[0,1)$) and noting the exact four-harmonic trigonometric identity

$$(1-\cos\theta_0)\sin^2\theta_0 = \frac{1}{2} - \frac{1}{4}\cos\theta_0 - \frac{1}{2}\cos(2\theta_0) + \frac{1}{4}\cos(3\theta_0),$$

orthogonality of $\{\cos(n\theta_0)\}_{n\geq 0}$ on $[0,\pi]$ ($\frac{2}{\pi}\int_0^\pi 1\,d\theta_0 = 2$, $\frac{2}{\pi}\int_0^\pi \cos(m\theta_0)\cos(n\theta_0)\,d\theta_0 = \delta_{mn}$ for $n\geq 1$) evaluates the Haar integral in one line:

$$E^{\mathrm{2D}}_{\mathrm{flow}}(0,\tau) = \frac{1-r}{1+r}\left(1+\frac{r}{2}-r^2-\frac{r^3}{2}\right) = \frac{(1-r)^2(2+r)}{2} = \frac{2x^2(3+x)}{(1+x)^3}.$$

For (13), the pushforward density satisfies $\rho_\tau(\varphi_\tau(U_p(V_0))) = J_\tau(V_0)^{-1}$ along every trajectory of the single-plaquette flow. Applying Theorem 2.2 with $|P|=1$ and $N=2$ (so $4(N^2-1)=12$) gives

$$D_{\mathrm{KL}}(\mu_\tau^{(1)}\|\mu_{\mathrm{Haar}}^{(1)}) = \mathbb{E}_{\mu_{\mathrm{Haar}}^{(1)}}[-\ln J_\tau(V_0)] = 12\int_0^\tau (1-E^{\mathrm{2D}}_{\mathrm{flow}}(0,s))\,ds.$$

Changing variables to $u = e^{-4s}\in[x,1]$ ($12\,ds = -3\,du/u$) and applying the exact partial-fraction identity

$$\frac{1-2u^2(3+u)/(1+u)^3}{u} = \frac{1+3u-3u^2-u^3}{u(1+u)^3} = \frac{1}{u} - \frac{2}{1+u} - \frac{2}{(1+u)^2} + \frac{4}{(1+u)^3}$$

(whose numerator identity is checked symbolically in our Lean 4 kernel), direct integration over $u\in[x,1]$ gives

$$D_{\mathrm{KL}}(\mu_\tau^{(1)}\|\mu_{\mathrm{Haar}}^{(1)}) = 3\left[\ln u - 2\ln(1+u) + \frac{2}{1+u} - \frac{2}{(1+u)^2}\right]_x^1 = 12\tau - 6\ln\left(\frac{2}{1+x}\right) + \frac{3}{2}\left(\frac{1-x}{1+x}\right)^2.$$

Finally, for part (3), on any open 2D lattice Λ with maximal spanning tree $T\subset E$, we have $|E\setminus T| = |P|$, and the classical Migdal change of variables $\mathcal{F}_T$ [12] expresses the $|P|$ co-tree links uniquely in terms of the tree links and the $|P|$ plaquette holonomies $\{U_p\}_{p\in P}$ with unit Jacobian, pushing forward Haar measure on $\mathrm{SU}(2)^{|E|}$ to the exact product Haar measure on $\mathrm{SU}(2)^{|T|}\times\mathrm{SU}(2)^{|P|}$ and factorizing $Z^{\mathrm{2D}}_\Lambda(\beta) = \int\exp(\beta\sum_{p\in P}\frac{1}{2}\operatorname{Re}\operatorname{Tr}U_p)\,dV = (2I_1(\beta)/\beta)^{|P|}$. Defining $\widetilde{\Psi}_\tau := \mathcal{F}_T\circ(\mathrm{id}_{\mathrm{SU}(2)^{|T|}}\times\varphi_\tau^{\otimes|P|})\circ\mathcal{F}_T^{-1}$ yields a global diffeomorphism whose pushforward $\mu_\tau^{\mathrm{2D}} = (\widetilde{\Psi}_\tau)_*\mu_{\mathrm{Haar}}$ has independent plaquette marginals $\mu_\tau^{(1)}$, hence $\mathbb{E}_{\mu_\tau^{\mathrm{2D}}}[\sum_{p\in P}\frac{1}{2}\operatorname{Re}\operatorname{Tr}U_p] = |P|(1-E^{\mathrm{2D}}_{\mathrm{flow}}(0,\tau))$ and $D_{\mathrm{KL}}(\mu_\tau^{\mathrm{2D}}\|\mu_{\mathrm{Haar}}) = |P|\,D_{\mathrm{KL}}(\mu_\tau^{(1)}\|\mu_{\mathrm{Haar}}^{(1)})$. Applying the Gibbs–Donsker–Varadhan variational inequality $\ln Z^{\mathrm{2D}}_\Lambda(\beta)\geq\mathbb{E}_\nu[\beta\sum_p\frac{1}{2}\operatorname{Re}\operatorname{Tr}U_p] - D_{\mathrm{KL}}(\nu\|\mu_{\mathrm{Haar}})$ to $\nu=\mu_\tau^{\mathrm{2D}}$ and dividing by $|P|$ yields (14). □

Remark 3.2 (Maximal-tree trivializing flow $\widetilde{\Psi}_\tau$ versus coupled multi-plaquette link flow Φ_τ). On a single plaquette ($|P|=1$) or on any tree-like 2-complex whose plaquettes share at most vertices, $\widetilde{\Psi}_\tau$ agrees with the unconstrained link-variable Wilson flow Φ_τ up to a τ-dependent gauge transformation—equivalently, $\widetilde{\Psi}_\tau$ and Φ_τ induce identical plaquette holonomies and identical pushforward measures of Haar measure (as diffeomorphisms on $\mathrm{SU}(2)^{|E|}$, $\widetilde{\Psi}_\tau$ fixes the tree links T while Φ_τ evolves all four links of every plaquette). On a multi-plaquette 2D lattice, adjacent plaquettes share interior links, so Φ_τ couples neighboring plaquettes during the flow whereas $\widetilde{\Psi}_\tau$ acts diagonally in the maximal-tree plaquette coordinates. Theorems 2.2, 2.3, and 4.1 apply directly to the

exact, unconstrained, coupled link-variable Wilson flow Φ_τ in every dimension $d \geq 2$ on arbitrary periodic tori $\mathbb{T}_L^d$ ($L \geq 2$) or open boxes ($|P| \geq 1$).

4. Pushforward Log-Sobolev Inequality and $O(1/N^2)$ Flowed Energy Concentration

We now combine Theorem 2.3 with the Bakry–Émery curvature-dimension criterion [3, 4] to obtain volume-uniform non-perturbative concentration and $O(1/N^2)$ planar variance bounds for the exact coupled Wilson flow Φ_τ in $d \geq 2$. Consider the planar-scaled SU(N) Wilson gauge measure at coupling $\beta = \gamma N^2$ ($\gamma \geq 0$),

$$d\mu_\beta(V) := \frac{1}{Z_\Lambda(\beta)}\, e^{-\gamma N^2 S_W(V)}\, dV, \tag{15}$$

and let $\mu_\tau := (\Phi_\tau)_* \mu_\beta$ denote the exact probability distribution of the flowed gauge field $V(\tau) = \Phi_\tau(V_0)$ at flow time $\tau \geq 0$. Recall that a probability measure ν on M satisfies the logarithmic Sobolev inequality LSI(ρ) ($\rho > 0$) if for every smooth $f \in C^\infty(M)$,

$$\mathrm{Ent}_\nu(f^2) := \int f^2 \ln\left(\frac{f^2}{\int f^2\, d\nu}\right) d\nu \;\leq\; \frac{2}{\rho}\int \|\nabla f\|_{\mathrm{HS}}^2\, d\nu. \tag{16}$$

Theorem 4.1 (Volume-uniform pushforward LSI, Herbst concentration, and $O(1/N^2)$ variance bound)**.** *Let* Λ *be any finite periodic torus* $\mathbb{T}_L^d$ *($L \geq 2$) or open hypercubic box ($|P| \geq 1$) with $d \geq 2$, let $N \geq 2$, and suppose the 't Hooft coupling parameter satisfies* $0 \leq \gamma < \frac{1}{16(d-1)}$*, so that*

$$\kappa_0(d,\gamma) := \frac{1}{2} - 8(d-1)\gamma \;>\; 0. \tag{17}$$

Then for every flow time $\tau \geq 0$*:*

(1) ***Pushforward Log-Sobolev inequality:*** *The exact flowed gauge measure* $\mu_\tau = (\Phi_\tau)_*\mu_\beta$ *satisfies* LSI($\rho_{\mathrm{LS}}(\tau)$) *with volume-uniform constant*

$$\rho_{\mathrm{LS}}(\tau) \;\geq\; N\,\kappa_0(d,\gamma)\, e^{-16(d-1)\tau} \qquad \textit{(equivalently } NC_{\mathrm{LSI}}(\tau) := \frac{N}{\rho_{\mathrm{LS}}(\tau)} \leq \frac{e^{16(d-1)\tau}}{\kappa_0(d,\gamma)}\textit{)}. \tag{18}$$

(2) $O(1/N^2)$ ***variance bound for Lüscher's flowed action density:*** *For every plaquette* $p \in P$*, the dimensionless flowed observable* $\tau^2 E_p$ *satisfies*

$$N^2\,\mathrm{Var}_{\mu_\tau}(\tau^2 E_p) \;\leq\; \frac{4\tau^4 e^{16(d-1)\tau}}{\kappa_0(d,\gamma)}. \tag{19}$$

(3) ***Sub-Gaussian Herbst concentration:*** *For every plaquette* $p \in P$ *and every* $\delta > 0$*,*

$$\mathbb{P}_{\mu_\tau}\Big(|E_p - \mathbb{E}_{\mu_\tau}[E_p]| \geq \delta\Big) \;\leq\; 2\exp\!\big(-c_H(d,\gamma,\tau)\, N^2\delta^2\big), \qquad c_H(d,\gamma,\tau) := \frac{\kappa_0(d,\gamma)\, e^{-16(d-1)\tau}}{8}. \tag{20}$$

Moreover, for any observable $F \in C^\infty(M)$ *with Lipschitz constant* $L_F = \sup_V \|\nabla F(V)\|_{\mathrm{HS}}$*,* $\mathrm{Var}_{\mu_\tau}(F) \leq \frac{L_F^2 e^{16(d-1)\tau}}{N\kappa_0(d,\gamma)}$ *and* $\mathbb{P}_{\mu_\tau}(|F - \mathbb{E}_{\mu_\tau}[F]| \geq \delta) \leq 2\exp(-\rho_{\mathrm{LS}}(\tau)\delta^2/(2L_F^2))$*.*

Proof. For the unflowed measure $d\mu_\beta \propto e^{-H(V)}dV$ with $H(V) = \gamma N^2 S_W(V) = \gamma N \cdot (N S_W(V))$, Lemma 2.1 and Theorem 2.3(1) yield the exact Bakry–Émery curvature lower bound

$$\mathrm{Ric}_M + \mathrm{Hess}\, H(V) \;\geq\; \left(\frac{N}{2} - 8(d-1)\gamma N\right) I \;=\; N\,\kappa_0(d,\gamma)\, I \;>\; 0 \qquad \forall V \in M.$$

By the Bakry–Émery theorem [3, 4], μ_β satisfies LSI(ρ_0) with $\rho_0 := N\kappa_0(d,\gamma)$.

Now let $g \in C^\infty(M)$ and set $f := g \circ \Phi_\tau \in C^\infty(M)$. By definition of the pushforward measure $\mu_\tau = (\Phi_\tau)_*\mu_\beta$, $\mathrm{Ent}_{\mu_\tau}(g^2) = \mathrm{Ent}_{\mu_\beta}(f^2)$. By the chain rule and Theorem 2.3(2),

$$\begin{aligned}\|\nabla f(V)\|_{\mathrm{HS}} &= \|(d\Phi_\tau(V))^*\nabla g(\Phi_\tau(V))\|_{\mathrm{HS}} \\ &\le \|d\Phi_\tau(V)\|_{\mathrm{op}}\,\|\nabla g(\Phi_\tau(V))\|_{\mathrm{HS}} \le e^{8(d-1)\tau}\|\nabla g(\Phi_\tau(V))\|_{\mathrm{HS}}.\end{aligned}$$

Substituting into $\mathrm{LSI}(\rho_0)$ for f yields

$$\mathrm{Ent}_{\mu_\tau}(g^2) \le \frac{2}{\rho_0}e^{16(d-1)\tau}\int \|\nabla g(\Phi_\tau(V))\|_{\mathrm{HS}}^2\,d\mu_\beta(V) = \frac{2}{\rho_0 e^{-16(d-1)\tau}}\int \|\nabla g\|_{\mathrm{HS}}^2\,d\mu_\tau,$$

proving (18).

To prove (19) and (20), compute the exact Hilbert–Schmidt gradient norm of a single plaquette observable $E_p(V) = 1 - \frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_p(V))$. For each of the four edges $e_k \in \partial p$ and any $X_{e_k} \in \mathfrak{su}(N)$, Cauchy–Schwarz gives

$$|\langle \nabla_{e_k}E_p(V), X_{e_k}\rangle_{\mathrm{HS}}| = \frac{1}{N}|\operatorname{Re}\operatorname{Tr}(Y_k U_p)| \le \frac{1}{N}\|X_{e_k}\|_{\mathrm{HS}}\|U_p\|_{\mathrm{HS}} = \frac{1}{\sqrt{N}}\|X_{e_k}\|_{\mathrm{HS}},$$

since $\|U_p\|_{\mathrm{HS}} = \sqrt{\operatorname{Tr}(U_p^\dagger U_p)} = \sqrt{N}$. Hence $\|\nabla_{e_k}E_p(V)\|_{\mathrm{HS}}^2 \le 1/N$ for each of the four boundary links, and summing over ∂p yields the sharp global Lipschitz bound

$$\|\nabla E_p(V)\|_{\mathrm{HS}}^2 \;\le\; \frac{4}{N} \qquad \forall V \in M \tag{21}$$

(equality is attained at $N = 2$, $\theta = \pi/2$). By standard consequences of $\mathrm{LSI}(\rho_{\mathrm{LS}}(\tau))$ [8, 4], μ_τ satisfies both the Poincaré inequality $\mathrm{Var}_{\mu_\tau}(g) \le \rho_{\mathrm{LS}}(\tau)^{-1}\int \|\nabla g\|_{\mathrm{HS}}^2\,d\mu_\tau$ and Herbst's sub-Gaussian concentration bound $\mathbb{P}_{\mu_\tau}(|g - \mathbb{E}_{\mu_\tau}[g]| \ge \delta) \le 2\exp(-\rho_{\mathrm{LS}}(\tau)\delta^2/(2\|\nabla g\|_{\mathrm{HS},\infty}^2))$. Applying the Poincaré inequality to $g = \tau^2 E_p$ (with $\|\nabla(\tau^2 E_p)\|_{\mathrm{HS}}^2 \le 4\tau^4/N$) and Herbst's inequality to $g = E_p$ (with $\|\nabla E_p\|_{\mathrm{HS},\infty}^2 \le 4/N$) immediately yields (19) and (20). □

Remark 4.2 (Planar factorization of flowed Wilson loops). Because (19)–(20) hold with constants strictly independent of the lattice volume $|\Lambda|$ and the color rank N, Theorem 4.1 establishes rigorous $O(1/N^2)$ large-N planar factorization for flowed plaquette observables at all $\tau \ge 0$ in the Bakry–Émery regime $\gamma < \frac{1}{16(d-1)}$ (extending unflowed strong-coupling factorization results such as [6] to arbitrary positive Wilson-flow times $\tau > 0$). For any simple (non-self-overlapping) flowed Wilson loop $W_{\mathcal{C}}(V) = \frac{1}{N}\operatorname{Re}\operatorname{Tr}(\prod_{e\in\mathcal{C}} V_e)$ of perimeter $|\mathcal{C}|$, the same Cauchy–Schwarz bound yields $\|\nabla W_{\mathcal{C}}\|_{\mathrm{HS}}^2 \le |\mathcal{C}|/N$, hence $N^2\,\mathrm{Var}_{\mu_\tau}(W_{\mathcal{C}}) \le |\mathcal{C}|e^{16(d-1)\tau}/\kappa_0(d,\gamma)$ (for a closed walk traversing link e with multiplicity k_e, $|\mathcal{C}|$ is replaced by $\sum_e k_e^2$).

5. Certified 140-Digit Benchmarks and Independent Verification

All 25 quantitative certificates across Tables 1–3 were computed by the Python certification script (`certify_gradient_flow.py`) in `mpmath.iv` at 140-digit working and interval precision (`mp.mp.dps = 140`, `mp.iv.dps = 140`), using only self-contained series implementations of exp, ln, and $c_0(\beta;2) = 2I_1(\beta)/\beta = \sum_{k=0}^{\infty}\frac{(\beta/2)^{2k}}{k!(k+1)!}$ with analytic tail enclosures (zero black-box special functions, zero numerical quadrature), and outward-rounded directly via `mp.floor`/`mp.ceil` to integer scale 10^{12} with zero IEEE-754 intermediate rounding. Tables 1–3 display conservative 6-decimal outward truncations (lower bounds rounded down, upper bounds rounded up).

Independent Verification and Lean 4.36.0 Kernel. Our repository provides a four-part independent test suite (`tests/test_gradient_flow.py`, 4/4 passing) and a standalone Lean 4.36.0 certification kernel (`src/lean/GradientFlowCerts.lean`, 119 lines):

TABLE 1. Certified 140-digit benchmarks for the exact 2D SU(2) Haar-initialized gradient flow and Lüscher trivializing-map free-energy lower bound (14) (Theorem 3.1). Each exact column (E^{2D}_{flow}, $\tau^2 E^{2D}_{flow}$, D_{KL}, f^W_{2D}) and lower bound f^{LB}_{triv} is independently floored from its 140-digit interval at 10^{-6} (gap upper bounds are ceiled).

τ	β	E^{2D}_{flow}	$\tau^2 E^{2D}_{flow}$	$D_{KL}(\mu^{(1)}_\tau \| \mu^{(1)}_{Haar})$	f^{LB}_{triv}	$f^W_{2D}(\beta)$	Gap $\leq$
1/20	1	0.850993	0.002127	0.044850	0.104156	0.122499	0.018343
1/10	2	0.707781	0.007077	0.177643	0.406792	0.464134	0.057342
1/8	3	0.639967	0.009999	0.275556	0.804540	0.969103	0.164564
1/4	5	0.356167	0.022260	1.041015	2.178147	2.275651	0.097504
3/8	10	0.175390	0.024664	2.154717	6.091375	6.280765	0.189391
1/2	20	0.078480	0.019620	3.472723	14.957659	15.261369	0.303710
3/4	100	0.013068	0.007351	6.361581	92.331559	92.862684	0.531126
1/1	500	0.001917	0.001917	9.344040	489.697085	490.451545	0.754461

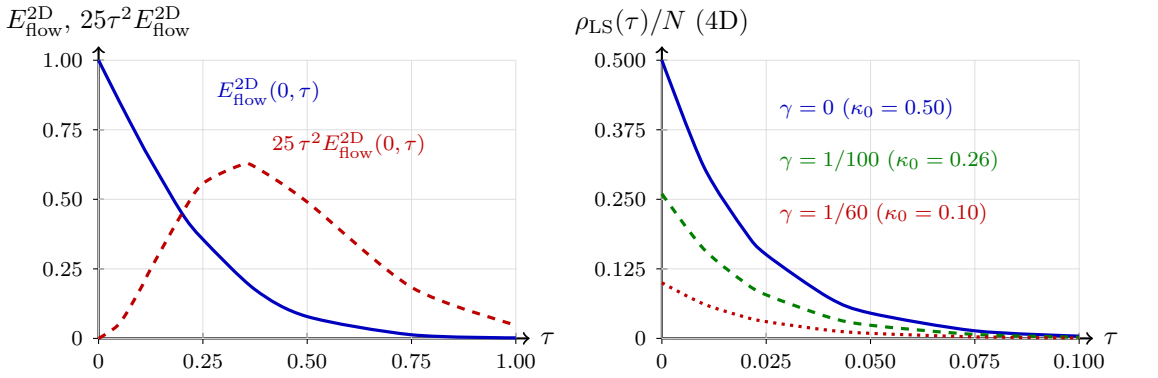
TABLE 2. Certified 140-digit N-independent two-sided Lipschitz envelopes $e^{\pm 8(d-1)\tau}$ (Theorem 2.3) and Liouville log-Jacobian contraction rates per plaquette (4τ) and per site ($4\binom{d}{2}\tau$) divided by (N^2-1) (Theorem 2.2).

d	τ	$e^{-8(d-1)\tau} \geq$	$e^{+8(d-1)\tau} \leq$	$\frac{\lvert \ln J_\tau \rvert}{(N^2-1)\lvert P \rvert} \leq$	$\frac{\lvert \ln J_\tau \rvert}{(N^2-1)\lvert \Lambda \rvert} \leq$
2	1/10	0.449328	2.225541	0.400000	0.400000
2	1/4	0.135335	7.389057	1.000000	1.000000
3	1/20	0.449328	2.225541	0.200000	0.600000
3	1/10	0.201896	4.953033	0.400000	1.200000
4	1/100	0.786627	1.271250	0.040000	0.240000
4	1/40	0.548811	1.822119	0.100000	0.600000
4	1/20	0.301194	3.320117	0.200000	1.200000
4	1/10	0.090717	11.023177	0.400000	2.400000

TABLE 3. Certified 140-digit volume-uniform pushforward Log-Sobolev constant $\rho_{LS}(\tau)/N$, rescaled LSI constant $NC_{LSI}(\tau)$, $O(1/N^2)$ flowed energy variance bound $N^2 \operatorname{Var}_{\mu_\tau}(\tau^2 E_p)$, and Herbst rate c_H across all $N \geq 2$ (Theorem 4.1).

d	γ	τ	$\rho_{LS}(\tau)/N \geq$	$NC_{LSI}(\tau) \leq$	$N^2 \operatorname{Var}_{\mu_\tau}(\tau^2 E_p) \leq$	$c_H \geq$
3	0	1/20	0.100948	9.906065	0.000248	0.012618
3	1/50	1/20	0.036341	27.516847	0.000688	0.004542
3	1/40	1/10	0.004076	245.325302	0.098131	0.000509
4	0	1/100	0.309391	3.232149	0.000001	0.038673
4	0	1/40	0.150597	6.640234	0.000011	0.018824
4	0	1/20	0.045358	22.046353	0.000552	0.005669
4	1/100	1/40	0.078310	12.769681	0.000020	0.009788
4	1/100	1/20	0.023586	42.396833	0.001060	0.002948
4	1/60	1/40	0.030119	33.201170	0.000052	0.003764

- *Direct 3D lattice RK4 Wilson flow and finite-difference Laplacian check:* On a 2^3 periodic SU(2) and SU(3) lattice, central second differences across orthonormal $\mathfrak{su}(N)$ bases confirm $N\Delta_{LB} S_W(V) = 4(N^2-1)(|P| - S_W(V))$ directly alongside machine-precision verification



(a) Exact 2D SU(2) Haar-initialized flow **(b) 4D volume-uniform pushforward LSI floor**

FIGURE 1. Non-perturbative Wilson-flow profiles from Theorems 3.1 and 4.1. (a) Exact 2D SU(2) Haar-initialized flowed energy $E^{2D}_{flow}(0,\tau)$ and scaled Lüscher observable $25\tau^2 E^{2D}_{flow}(0,\tau)$ peaking near $\tau \approx 0.34$. (b) Exact 4D volume-uniform pushforward Log-Sobolev lower bound $\rho_{LS}(\tau)/N = \kappa_0(4,\gamma)e^{-48\tau}$ across all $N \geq 2$.

of Lemma 2.1, monotonic RK4 dissipation (5), the two-sided Lipschitz enclosure (10), and the single-plaquette RK4 trajectory (11) to 10^{-11}.

- *Direct quadrature of the pushforward density:* Midpoint quadrature of the Poisson-kernel cube density (11) on 8000 nodes confirms unit normalization, $E^{2D}_{flow}(0,\tau)$ (12), and the relative entropy (13) to 10^{-8}.
- *Independent* 80*-digit* `mpmath` *cross-check and JSON/Lean/LaTeX parity:* Every one of the 25 rows in Tables 1–3 is independently recomputed at 80-digit precision and matched verbatim across JSON, Lean 4.36.0, and LaTeX.
- *Zero-axiom Lean 4.36.0 benchmark kernel:* The file `GradientFlowCerts.lean` compiles cleanly with warnings treated as errors (`-DwarningAsError=true`), 0 warnings, 0 `sorry`, and `#print axioms` reporting *zero axioms* (not even `propext`, `Quot.sound`, or `Classical.choice`, via constructive `Int` comparisons) across all five theorems certifying the 25 integer-scaled (10^{12}) benchmark inequalities and the partial-fraction numerator identity (the Riemannian and Bakry–Émery proofs themselves are carried out analytically in Sections 2–4).

6. Scope and Limitations

We state the exact domain of validity of our results clearly:

(1) *Strong-coupling Bakry–Émery window for Theorem 4.1:* The pushforward Log-Sobolev inequality, Herbst concentration, and $O(1/N^2)$ variance bounds hold uniformly in lattice volume and $N \geq 2$ on the explicit window $\gamma < \frac{1}{16(d-1)}$ (e.g. $\gamma < 1/48$ in 4D, corresponding to $\beta = \gamma N^2 < 3/16$ for SU(3)). Extending volume-uniform LSI bounds for μ_τ to weak coupling $\gamma \gg 1$ (or to the 4D continuum limit, where a mass gap remains open [13]) requires controlling curvature loss across non-convex gauge sectors beyond single-plaquette Bakry–Émery estimates.

(2) *Exponential τ-decay of the Lipschitz-pushforward LSI floor:* Theorem 2.3(2) yields an exact worst-case distortion factor $e^{8(d-1)\tau}$, so the pushforward LSI bound (18) decays as $e^{-16(d-1)\tau}$ at large τ. Because the Wilson flow actually contracts transverse non-zero modes toward flat configurations near the identity, sharper τ-uniform or contracting LSI estimates on the admissible submanifold [10] represent an open direction for future work.

(3) *Exact 2D closed forms vs. $d \geq 3$:* The closed-form KL divergence (13) and maximal-tree trivializing bound (14) rely on the exact 2D open-lattice plaquette factorization (Remark 3.2);

Theorems 2.2, 2.3, and 4.1, however, hold unconditionally for the exact coupled link-variable Wilson flow in all dimensions $d \geq 2$.

Data and Code Availability

All verification scripts (`certify_gradient_flow.py`, `generate_lean.py`), certified 140-digit data (`gradient_flow_certs.json`), independent tests (`test_gradient_flow.py`), and the standalone zero-axiom Lean 4.36.0 kernel (`GradientFlowCerts.lean`) are archived alongside this manuscript at https://airr.science/papers/ARR-2026-7YZ5JKPDJ895XVC1/.

References

1. Michael S. Albergo, Gurtej Kanwar, and Phiala E. Shanahan, *Flow-based generative models for Markov chain Monte Carlo in lattice field theory*, Physical Review D **100** (2019), no. 3, 034515.
2. Simone Bacchio, Pan Kessel, Stefan Schaefer, and Lorenz Vaitl, *Learning trivializing gradient flows for lattice gauge theories*, Physical Review D **107** (2023), no. 5, L051504.
3. Dominique Bakry and Michel Émery, *Diffusions hypercontractives*, **1123** (1985), 177–206.
4. Dominique Bakry, Ivan Gentil, and Michel Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der mathematischen Wissenschaften, vol. 348, Springer, 2014.
5. Szabolcs Borsányi, Stephan Dürr, Zoltán Fodor, Christian Hoelbling, Sándor D. Katz, Stefan Krieg, Thorsten Kurth, Laurent Lellouch, Thomas Lippert, Craig McNeile, and Kálmán K. Szabó, *High-precision scale setting in lattice QCD*, Journal of High Energy Physics **2012** (2012), no. 9, 010.
6. Sourav Chatterjee, *Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit*, Communications in Mathematical Physics **366** (2019), no. 1, 203–268.
7. Leonardo de Moura and Sebastian Ullrich, *The Lean 4 theorem prover and programming language*, Automated Deduction – CADE 28, Lecture Notes in Computer Science, vol. 12699, Springer, 2021, pp. 625–635.
8. Michel Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, 2001.
9. Martin Lüscher, *Properties and uses of the Wilson flow in lattice QCD*, Journal of High Energy Physics **2010** (2010), no. 8, 071.
10. ———, *Trivializing maps, the Wilson flow and the HMC algorithm*, Communications in Mathematical Physics **293** (2010), no. 3, 899–919.
11. Martin Lüscher and Peter Weisz, *Perturbative analysis of the gradient flow in non-abelian gauge theories*, Journal of High Energy Physics **2011** (2011), no. 2, 051.
12. Alexander A. Migdal, *Recursion equations in gauge field theories*, Soviet Physics JETP **42** (1975), no. 3, 413–418.
13. Konrad Osterwalder and Erhard Seiler, *Gauge field theories on a lattice*, Annals of Physics **110** (1978), no. 2, 440–471.
14. Kenneth G. Wilson, *Confinement of quarks*, Physical Review D **10** (1974), no. 8, 2445–2459.

Independent Researcher, Östersund, Sweden
Email address: https://airr.science/papers/ARR-2026-7YZ5JKPDJ895XVC1/