

Undecidable projection costs for quantum channels with finite tracial ancillas

Lluis Eriksson

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Abstract

We transfer a quantitative unitary-relation construction to a quadratic optimization problem for quantum channels. The input is a real rational orthogonal projection on the Choi space, together with an explicit rational bistochastic Choi state supported exactly on its kernel. The minimum cost over all bistochastic channels is therefore zero. Nevertheless, minimizing the same cost over channels implemented by a unitary interaction with an unrestricted finite tracial ancilla is undecidable at one fixed additive precision. The objective is a projective measurement probability, not a determinant. Its equivalent matrix formulation is the squared distance between a unitary and an amplified linear subspace. This distance at fixed dimension is the Kraus-space quantity studied by Yu, Duan and Xu; we examine its stabilization under a completely depolarizing ancillary channel. We prove the transfer with explicit constants, derive the absence of a recursive uniform ancilla bound, and give the exact answer when the subspace projection is completely positive. The source hardness and the relation compiler are inherited results; no general novelty claim about noncomputable tracial optimization is made.

1 The channel test and the main result

Let $\tau_d = \text{Tr}/d$, and put $\|X\|_{2,d}^2 = \tau_d(X^*X)$. All matrix-space norms below use this normalized trace. Traces of density matrices are ordinary, unnormalized traces. Our vectorization convention is

$$\text{vec}(K) = \sum_{i,j} K_{ij} |i\rangle_{\text{out}} \otimes |j\rangle_{\text{in}}, \quad J(T) = \sum_{j,k} T(E_{jk}) \otimes E_{jk}.$$

For $U \in \mathcal{U}(Dm)$ define

$$T_U(X) = (\text{id}_D \otimes \tau_m)(U(X \otimes I_m)U^*), \quad \rho_U = J(T_U)/D. \quad (1)$$

These channels are completely positive, unital and trace preserving. The ancilla is maximally mixed; allowing an arbitrary pure ancilla would give a different class.

Given a Hermitian projection $H \in M_{D^2}(\mathbb{Q})$, set

$$v_m(H) = \min_{U \in \mathcal{U}(Dm)} \text{Tr}(H\rho_U), \quad v_\infty(H) = \inf_{m \geq 1} v_m(H). \quad (2)$$

Operationally, prepare a maximally entangled input and reference, apply the channel to the input, and measure the two-outcome projective test $\{H, I - H\}$. The cost is the probability of the outcome H . We do not impose restrictions on the circuit implementing U .

Theorem 1. *There is a computable map from Turing machines T to pairs of real rational matrices $(H_T, \rho_{0,T})$, of a finite dimension D depending on T , such that*

$$H_T^* = H_T = H_T^2, \quad \rho_{0,T} \succeq 0, \quad \text{Tr}_{\text{in}} \rho_{0,T} = \text{Tr}_{\text{out}} \rho_{0,T} = I_D/D, \quad (3)$$

$$\text{supp}(\rho_{0,T}) = \ker H_T, \quad (4)$$

and, with $\gamma = 1/36\,288\,000\,000$,

$$T \text{ halts} \implies v_\infty(H_T) \leq \gamma/4, \quad (5)$$

$$T \text{ does not halt} \implies v_\infty(H_T) \geq \gamma. \quad (6)$$

Consequently additive approximation to error $\gamma/8$ is undecidable on these promised inputs. Yet the semidefinite relaxation

$$\min_{\rho \succeq 0, \text{Tr}_{\text{in}} \rho = \text{Tr}_{\text{out}} \rho = I_D/D} \text{Tr}(H_T \rho) = 0 \quad (7)$$

has the displayed rational optimizer $\rho_{0,T}$.

The theorem concerns optimization of a test over a class of channels. It does not decide whether the particular channel with state $\rho_{0,T}$ belongs to that class. The YES alternative gives a small cost, not exact membership in a zero-cost face. The constant is approximately $2.76 \cdot 10^{-11}$ and is not a practical numerical threshold. The reduction is computable; expansion of its source games need not run in polynomial time in the machine description.

Lee and Watrous [3] establish NP-hardness for unitary quadratic minimization in a dimension specified as part of the finite input. Here each fixed- m problem is decidable, and noncomputability enters through unrestricted auxiliary dimension. Yu, Duan and Xu [4, Lemma 2] already study the fixed-dimensional Kraus-space distance used below. Factorizable channels and their closures are established objects [5, 6, 7]. General noncomputability of tracial optimization is established by Goldbring and Hart [8], with related fixed-precision results in [9]. The proposed contribution is the simultaneous restriction to one rational projection cost, a fixed gap, and the explicit zero-cost bistochastic certificate in (4).

2 Subspace distance and its channel interpretation

For a nonzero complex linear subspace $S \subseteq M_D(\mathbb{C})$ let Π_S be its Hilbert–Schmidt orthogonal projection and define

$$h_m(S) = \min_{U \in \mathcal{U}(Dm)} \|U - (\Pi_S \otimes \text{id}_{M_m})(U)\|_{2,Dm}^2, \quad h_\infty(S) = \inf_m h_m(S). \quad (8)$$

If $B_1, \dots, B_s$ is a basis with $\tau_D(B_\ell^* B_k) = \delta_{\ell k}$, its vectorized projection and complementary cost are

$$P_S = \frac{1}{D} \sum_{\ell} \text{vec}(B_\ell) \text{vec}(B_\ell)^*, \quad H_S = I_{D^2} - P_S. \quad (9)$$

Real rational B_ℓ give real rational H_S . No rational orthonormal basis of the complementary space is needed. Positivity of the matrix P_S on vectorized Hilbert space must not be confused with complete positivity of the map Π_S .

Lemma 2. *At each finite m ,*

$$h_m(S) = \min_{X \in S \otimes M_m} \tau_{Dm}(|X| - I)^2, \quad (10)$$

$$1 - h_m(S) = \max_{\substack{X \in S \otimes M_m \\ \|X\|_{2,Dm} = 1}} (\tau_{Dm}|X|)^2. \quad (11)$$

Moreover $v_m(H_S) = h_m(S)$.

Proof. The squared distance from a fixed X to the unitary group equals $\tau(|X| - I)^2$: a unitary polar factor attains it, also when X is singular by extending its partial isometry. Minimizing $\|X - U\|_2^2$ first over $X \in S \otimes M_m$ instead gives the projection in (8). This proves (10); as minimizing X is the projection of a minimizing unitary.

Orthogonality gives

$$1 - h_m = \max_U \|(\Pi_S \otimes \text{id})(U)\|_2^2.$$

Hilbert-space duality and $\max_U |\tau(U^*X)| = \tau|X|$ then give (11), by interchanging two maxima over compact sets.

For the last claim, let $U_{ab} \in M_D$ be the ancillary blocks, namely $(U_{ab})_{ij} = U_{(i,a),(j,b)}$. Then

$$\rho_U = \frac{1}{Dm} \sum_{a,b} \text{vec}(U_{ab}) \text{vec}(U_{ab})^*, \quad \text{Tr}(H_S \rho_U) = \frac{1}{Dm} \sum_{a,b} \|(I - \Pi_S)(U_{ab})\|_F^2.$$

The latter is precisely the squared norm in (8). After reordering the indices of $\text{vec}(U)$ as $((i,j), (a,b))$, the quadratic form is $\text{vec}(U)^*(H_S \otimes I_{m^2}) \text{vec}(U)/(Dm)$. The vectorized ancillary identity has size m^2 , not m . $\square$

For a channel Φ write $K(\Phi)$ for its Kraus operator span. The quantity on the right of (10) at $m = 1$ is exactly C_Φ of [4] when $S = K(\Phi)$. The completely depolarizing channel $\Delta_m(X) = \tau_m(X)I_m$ has Kraus span M_m , so

$$h_m(K(\Phi)) = C_{\Phi \otimes \Delta_m}.$$

Thus the objective is a stabilization of an existing quantity. The Schur-channel tensoring result in [4, Theorem 2] concerns a diagonal Kraus span, rather than M_m , and does not provide a stabilization bound here.

Let $\mathcal{F}_{\text{matr}}(D)$ be the union of the channels (1), and $\mathcal{F}_{\text{fin}}(D)$ the channels factoring through finite-dimensional tracial C^* -algebras. The standard identification [7, Proposition 3.8] is

$$\mathcal{F}_{\text{fin}}(D) = \text{conv } \mathcal{F}_{\text{matr}}(D).$$

Indeed an ancillary direct sum $\bigoplus_j M_{m_j}$ with trace $\sum_j \lambda_j \tau_{m_j}$ gives the corresponding convex combination of matrix-ancilla channels, and conversely. Consequently

$$v_\infty(H) = \min_{\Phi \in \mathcal{F}_{\text{fin}}(D)} \text{Tr}(HJ(\Phi)/D). \quad (12)$$

Linearity preserves the infimum under convexification, and continuity preserves it under closure; the closure is compact. This minimum need not be realized by one finite ancilla. We do not replace this closure by all factorizable channels through arbitrary finite von Neumann algebras [5, 6].

3 The quantitative relation construction

We use the source-game and rational block construction from [1], whose hardness input is Taller–Vidick [2, Theorem 5.2]. We state precisely the imported interface and reprove the quadratic transfer. This paper does not claim a new machine-to-game theorem.

For $\varepsilon = 2^{-40}$ the construction produces a finite uniform context list. In the YES case it admits finite-dimensional synchronous strategies with errors approaching a value at most $\varepsilon/4$; in the NO case every such strategy has error at least $1/1296$. Each context gives seven signed unitary words, of length at most four; let R be their total number. For unitary generators define their mean squared relation error by

$$\xi = \frac{1}{R} \sum_{r=1}^R \|V_{i_1(r)} \cdots V_{i_{k(r)}(r)} - \sigma_r I\|_{2,m}^2.$$

A game strategy of error e gives generators with $\xi \leq 34e$, and generators with error ξ give a game strategy of error at most 50ξ , in the same dimension. Positional normalization in [1] handles repeated questions, rational sampling weights and distinct answer positions; those steps are part of the imported construction.

The construction yields real rational A_ℓ , positive integers w_ℓ, t , and $D = t^2$, with

$$\sum_{\ell} A_{\ell} A_{\ell}^* = \sum_{\ell} A_{\ell}^* A_{\ell} = I_D, \quad \text{Tr}(A_{\ell}^* A_k) = \delta_{\ell k} w_{\ell}^2, \quad \sum_{\ell} w_{\ell}^2 = D, \quad (13)$$

$$\rho := R/D \in [1/8000, 1/200]. \quad (14)$$

Set $B_{\ell} = tA_{\ell}/w_{\ell}$ and $S = \text{span}_{\mathbb{C}}\{B_{\ell}\}$. The matrices A_{ℓ} have disjoint entry supports. Their pencil $L_A(Z) = \sum_{\ell} A_{\ell} \otimes Z_{\ell}$ is a direct sum of singleton labels and 25 copies of each two-by-two block

$$G(U, V, W, Z) = \begin{pmatrix} (3/5)U & (4/5)W \\ (4/5)Z & -(3/5)V \end{pmatrix}. \quad (15)$$

Label ℓ has at least $w_{\ell}^2/2$ singleton occurrences. Every original relation has at most six assigned gate occurrences, including its own padding-square gate; these occurrence sets are disjoint across relations. A common reference label makes the construction homogeneous. Telescoping after fixing that reference bounds each word residual by the sum of its assigned gate residuals. The use of separate occurrences, even for repeated labels, is essential for the averaged estimate below.

For completeness, assign exact products to the prefix labels and identities to the reference, padding and free labels. Every gate is then unitary except the final gate for each original relation. The residual of that final gate is the original word residual, up to unitary multiplication. All these properties, including the computable rational normalization, are established in [1, Sections 2–5].

Lemma 3 (Quadratic transfer). *For this constructed space, a unitary relation assignment of mean error ξ gives*

$$h_m(S) \leq 8\rho\xi. \quad (16)$$

Conversely, every $W \in S \otimes M_m$ with $\|W\|_2 = 1$ and $\Delta = \text{dist}_{2,Dm}(W, \mathcal{U}(Dm))$ gives a synchronous game strategy of error at most

$$50(9 + 6\sqrt{2})\Delta^2/\rho. \quad (17)$$

Proof. For unitary labels in (15), put $c = 12/25$ and $E = UZ^* - WV^*$. Direct multiplication gives

$$GG^* = \begin{pmatrix} I & cE \\ cE^* & I \end{pmatrix}, \quad \|GG^* - I\|_{2,2m}^2 = c^2\|E\|_{2,m}^2.$$

Since $\|E\|_{\text{op}} \leq 2$, the singular values of G lie in $[1/5, 7/5]$. For any such singular value s , $(s-1)^2 \leq (25/36)(s^2-1)^2$. Therefore

$$\text{dist}_{2,2m}(G, \mathcal{U}(2m))^2 \leq \frac{4}{25}\|E\|_{2,m}^2.$$

Choose the completeness labels described above and $W = L_A(V) \in S \otimes M_m$. The 25 copies of each final gate have total scalar size 50; all other blocks have zero polar distance. Thus

$$h_m(S) \leq \text{dist}_{2,Dm}(W, \mathcal{U}(Dm))^2 \leq \frac{50}{D} \frac{4}{25} \sum_r e_r^2 = 8\rho\xi,$$

where the first inequality follows by using this W in the joint distance minimization.

For the reverse direction write $W = L_A(Z)$. The Gram identity gives $\sum_\ell (w_\ell^2/D) \|Z_\ell\|_2^2 = 1$. Because W is block diagonal, a nearest polar unitary P may be chosen block diagonal. For each label choose a polar unitary V_ℓ of Z_ℓ . The singleton proportion yields

$$\sum_\ell \frac{w_\ell^2}{D} \|Z_\ell - V_\ell\|_2^2 \leq 2\Delta^2.$$

For $K = L_A(V)$, disjoint supports imply $\|K - W\|_2 \leq \sqrt{2}\Delta$. Hence $\|K - P\|_2 \leq (1 + \sqrt{2})\Delta$. All gate and singleton blocks satisfy $\|K\|_{\text{op}} \leq 7/5$, giving

$$\|KK^* - I\|_2^2 \leq C_0\Delta^2, \quad C_0 = \frac{144}{25}(3 + 2\sqrt{2}).$$

If e_g is the residual norm of gate g with these unitary labels, its 25 copies contribute $50c^2e_g^2/D$ to that squared defect. The disjoint gate-occurrence sets and Cauchy–Schwarz in the at-most-six-term telescoping bounds give

$$\xi \leq \frac{6}{R} \sum_g e_g^2 \leq \frac{6C_0}{50c^2} \frac{\Delta^2}{\rho} = (9 + 6\sqrt{2}) \frac{\Delta^2}{\rho}.$$

Apply the imported game-rounding estimate $e \leq 50\xi$. This argument requires no determinant bound or maximizing determinant witness. $\square$

4 Fixed precision and an explicit bistochastic certificate

Proof of Theorem 1. Use the preceding construction and define $H_T = H_S$ by (9). In a YES instance, take strategies approaching the stated completeness value. Equation (16) gives

$$h_\infty(S) \leq 8\rho \frac{34\varepsilon}{4} \leq \frac{17}{50} 2^{-40} < \gamma/4.$$

The argument uses an infimum and does not assume the optimal strategy is attained.

For soundness suppose $h_m(S) < \gamma$ for some m , and choose a minimizing unitary U in (8). Set

$$X = (\Pi_S \otimes \text{id})(U), \quad a = \|X\|_2^2 = 1 - h_m(S), \quad W = X/\sqrt{a}.$$

Then $\|W\|_2 = 1$ and orthogonality gives $\text{Re } \tau(U^*W) = \sqrt{a}$. Consequently

$$\text{dist}(W, \mathcal{U}(Dm))^2 \leq \|W - U\|_2^2 = 2 - 2\sqrt{1 - h_m(S)} \leq 2h_m(S). \quad (18)$$

Although this U need not be block diagonal, a nearest polar unitary of W is; replacing U by that polar unitary only improves the distance. Lemma 3 gives a game strategy of error

$$e \leq 100(9 + 6\sqrt{2})h_m(S)/\rho < 1750h_m(S)/\rho < 1750 \cdot 8000\gamma = \frac{1}{2592}.$$

This contradicts the NO lower bound $1/1296$. Hence all $h_m(S) \geq \gamma$ in a NO instance. Lemma 2 transfers the conclusion to v_∞ .

Finally let $T_0(X) = \sum_\ell A_\ell X A_\ell^*$ and

$$\rho_0 = \frac{1}{D} \sum_\ell \text{vec}(A_\ell) \text{vec}(A_\ell)^* = \frac{1}{D} \sum_\ell \frac{w_\ell^2}{D} \text{vec}(B_\ell) \text{vec}(B_\ell)^*.$$

The two identities in (13) give both marginals I_D/D . In our convention the input marginal uses $(\sum_\ell A_\ell^* A_\ell)^T$, also I_D . All entries are real rational. Since every weight is positive and the B_ℓ are independent, the support is exactly $\text{vec}(S) = \ker H_T$. This proves (4) and makes ρ_0 a zero-cost optimizer of (7). An additive $\gamma/8$ approximation separates the alternatives by comparison with $5\gamma/8$, and would decide halting. $\square$

The zero relaxation value is a promise satisfied in both alternatives, not a test that detects the YES case. In a NO instance every approximately matrix-factorizable channel has test cost at least γ , even though the rational bistochastic channel T_0 has cost zero. This separates optimization over two distinct feasible channel classes.

5 Ancilla bounds, amplification and a computable boundary

Corollary 4. *No total recursive function of the encoded input length uniformly bounds an ancilla size m for which $v_m(H) \leq v_\infty(H) + \gamma/8$ on the inputs of Theorem 1.*

Proof. At fixed m , unitarity is a finite system of real polynomial equations and the objective is quadratic with rational coefficients. The feasible set is compact. Real quantifier elimination therefore computes v_m to arbitrary rational precision, uniformly in the finite input.

If a proposed bound is $b(L)$, compute every v_m , $1 \leq m \leq b(L)$, to error $\gamma/16$ and take their minimum. In YES instances this is at most $7\gamma/16$; in NO instances it is at least $15\gamma/16$. Comparison with $3\gamma/4$ decides halting. $\square$

Since $U \mapsto U \otimes I_k$ preserves normalized cost, $h_{mk}(S) \leq h_m(S)$. The sequence $h_{n!}(S)$ decreases to $h_\infty(S)$. Indeed every fixed m divides all sufficiently large factorials. Finite-dimensional quantifier elimination and rational error bounds consequently provide upper semi-computability of h_∞ ; Theorem 1 rules out a uniform computable convergence modulus at the displayed precision. This does not assert that every individual input has a noncomputable value.

For any fixed k , associativity and a permutation of tensor factors give

$$h_m(S \otimes M_k) = h_{km}(S), \quad h_\infty(S \otimes M_k) = h_\infty(S).$$

For the second identity, the infimum over multiples is no smaller than the unrestricted one, while replication of any witness into a multiple of k gives the reverse inequality. This invariance does not provide a finite stabilization dimension.

Proposition 5 (Completely positive subspace projection). *If Π_S is completely positive as a map on M_D , then for every m ,*

$$h_m(S) = 1 - \|\Pi_S(I_D)\|_{2,D}^2.$$

Thus this class has an exact computable value whenever Π_S is rational.

Proof. Put $\Psi = \Pi_S \otimes \text{id}_{M_m}$ and $A = \Psi(I) = \Pi_S(I_D) \otimes I_m$. For unitary U , complete positivity sends the positive block matrix $\begin{pmatrix} I & U \\ U^* & I \end{pmatrix}$ to $\begin{pmatrix} A & \Psi(U) \\ \Psi(U)^* & A \end{pmatrix}$. Block positivity implies $\Psi(U) = A^{1/2}CA^{1/2}$ for a contraction C on the support of A ; the statement for singular A follows by restriction to its support, or by an $A + \eta I$ limit. Schatten Hölder gives

$$\|\Psi(U)\|_2 \leq \|A^{1/2}\|_4^2 \|C\|_{\text{op}} \leq \|A\|_2.$$

Equality is attained at $U = I$. Apply the projection identity in Lemma 2 and normalized trace under tensoring. $\square$

This is a standard complete-positivity norm argument, consistent with the multiplicativity results in [10]; it is included to specify the boundary of the hardness claim. Our construction promises positivity of H as a matrix, not complete positivity of Π_S . Likewise, if S contains a unitary then $h_m(S) = 0$ for all m . In particular adjoining the identity to S would trivialize this problem. No stabilization theorem for operator norms can be applied without checking these hypotheses and the normalized Schatten norm used here.

6 Verification, dependencies and remaining questions

The transfer is analytic and uniform in m . Finite complex-matrix diagnostics check projection, polar-distance, Choi, marginal and amplification identities; rational arithmetic checks the projection certificate and the gap margins. Such tests do not prove source-game undecidability, replace the cited relation construction, or constitute a proof-assistant verification. The accompanying research record identifies exact files and rerun results.

The dependency on [1] is substantial: its rational compiler, singleton proportions, question normalization and game-rounding constants are reused. The forward estimate here is proved directly from the gate singular values, and the reverse estimate applies to a projected arbitrary unitary. The channel formulation and the zero-relaxation certificate make the consequence externally interpretable, but do not remove that dependence. The interpretations of (10) and (12) are established background, explicitly credited above. Targeted literature checking has not certified worldwide priority.

The theorem gives no fixed value of D , no bounded circuit architecture, no laboratory resource estimate, and no communication-capacity result. It also does not show undecidability of mixed-unitary membership at a fixed finite dimension. Determining the least number of Kraus directions needed for this phenomenon, or useful restrictions beyond complete positivity of Π_S with effective ancilla bounds, remain separate questions.

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