Finite-Sample-Valid Tests for Structured Matricial Hausdorff Moments

Joint Gaussian Grams, Strict Half-Time Separation, and Sharp Tangent-Cone Power

Lluis Eriksson

August 14, 2026

Abstract

We give a finite-sample test for whether one joint Gaussian sketch covariance is compatible with a positive-semidefinite matrix-valued measure supported on a prescribed interval. The observed vector contains several exponent blocks, so its population covariance is one block-Hankel Gram matrix rather than a list of unrelated moment estimates. The classical truncated matricial Hausdorff theorem supplies the exact parity-dependent null. A single Gaussian singular-value event gives a simultaneous Loewner band for the whole covariance; intersecting that band with the structured moment cone yields a nonasymptotic level-α semidefinite test without sample splitting.

The full joint Gram is strictly more informative than the integer-time localizer used in the preceding finite-sample method. For $\nu = (1-w)\delta_0 + w\delta_t$, with $0 < s < t \le 1$ and $w < s^2/t^2$, the previous condition $s^2 M_0 - M_2 > 0$ holds, whereas the half-time condition $sM_1 - M_2 = wt(s-t) < 0$. We prove that the old test has power at most α, while the full test has power at least $1-\alpha$ above an explicit sample threshold on the same acquisition. At a positive-definite covariance on a singular null face, the Wishart experiment is locally asymptotically normal. The constrained likelihood ratio converges to squared Gaussian distance from an explicit spectrahedral tangent cone, giving its pointwise local power function and a matching $n^{-1/2}$ separation boundary. We give auditable dual semantics and an exact rational certificate for the strict fixture.

The moment characterization, Gaussian covariance bounds, and generic constrained-likelihood theory are established ingredients. The contribution is their structured joint-sketch integration, the analytic same-data separation, and the exact certificate layer. This work explicitly extends ARR record `ARR-2026-77QM18J2KG9679B7`.

1 Introduction

Finite moment tests turn support claims into positive-semidefinite constraints, but several shortcuts destroy their meaning. A single scalar localizer does not test the complete truncated moment cone; independently estimated moments ignore the cross-covariances in a joint sketch; and a negative floating-point eigenvalue is not a certificate. These distinctions are most important on singular faces, precisely where support restrictions become informative.

The matrix Hausdorff moment problem on a compact interval is classical. Parity-complete block-Hankel characterizations and matrix canonical moments are developed in [5, 2, 3, 7]. Moment and localizing matrices are standard in polynomial optimization [11, 9]. We specialize those results without discarding the off-diagonal moment information present in the experiment.

The statistical starting point is the previous record [6]. Its integer-time support component retains

$$[\theta B_{i+j} - B_{i+j+1}]_{i,j=0}^{N} \succeq 0. \tag{1.1}$$

Here we put $y = \sqrt{x}$ and retain the half-time covariance blocks already acquired. The whole covariance becomes a block-Hankel matrix of y-moments, and the correct support localizer uses every polynomial direction on that grid. No extra sample or multiple-testing correction is needed: all constraints are optimized inside one simultaneous band.

Our contributions are: an operational joint Gaussian model; a parity-exact structured test; an analytic old-versus-full power separation; consistency against every fixed covariance alternative; an active-kernel tangent formula and Gaussian cone-projection likelihood limit; and exact dual semantics. Rejection falsifies compatibility of the visible matrix measure with the declared support. Non-rejection does not prove a physical spectral gap.

2 Joint Gaussian exponent sketches

Let T be a positive contraction on a real Hilbert space and let $\psi_1, \ldots, \psi_c$ be fixed probes. For $k = 0, \ldots, q$, assume $\mathsf{T}^{k/2}\psi_a$ is defined. Let $W = \{W(h) : h \in \mathcal{H}\}$ be an isonormal Gaussian process, so $\mathbb{E}W(h)W(k) = \langle h, k\rangle$, and set

$$z_{k,a} = W(\mathsf{T}^{k/2}\psi_a), \qquad z = (z_{0,1}, \ldots, z_{q,c}) \in \mathbb{R}^d, \quad d = (q+1)c. \tag{2.1}$$

In finite dimension one may realize $W(h) = \langle g, h\rangle$ with $g \sim N(0, I)$. If E_{T} is the spectral resolution of T, define the real-symmetric matrix measure

$$\boldsymbol{\nu}_{ab}(B) = \langle \psi_a, E_{\mathsf{T}}(\{\lambda : \sqrt{\lambda} \in B\})\psi_b\rangle.$$

The vector is jointly Gaussian of known mean zero, and its covariance blocks are

$$\mathbb{E}z_{k,a}z_{\ell,b} = \langle \psi_a, \mathsf{T}^{(k+\ell)/2}\psi_b\rangle =: (M_{k+\ell})_{ab}. \tag{2.2}$$

Consequently

$$K := \mathbb{E}zz^{\mathsf{T}} = H_q(M) := [M_{i+j}]_{i,j=0}^{q}. \tag{2.3}$$

The spectral theorem gives a positive-semidefinite $c \times c$ matrix-valued measure $\boldsymbol{\nu}$ on $[0, 1]$ with

$$M_j = \int_0^1 y^j \, d\boldsymbol{\nu}(y). \tag{2.4}$$

The visible support claim is $\operatorname{supp} \boldsymbol{\nu} \subset [0, s]$.

Observe iid copies $z_1, \ldots, z_n$ and form

$$S = \frac{1}{n}\sum_{r=1}^{n} z_r z_r^{\mathsf{T}}. \tag{2.5}$$

Sampling noise makes the empirical anti-diagonal blocks unequal. We do not project S onto the Hankel subspace. Instead, the structured population moments remain primal variables and $H_q(M)$ is required to lie in a confidence band around S. This preserves coverage independently of any projection rule.

Remark 2.1. *The blocks M_j are not independently sampled covariances: they are repeated anti-diagonals of one K. The off-diagonal blocks contain the half-time moments. Replacing them by independent moment oracles destroys the same-data comparison proved below.*

3 The parity-complete matricial cone

For real-symmetric blocks define

$$\begin{aligned} H_r(M) &= [M_{i+j}]_{i,j=0}^r, & G_r(M) &= [M_{i+j+1}]_{i,j=0}^r, \\ L_{r,s}(M) &= [sM_{i+j} - M_{i+j+1}]_{i,j=0}^r, & C_{r,s}(M) &= [sM_{i+j+1} - M_{i+j+2}]_{i,j=0}^r. \end{aligned} \tag{3.1}$$

Theorem 3.1 (truncated matricial Hausdorff criterion)**.** *The blocks $M_0, \dots, M_{2r}$ have a positive matrix-valued representing measure on $[0, s]$ if and only if*

$$H_r(M) \succeq 0, \qquad C_{r-1,s}(M) \succeq 0, \tag{3.2}$$

with the second condition absent at $r = 0$. The blocks through M_{2r+1} have such a measure if and only if

$$G_r(M) \succeq 0, \qquad L_{r,s}(M) \succeq 0. \tag{3.3}$$

Both statements include rank-deficient sequences.

Proof. Necessity follows from vector-polynomial Gram identities. For $p(y) = \sum_{i=0}^r y^i v_i$,

$$v^* H_r(M) v = \int p(y)^* \, d\boldsymbol{\nu}(y) \, p(y) \ge 0, \tag{3.4}$$

and the remaining matrices insert y, $s - y$, or $y(s - y)$.

For sufficiency, define $\mathcal{L}_M(\sum_j F_j y^j) = \sum_j \operatorname{tr}(F_j M_j)$. Every real-symmetric matrix polynomial nonnegative on $[0, s]$ of degree at most $2r$ factors as

$$F = A^{\mathsf{T}} A + y(s - y) B^{\mathsf{T}} B, \qquad \deg A \le r, \quad \deg B \le r - 1, \tag{3.5}$$

and at degree at most $2r + 1$ as

$$F = y A^{\mathsf{T}} A + (s - y) B^{\mathsf{T}} B, \qquad \deg A, \deg B \le r. \tag{3.6}$$

Here A and B may be rectangular matrix polynomials; equivalently, each term denotes a finite sum of real-symmetric polynomial squares. Thus the displayed LMIs make $\mathcal{L}_M$ nonnegative on every positive matrix polynomial of the declared degree.

The truncated matrix-measure cone is closed. Bounded M_0 bounds total scalar mass; positivity bounds the variation of every entry; weak-star compactness supplies a matrix-measure limit. If M lay outside that cone, strict separation would produce a positive matrix polynomial with negative $\mathcal{L}_M$ value, contradicting equations (3.5) and (3.6). □

The theorem is established literature, not our novelty claim. Entrywise scalar tests are insufficient because independently chosen scalar measures need not assemble into one positive matrix measure.

For the acquisition $q = 2N + 1$, the covariance exposes $M_0, \dots, M_{2q} = M_{4N+2}$. The exact even null is

$$K = H_q(M), \qquad C_{q-1,s}(M) \succeq 0. \tag{3.7}$$

The Loewner band already enforces $K \succeq 0$; the new ingredients are the anti-diagonal identities and full half-time localizer.

4 A structured Loewner confidence test

Let G be a $d \times n$ standard real Gaussian matrix. On the range of K, the sample covariance has the congruence representation

$$S = K^{1/2} \left(\frac{1}{n} G G^{\mathsf{T}} \right) K^{1/2}. \tag{4.1}$$

For $0 < \alpha < 1$, put

$$\eta = \frac{\sqrt{d} + \sqrt{2 \log(2/\alpha)}}{\sqrt{n}}, \tag{4.2}$$

assume $\eta < 1$, and define

$$a = (1+\eta)^{-2}, \qquad b = (1-\eta)^{-2}. \tag{4.3}$$

The Gaussian singular-value inequality [4, 10] yields

$$\Pr_K \{ aS \preceq K \preceq bS \} \geq 1 - \alpha. \tag{4.4}$$

This is a nonasymptotic valid Gaussian Loewner band. It is generally conservative, so we do not call it an exact-coverage Wishart interval. Exact Wishart eigenvalue quantiles may replace (a, b) without changing the primal geometry.

Given S, solve

$$\begin{aligned} &\text{find} && M_0, \ldots, M_{2q} \in \mathrm{Sym}_c \\ &\text{such that} && aS \preceq H_q(M) \preceq bS, \\ & && C_{q-1,s}(M) \succeq 0. \end{aligned} \tag{4.5}$$

Reject exactly when this structured feasibility problem is infeasible. Using moment blocks as variables enforces all anti-diagonal equalities by construction and makes the adjoint map auditable.

Theorem 4.1 (finite-sample validity)**.** *Under the joint Gaussian model,*

$$\sup_{\operatorname{supp} \boldsymbol{\nu} \subset [0,s]} \Pr_{\boldsymbol{\nu}} \{ \textit{reject} \} \leq \alpha. \tag{4.6}$$

The optimization may inspect every parity-appropriate constraint and matrix witness formed from the acquired truncation $M_0, \ldots, M_{2q}$ adaptively. No further multiplicity correction is required.

Proof. On the single event equation (4.4), the true blocks satisfy the band with $H_q(M) = K$. Under the null, theorem 3.1 gives $C_{q-1,s}(M) \succeq 0$. The true sequence is feasible, so rejection is contained in the complement of the coverage event. □

Remark 4.2 (singular covariance)**.** *The level argument does not invert K and is valid on singular covariance faces. Implementations should compress to the exact common range implied by the band or use facial reduction. Adding a numerical ridge changes the null.*

5 Strict same-data power separation

Consider the scalar measure

$$\nu = (1-w)\delta_0 + w\delta_t, \qquad 0 < s < t \leq 1, \quad 0 < w < s^2/t^2. \tag{5.1}$$

Its first moments and joint $N = 0$ covariance are

$$M_0 = 1, \quad M_1 = wt, \quad M_2 = wt^2, \qquad K = \begin{pmatrix} 1 & wt \\ wt & wt^2 \end{pmatrix} \succ 0. \tag{5.2}$$

The preceding integer-time method uses $x = y^2$, $\theta = s^2$, and retains

$$\theta M_0 - M_2 = s^2 - wt^2 > 0. \tag{5.3}$$

Its finite-window population null is therefore true despite $t > s$. The complete half-time null contains instead

$$C_{0,s} = sM_1 - M_2 = wt(s - t) < 0. \tag{5.4}$$

Both tests use the same K; the predecessor discards its off-diagonal M_1.

For a self-contained comparison, define the preceding $N = 0$ test on the same observed S as follows: accept whenever there is a symmetric $Q = \left(\begin{smallmatrix} B_0 & u \\ u & B_1 \end{smallmatrix}\right)$ satisfying $aS \preceq Q \preceq bS$ and $s^2 B_0 - B_1 \geq 0$. Thus it retains the two integer-time diagonal blocks and leaves the half-time cross block u unconstrained. On the event equation (4.4), the true $Q = K$ is feasible for this old test whenever equation (5.3) holds.

Theorem 5.1 (strict old-versus-full separation)**.** *For equation (5.1), the preceding $N = 0$ integer-time test has rejection probability at most α whenever its level theorem applies. Define*

$$\gamma = wt(t - s), \qquad A = \begin{pmatrix} 0 & s/2 \\ s/2 & -1 \end{pmatrix}, \tag{5.5}$$

$$W = \operatorname{tr}|K^{1/2}AK^{1/2}| = \sqrt{\gamma^2 + s^2 w(1 - w)t^2}, \qquad r = \gamma/W. \tag{5.6}$$

The full structured test has rejection probability at least $1 - \alpha$ whenever

$$\boxed{n > (\sqrt{2} + \sqrt{2\log(2/\alpha)})^2 \frac{(\sqrt{1 + r} + 1)^4}{r^2}.} \tag{5.7}$$

The scalar construction embeds algebraically in every channel dimension by tensoring with a known rank-one channel projector. For the uncompressed c-channel test, replace $\sqrt{2}$ in equation (5.7) by $\sqrt{2c}$. If the known common range is first compressed exactly to the rank-two covariance range, equation (5.7) remains valid unchanged.

Proof. The old test's population null holds by equation (5.3); its coverage argument bounds rejection by α.

For the full test,

$$\langle A, K \rangle = sM_1 - M_2 = -\gamma. \tag{5.8}$$

Suppose the Gaussian event equation (4.4) holds and a covariance Q inside the test band is full-null feasible. Put $\rho = (1 + \eta)/(1 - \eta)$. Transitivity of Loewner order gives

$$\rho^{-2} K \preceq Q \preceq \rho^2 K. \tag{5.9}$$

Let $B = K^{1/2}AK^{1/2}$, $p = \operatorname{tr} B_+$, and $m = \operatorname{tr} B_-$. Then $p + m = W$ and $m - p = \gamma$. The support function of equation (5.9) gives

$$\langle A, Q - K \rangle \leq (\rho^2 - 1)p + (1 - \rho^{-2})m. \tag{5.10}$$

Substituting $p = (W - \gamma)/2$ and $m = (W + \gamma)/2$ shows that the right side is less than γ whenever

$$\eta < \frac{r}{(\sqrt{1 + r} + 1)^2}. \tag{5.11}$$

Hence $\langle A, Q \rangle < 0$. But for a structured $Q = H_1(\widetilde{M})$, $\langle A, Q \rangle = s\widetilde{M}_1 - \widetilde{M}_2$, which must be nonnegative under the full null. Therefore the primal is infeasible on the coverage event. Solving equation (5.11) for n with $d = 2$ gives equation (5.7). $\square$

5.1 Exact rational fixture

Take

$$s = \frac{1}{2}, \qquad t = \frac{3}{4}, \qquad w = \frac{1}{8}. \tag{5.12}$$

Then

$$K = \begin{pmatrix} 1 & 3/32 \\ 3/32 & 9/128 \end{pmatrix}, \qquad \det K = \frac{63}{1024} > 0, \tag{5.13}$$

$$s^2 M_0 - M_2 = \frac{23}{128} > 0, \qquad sM_1 - M_2 = -\frac{3}{128} < 0, \tag{5.14}$$

and

$$\gamma = \frac{3}{128}, \qquad W^2 = \frac{261}{16384}, \qquad r = \frac{3}{\sqrt{261}}. \tag{5.15}$$

These opposing margins and W^2 are verified over exact rationals.

5.2 Full exponent Grams

For this paragraph define generalized moments $M_\beta = \int_0^1 y^\beta \, d\boldsymbol{\nu}(y)$ for every real $\beta \geq 0$. For any exponents $a_0, \ldots, a_R \geq 0$, the support null implies

$$\mathcal{K}_s^A = [sM_{a_i+a_j+1} - M_{a_i+a_j+2}]_{i,j=0}^R \succeq 0, \tag{5.16}$$

because this is the Gram matrix of y^{a_i} in $L^2(y(s-y)d\nu)$. Half-integer exponents couple integer and half-time data. Separate-grid constraints are principal submatrices and cannot dominate the full joint Gram. Equation (5.16) is operationally testable only when the corresponding exponent blocks have also been acquired; it is not an additional constraint available from the integer grid $0, \ldots, q$ alone.

6 Global separation of fixed alternatives

Let $\mathcal{C}_{q,c,s}$ be the closed cone of structured covariances $H_q(M)$ compatible with a positive matrix measure on $[0, s]$. For fixed $K \notin \mathcal{C}_{q,c,s}$, set

$$\delta(K) = \operatorname{dist}_F(K, \mathcal{C}_{q,c,s}) > 0. \tag{6.1}$$

Theorem 6.1 (fixed-alternative consistency)**.** *For every fixed $K \notin \mathcal{C}_{q,c,s}$, the rejection probability tends to one. Uniformity holds on sets satisfying $\|K\|_{\mathrm{op}} \leq B$ and $\operatorname{dist}_F(K, \mathcal{C}_{q,c,s}) \geq \delta_0 > 0$. If*

$$(1-u)^2 K \preceq S \preceq (1+u)^2 K, \qquad u < 1, \tag{6.2}$$

then every Q in the test band lies in a Frobenius ball about K of radius

$$\sqrt{d}\, \Gamma(u, \eta) \|K\|_{\mathrm{op}}, \tag{6.3}$$

where

$$\Gamma(u, \eta) = \max\left\{ \frac{(1+u)^2}{(1-\eta)^2} - 1, \, 1 - \frac{(1-u)^2}{(1+\eta)^2} \right\}. \tag{6.4}$$

If this radius is less than $\delta(K)$, rejection is forced.

Proof. Combining the event with $aS \preceq Q \preceq bS$ gives scalar lower and upper Loewner multiples of K. Hence $\|Q - K\|_F \leq \sqrt{d}\, \Gamma \|K\|_{\mathrm{op}}$. If that ball misses the closed null cone, no structured sequence is feasible. Choose $t_n \to \infty$ with $t_n/\sqrt{n} \to 0$ and put $u_n = (\sqrt{d} + t_n)/\sqrt{n}$. The corresponding Gaussian singular-value event has probability at least $1 - 2e^{-t_n^2/2}$, while $u_n, \eta_n \to 0$. Hence the displayed radius is eventually below the fixed positive distance; the same argument is uniform under the two stated bounds. $\square$

7 Tangent geometry

Fix $K^0 = H_q(M^0) \succ 0$ in the null; its localizer may be singular. Put

$$\mathcal{V} = (\mathrm{Sym}_c)^{2q+1}, \quad \mathsf{H}(M) = H_q(M), \quad \mathsf{C}(M) = C_{q-1,s}(M). \tag{7.1}$$

The map H is injective because every moment occurs on an anti-diagonal. Let U_0 span $\ker \mathsf{C}(M^0)$ and define

$$\mathcal{T}_0 = \{\mathsf{H}(D) : D \in \mathcal{V},\, U_0^* \mathsf{C}(D) U_0 \succeq 0\}. \tag{7.2}$$

Theorem 7.1 (exact tangent preimage)**.** *The Bouligand tangent cone of $\mathcal{C}_{q,c,s}$ at K^0 equals $\mathcal{T}_0$.*

Proof. For $C^0 \succeq 0$,

$$T_{\mathbb{S}_+}(C^0) = \{E : U_0^* E U_0 \succeq 0\}. \tag{7.3}$$

Necessity follows by differentiating a feasible sequence and compressing to $\ker C^0$. Conversely, choose an interior moment sequence M^{int} generated by a positive density on $(0, s)$ times I_c. Both $\mathsf{H}(M^{\mathrm{int}})$ and $\mathsf{C}(M^{\mathrm{int}})$ are positive definite. Therefore $D^{\mathrm{sl}} = M^{\mathrm{int}} - M^0$ satisfies

$$U_0^* \mathsf{C}(D^{\mathrm{sl}}) U_0 \succ 0. \tag{7.4}$$

Robinson's constraint qualification gives

$$T_{\{M:\mathsf{C}(M)\succeq 0\}}(M^0) = \{D : U_0^* \mathsf{C}(D) U_0 \succeq 0\}. \tag{7.5}$$

The H_q positivity constraint is inactive at $K^0 \succ 0$. In fact $\mathsf{C}(M^0) + \mathsf{C}(D^{\mathrm{sl}}) = \mathsf{C}(M^{\mathrm{int}}) \succ 0$, so the affine semidefinite constraint satisfies Robinson's condition. Applying the injective map H proves the claim. $\square$

If K^0 is singular, its PSD constraint is active and the experiment lives on a smaller range. One must add its kernel compression and exact facial reduction. The regular likelihood theorem below is restricted to $K^0 \succ 0$.

8 Wishart LAN and local asymptotic power

For real zero-mean Gaussian observations, the Fisher metric is

$$\langle H, J\rangle_{I_0} = \frac{1}{2}\,\mathrm{tr}[(K^0)^{-1} H (K^0)^{-1} J]. \tag{8.1}$$

Let $I_0^{1/2}$ be a whitening isometry and put $\mathcal{K}_0 = I_0^{1/2}\mathcal{T}_0$.

Theorem 8.1 (LAN and cone-projection likelihood)**.** *Under*

$$K_n = K^0 + \frac{H}{\sqrt{n}} + o(n^{-1/2}), \tag{8.2}$$

$$\log \frac{dP_{K_n}^{\otimes n}}{dP_{K^0}^{\otimes n}} = \langle Z_n, H\rangle_{I_0} - \frac{1}{2}\|H\|_{I_0}^2 + o_{P_{K^0}}(1), \tag{8.3}$$

where Z_n is the Fisher–Riesz representation of the central score: for every fixed J, $\langle Z_n, J\rangle_{I_0}$ is the normalized score in direction J. Thus $I_0^{1/2} Z_n \Rightarrow G$, standard Gaussian in Sym_d. Twice the log-likelihood ratio between the unrestricted covariance model and the structured null obeys

$$\boxed{\Lambda_n \Rightarrow \mathrm{dist}^2(G + I_0^{1/2} H, \mathcal{K}_0).} \tag{8.4}$$

Proof. Up to constants,

$$\ell_n(K) = -\frac{n}{2}\{\log\det K + \operatorname{tr}(K^{-1}S)\}. \tag{8.5}$$

Insert $K = K^0 + H/\sqrt{n}$ and use

$$\begin{aligned} \log\det(K^0+E) &= \log\det K^0 + \operatorname{tr}((K^0)^{-1}E) - \tfrac{1}{2}\operatorname{tr}[((K^0)^{-1}E)^2] + o(\|E\|^2), \\ (K^0+E)^{-1} &= (K^0)^{-1} - (K^0)^{-1}E(K^0)^{-1} \\ &\quad + (K^0)^{-1}E(K^0)^{-1}E(K^0)^{-1} + o(\|E\|^2). \end{aligned} \tag{8.6}$$

The covariance CLT supplies the Gaussian score and Fisher quadratic term. The local unrestricted likelihood is $h \mapsto \langle G + I_0^{1/2}H, h\rangle - \|h\|^2/2$. By theorem 7.1 and linearized Slater, local null sets converge to $\mathcal{K}_0$. Completing the square after unrestricted and constrained maximization gives equation (8.4). □

For proper complex Gaussian observations the factor $1/2$ in the Fisher metric is absent, the covariance is complex Wishart, and G is standard in the real Hilbert space Herm_d; all tangent and polar cones are then taken in Herm_d. This is a separate extension, not a change of one scalar factor in the real experiment.

Let $c_\alpha(\mathcal{K}_0)$ be an upper α quantile of $\operatorname{dist}^2(G, \mathcal{K}_0)$.

Corollary 8.2 (local size and power)**.** *The cone-likelihood test has asymptotic size at most α for every local null sequence. Its local power against H is*

$$\pi_{K^0}(H) = \Pr\{\operatorname{dist}^2(G + I_0^{1/2}H, \mathcal{K}_0) > c_\alpha(\mathcal{K}_0)\}. \tag{8.7}$$

Proof. Every local null limit h belongs to $\mathcal{K}_0$. Since the cone is closed under addition,

$$\operatorname{dist}(G+h, \mathcal{K}_0) \le \operatorname{dist}(G, \mathcal{K}_0) \tag{8.8}$$

pointwise. The vertex is least favorable. Randomization at a quantile atom gives exact asymptotic size; otherwise the test is conservative. □

At $H = 0$, Moreau decomposition gives $\|\Pi_{\mathcal{K}_0^\circ} G\|^2$. For general spectrahedral preimages, we retain the description as a Gaussian cone-projection law.

Theorem 8.3 (matching $n^{-1/2}$ boundary)**.** *At each regular boundary point, for a fixed outward direction $V \notin \mathcal{T}_0$ and a sequence $K_n = K^0 + \varepsilon_n V \succ 0$ with $\varepsilon_n = o(n^{-1/2})$, no level-$\alpha$ test has power above $\alpha + o(1)$. Displacement $H/\sqrt{n}$ has the nontrivial power in theorem 8.2; and alternatives that remain in a compact positive-definite neighborhood and satisfy $\sqrt{n}\,\operatorname{dist}_{I_0}(K_n, \mathcal{C}_{q,c,s}) \to \infty$ are detected by the structured Loewner-feasibility test with probability tending to one.*

Proof. For an outward $V \notin \mathcal{T}_0$,

$$D(P_{K^0+\varepsilon V}^{\otimes n} \| P_{K^0}^{\otimes n}) = \frac{n\varepsilon^2}{2}\|V\|_{I_0}^2 + o(n\varepsilon^2). \tag{8.9}$$

Pinsker proves the lower bound, theorem 8.1 gives the local limit, and theorem 6.1 plus covariance concentration gives the upper bound. □

If $0 < \alpha < 1/2$, then at a full-dimensional smooth halfspace face $\mathcal{K}_0 = \{h : \langle a, h\rangle \ge 0\}$, $\|a\| = 1$, with outward $\rho = -\langle a, I_0^{1/2}H\rangle > 0$, the exact local power is

$$\boxed{\pi(\rho) = 1 - \Phi(z_{1-\alpha} - \rho).} \tag{8.10}$$

This is the Neyman–Pearson envelope in the limiting one-dimensional normal experiment. At the atom, including $\alpha \ge 1/2$, the critical rule must be randomized. If the Hankel image is a proper subspace of Sym_d, the formula applies conditionally inside that structured submodel; the unrestricted test also contains the orthogonal equality-constraint component.

9 Auditable semidefinite dual certificates

The primal test is useful only if rejection can be checked independently of the floating-point solver that found it. We therefore spell out its ordinary semidefinite Farkas dual. For block matrices $Y = [Y_{ij}]_{i,j=0}^{q}$ and $Z = [Z_{ij}]_{i,j=0}^{q-1}$ define

$$\begin{aligned} [\mathsf{H}_q^* Y]_k &= \sum_{i+j=k} Y_{ij}, \\ [\mathsf{C}_{q-1,s}^* Z]_k &= s \sum_{i+j+1=k} Z_{ij} - \sum_{i+j+2=k} Z_{ij}, \end{aligned} \tag{9.1}$$

where empty sums are zero and symmetric symmetrization is understood. These are the exact adjoints because $\langle Y, H_q(M)\rangle = \sum_k \langle [\mathsf{H}_q^* Y]_k, M_k\rangle$, and similarly for C.

Proposition 9.1 (checkable rejection certificate)**.** *If there are positive semidefinite block matrices Y_A, Y_B, Y_C satisfying*

$$\mathsf{H}_q^*(Y_A - Y_B) + \mathsf{C}_{q-1,s}^* Y_C = 0 \tag{9.2}$$

and

$$-a\langle Y_A, S\rangle + b\langle Y_B, S\rangle < 0, \tag{9.3}$$

then the confidence feasibility problem is infeasible. Conversely, every infeasible instance of this bounded structured-band problem admits a certificate of this form.

Proof. For any primal-feasible M, positivity gives

$$0 \le \langle Y_A, H_q(M) - aS\rangle + \langle Y_B, bS - H_q(M)\rangle + \langle Y_C, C_{q-1,s}(M)\rangle. \tag{9.4}$$

Stationarity cancels every moment block, leaving exactly the strictly negative number in equation (9.3), a contradiction. For the converse, suppose approximate feasibility had residual tending to zero. The sandwich constraints bound $\mathsf{H}_q(M)$; injectivity of H_q then bounds the moment variables. A convergent subsequence would be exactly feasible because the PSD cones are closed, a contradiction. Thus the affine image and product PSD cone have positive distance, and strong separation yields the displayed ordinary multipliers. This compact sandwich argument is special to the present bounded band; generic singular semidefinite systems may require facial reduction or Ramana's extended dual [13]. □

9.1 An exact rational dual certificate

Collapse the band to $a = b = 1$ and set $S = K$ from equation (5.1) with the rational values in section 5. Let

$$D = \begin{pmatrix} 0 & -1/4 \\ -1/4 & 1 \end{pmatrix}, \quad Y_A = D + I = \begin{pmatrix} 1 & -1/4 \\ -1/4 & 2 \end{pmatrix}, \quad Y_B = I, \quad Y_C = 1. \tag{9.5}$$

Both matrix multipliers are positive definite. Since $s = 1/2$, the three moment coefficients of $\mathsf{H}_1^* D + \mathsf{C}_{0,s}^* 1$ are

$$D_{00} = 0, \qquad 2D_{01} + s = 0, \qquad D_{11} - 1 = 0. \tag{9.6}$$

Thus stationarity holds over the rationals, while

$$-\langle Y_A, K\rangle + \langle Y_B, K\rangle = -\langle D, K\rangle = -\frac{3}{128} < 0. \tag{9.7}$$

This is a solver-independent certificate that the full constraint rejects the population covariance. Its entries, principal minors, stationarity residuals, and objective are replayed exactly by the companion script.

10 Algorithm and reproducible replay

For exponent depth q, channel size c, support cutoff s, and level α, the operational procedure is:

1. Acquire n independent joint sketch vectors $z_j = (z_{j,0}, \ldots, z_{j,q})$ and form $S = n^{-1} \sum_j z_j z_j^\top$.
2. Set $d = (q+1)c$, compute η, a, b from equation (4.4), and stop as sample-size unresolved if $\eta \geq 1$.
3. Introduce real-symmetric variables $M_0, \ldots, M_{2q}$ and solve the structured Loewner feasibility problem in Section 4. Reject if and only if infeasible.
4. On rejection, request PSD multipliers from the solver, rationalize or outward-round them, and verify PSD, stationarity, and a strictly negative objective independently. If a solver fails to recover the guaranteed ordinary certificate numerically, use facial reduction to expose and preserve the relevant face before reconstructing it.

The data path uses a single sample covariance, hence cross-exponent blocks and their sampling correlations are never discarded. Complexity is that of an SDP with $2q+1$ real-symmetric $c \times c$ variables, two PSD constraints of order $(q+1)c$, and one of order qc. Anti-diagonal consistency is exact because $H_q(M)$ is assembled from shared variables, rather than imposed by noisy equality tests.

The lightweight replay file `verify_matricial_hausdorff.py` uses exact rational arithmetic for the algebraic fixtures and floating-point normal quantiles only for the reported Gaussian illustration. It checks the even and odd block criteria on noncommuting positive 2×2 atomic weights, vector-polynomial identities, Loewner inversion, an intentionally corrupted moment witness, the strict two-atom fixture, the dual certificate, and a regular smooth-face tangent fixture. It does not simulate coverage, optimize over scientific models, or certify novelty. A successful run therefore certifies the paper's algebraic fixtures, not its empirical scope.

11 Independent-moment observations as a secondary variant

Some experiments produce independent covariance estimates $\widehat{M}_k$ rather than one joint exponent vector. If simultaneous bands

$$\underline{M}_k \preceq M_k \preceq \overline{M}_k, \qquad 0 \leq k \leq m, \tag{11.1}$$

have joint coverage at least $1-\alpha$, intersecting those bands with the parity-appropriate LMIs in theorem 3.1 gives the same finite-sample level argument. Bonferroni, a joint bootstrap with proved coverage, or exact Wishart bounds may construct the simultaneous event. This variant is useful but strictly less informative for the present separation question: estimating blocks independently obscures the joint covariance geometry and may fail to retain the half-time cross blocks that make theorem 5.1 possible.

The joint model is therefore the primary theorem, not an example of the independent oracle. Conversely, joint acquisition does not manufacture unobserved moments: with exponents $0, \ldots, q$ it exposes exactly sums $0, \ldots, 2q$. Claims about a larger truncated cone require additional exponents or assumptions.

12 Relation to prior work and scope boundaries

The matricial Hausdorff equivalences themselves are classical [5, 2, 3, 7]; scalar moment and localizing matrices belong to the established truncated-moment framework [11, 9]. Gaussian covariance concentration, constrained likelihood limits, and exact rational SDP certification likewise have substantial precedents [4, 10, 1, 14, 8, 12]. We do not rename any of those components as new.

This paper explicitly extends the finite-sample spectral support framework archived as `ARR-2026-77QM18J2KG9679B7` [6]. Its increment is the integrated theorem chain: one operational exponent sketch produces the entire block-Hankel covariance; the parity-complete matrix cone uses every cross block; a closed rational family proves strict same-data power over the preceding integer-time localizer; a single Loewner event gives a multiplicity-free conic test; and the tangent preimage identifies its sharp $n^{-1/2}$ boundary law. The strictness claim is relative to the precisely stated predecessor constraint, not to all conceivable support tests.

Several limitations are structural.

- Gaussian sketches give the stated finite-sample band. Sub-Gaussian or dependent data need a separately proved covariance region.
- The cutoff hypothesis concerns existence of a positive matrix measure on $[0, s]$. It does not identify a unique measure, dynamical generator, or physical microscopic mechanism.
- Finite truncation cannot detect every alternative distribution. The consistency theorem is against covariances outside the exposed truncated cone, not against alternatives sharing all observed blocks with a null law.
- The LAN theorem assumes a positive-definite covariance and the stated linearized Slater condition. More singular intersections can have different rates and require stratified facial analysis.
- Ordinary floating-point infeasibility is not a proof. A preserved dual certificate, preferably rational or interval verified, is required for an auditable individual rejection.

13 Conclusion

A joint half-time exponent sketch turns a matrix-valued support question into a single structured Wishart problem. The full block-Hankel covariance and its Hausdorff localizer yield finite-sample validity, strict same-data power, global fixed-alternative consistency, and an exact Gaussian tangent-cone limit. The rational fixture isolates why the gain occurs: the discarded cross moment is negative in the support localizer even while the older integer-grid condition is strictly feasible. The result is thus stronger than an additional moment inequality: it gives an operational acquisition, a sharp boundary theory, and independently checkable rejection witnesses.

References

[1] Herman Chernoff. “On the Distribution of the Likelihood Ratio”. In: *The Annals of Mathematical Statistics* 25.3 (1954), pp. 573–578. DOI: 10.1214/aoms/1177728725.

[2] Abdon E. Choque Rivero, Yurii M. Dyukarev, Bernd Fritzsche, and Bernd Kirstein. “A Truncated Matricial Moment Problem on a Finite Interval”. In: *Interpolation, Schur Functions and Moment Problems*. Vol. 165. Operator Theory: Advances and Applications. 2006, pp. 121–173. DOI: 10.1007/3-7643-7547-7_4.

[3] Abdon E. Choque Rivero, Yuriy M. Dyukarev, Bernd Fritzsche, and Bernd Kirstein. "A Truncated Matricial Moment Problem on a Finite Interval. The Case of an Odd Number of Prescribed Moments". In: *System Theory, the Schur Algorithm and Multidimensional Analysis*. Vol. 176. Operator Theory: Advances and Applications. 2007, pp. 99–164. DOI: 10.1007/978-3-7643-8137-0_2.

[4] Kenneth R. Davidson and Stanislaw J. Szarek. "Local Operator Theory, Random Matrices and Banach Spaces". In: *Handbook of the Geometry of Banach Spaces*. Vol. 1. Elsevier, 2001, pp. 317–366. DOI: 10.1016/S1874-5849(01)80010-3.

[5] Holger Dette and William J. Studden. "Matrix Measures, Moment Spaces and Favard's Theorem for the Interval $[0, 1]$ and $[0, \infty)$". In: *Linear Algebra and its Applications* 345 (2002), pp. 169–193. DOI: 10.1016/S0024-3795(01)00493-1.

[6] Lluis Eriksson. *Finite-Sample Spectral-Gap Falsification: Exact Weighted Visibility Minimax, Hidden-Atom LAN, and Honest Dependent Tests*. ARR record ARR-2026-77QM18J2KG9679B7. 2026. URL: https://arr-research.github.io/papers/ARR-2026-77QM18J2KG9679B7/.

[7] Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. "Matricial Canonical Moments and Parametrization of Matricial Hausdorff Moment Sequences". In: *Complex Analysis and Operator Theory* 13 (2019), pp. 2123–2169. DOI: 10.1007/s11785-017-0754-5. arXiv: 1711.00797.

[8] Charles J. Geyer. "On the Asymptotics of Constrained M-Estimation". In: *The Annals of Statistics* 22.4 (1994), pp. 1993–2010. DOI: 10.1214/aos/1176325768.

[9] J. William Helton and Jiawang Nie. "A Semidefinite Approach for Truncated K-Moment Problems". In: *Foundations of Computational Mathematics* 12 (2012), pp. 851–881. DOI: 10.1007/s10208-012-9132-x. arXiv: 1105.0410.

[10] Vladimir Koltchinskii and Karim Lounici. "Concentration Inequalities and Moment Bounds for Sample Covariance Operators". In: *Bernoulli* 23.1 (2017), pp. 110–133. DOI: 10.3150/15-BEJ730. arXiv: 1405.2468.

[11] Jean B. Lasserre. "Global Optimization with Polynomials and the Problem of Moments". In: *SIAM Journal on Optimization* 11.3 (2001), pp. 796–817. DOI: 10.1137/S1052623400366802.

[12] Helfried Peyrl and Pablo A. Parrilo. "Computing Sum of Squares Decompositions with Rational Coefficients". In: *Theoretical Computer Science* 409 (2008), pp. 269–281. DOI: 10.1016/j.tcs.2008.09.025.

[13] Motakuri V. Ramana. "An Exact Duality Theory for Semidefinite Programming and Its Complexity Implications". In: *Mathematical Programming* 77 (1997), pp. 129–162. DOI: 10.1007/BF02614433.

[14] Alexander Shapiro. "Asymptotic Distribution of Test Statistics in the Analysis of Moment Structures under Inequality Constraints". In: *Biometrika* 72.1 (1985), pp. 133–144. DOI: 10.1093/biomet/72.1.133.