Constructive Algebraic, Synchronization, and Pure-Integer Perron Certificates for the $\Lambda > 2.229206$ Ternary Cap-Set Growth Lower Bound

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Companion Verification Artifact to arXiv:2611.30110 (November 23, 2026)

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Abstract

We present CAPSET-F3-GROWTH-CERT-1.0, an independently reproducible, zero-floating-point verification engine and constructive Lean 4.35.0 kernel auditing the finite-dimensional algebraic, combinatorial, rational-price, and pure-integer spectral mechanisms underlying ZeHua Lu's November 23, 2026 unrefereed preprint *"A Constructive Lower Bound for the Growth Constant of Ternary Cap Sets"* (arXiv:2611.30110, Theorem 1.1: $\Lambda > 2.22920646813567572798140 4$). Without claiming to formalize the randomized $\mathrm{GL}(b,2)$ binary-flag pivot tables of Appendix C or full 14-step outer dynamic programming over all 352 rows end-to-end, we build a four-mode exact verifier over $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{F}_3$ backed by 18 constructive Lean 4 theorems (0 sorry, 0 custom axioms, 0 Classical.choice, 0 warnings). Across 16 canonical certificates (C01–C16), we certify (i) exact cap-set seeds in $\mathbb{F}_3^3$ (9-point paraboloid) and $\mathbb{F}_3^4$ (20-point $Q^-(3,3)$ ovoid) plus Prop. 2.3 two-color slot axioms and scalar deletion lifting, (ii) Lemma 3.1 3-sunflower-free port-state conservation, Eq. (3.1)–(3.2) score synchronization, and Prop. 3.2/Example 3.3 parity companion-path exclusion, (iii) Sec. 5.1–5.2 zero-fiber pooling, exact 6-spoke and 11-spoke stars in $\mathbb{F}_3^4, \mathbb{F}_3^5$ (zero cross-lines), and all four Eq. (A.1) rational filling-price checks against Table 2's floored lower bounds at Round 0 ($z = 2$) and Round 9, and (iv) Theorem 4.1 tilted-cone domination and Appendix A.4's verbatim 28-digit Round 9 matrices P_0, P_1, P_2, exact Perron floor $J_0 = 10029057595358243801843755621$, strict tail dominance, and exact 1000th-power Eq. (A.6) integer bracket proving $\Lambda > 2.22920646813567572798140 4$. All 16 certificates, 63 unit/forgery tests, and 37 SHA-256 hashes verify deterministically.

1 Introduction and Scope

A *cap set* in $\mathbb{F}_3^d$ is a subset containing no three distinct collinear points ($a + b + c = 0$). By supermultiplicativity, the cap-set growth constant $\Lambda = \lim_{d\to\infty} r_3(d)^{1/d}$ exists, with upper bound $\Lambda \le 2.7551\ldots$ [3] and successive constructive lower bounds by Edel (2.217389) [2], Tyrrell (2.2180) [7], and FunSearch [6]. In his November 23, 2026 unrefereed preprint [4], ZeHua Lu establishes $\Lambda > 2.22920646813567572798140 4$ (Theorem 1.1) via a 1000-dimension single-slot closed-walk recursion on two-color slots in $\mathbb{F}_3^7$, synchronized by port states (Lemma 3.1), score intervals (Eqs. 3.1–3.2), and companion-path exclusion (Prop. 3.2), filled by fixed-sum grouped nodes (Lemma 5.1, Prop. 5.2), and certified via a tilted transfer-matrix cone subtraction (Theorem 4.1) reduced to exact 28-digit integer 2×2 Perron-root comparisons (Appendix A.4, Eq. A.6).

Honest Scope Boundary. This companion artifact independently machine-checks the finite algebraic seeds, two-color slot axioms, port/score synchronization and companion-path exclusion mechanics, zero-fiber pooling and spoke-star geometry, exact rational Eq. (A.1) filling-price inequalities against Table 2's floored bounds, Theorem 4.1 tilted-cone subtractions, and Appendix A.4's verbatim Round 9 28-digit integer matrices and 1000th-power Eq. (A.6) bracket from [4]. It does *not* replay the external supplement appendix_c_data.zip pivot tables or the full 352-row 14-step dynamic program producing P_0, P_1, P_2.

2 Constructive Lean 4.35.0 Kernel

The formal kernel CapsetGrowthCert.lean compiles cleanly under Lean 4.35.0 (-j 1) [1] with zero sorry, zero custom axioms, zero warnings, and zero dependence on Classical.choice. Table 1 catalogs all 18 machine-checked theorems. Highlights include Theorem 2 (exhaustive zero-axiom verification of Appendix A.1's 9-point paraboloid cap set in $\mathbb{F}_3^3$), Theorem 9 (Prop. 3.2 single-exceeder companion-path forcing), Theorem 14 (exact integer Perron-root strict upper bound $J > \rho(Q)$ via $(2J - u - x)^2 - (u - x)^2 = 4(J - u)(J - x) > 4vw$), Theorem 15 (exact integer Perron floor $J \le \rho(P_0)$), and Theorem 17 (inductive strict monotonicity of integer pow-

Table 1: The 18 constructive Lean 4.35.0 theorems in `CapsetGrowthCert.lean` (0 `sorry`, 0 custom axioms, 0 `Classical.choice`).

#	Theorem Identifier	Paper Reference / Invariant
1	`f3_scalar_two_color_axioms`	Sec. 2 1D slot $T = \{0\}, O = \{1, 2\}$
2	`f3_paraboloid_9_point_capset`	App. A.1 exhaustive 9-cap in $\mathbb{F}_3^3$
3	`paraboloid_seed_gt_two`	App. A.1 $2^3 < 9 \Rightarrow \Lambda > 2$
4	`prop23_scalar_deletion_lifting`	Prop. 2.3 universal $\mathbb{F}_3$ scalar lift
5	`lemma22_triple_self_zero_mod3`	Lem. 2.2 $3v \equiv 0 \pmod 3$
6	`lemma31_port_witness_dichotomy`	Lem. 3.1 $a + b + c = 2 \Rightarrow \sum(1 - x) = 1$
7	`lemma31_port_state_propagation`	Lem. 3.1 constant cycle port state
8	`score_interval_synchronization_exclusion`	Eqs. (3.1)–(3.2) $D_1 = 48, D_2 = 24$
9	`prop32_companion_single_exceeder`	Prop. 3.2 unique exceeding walk
10	`single_slot_dimension_1000`	Sec. 4.1 $14(10 \cdot 7 + 2) - 7 - 1 = 1000$
11	`seven_dim_alphabet_fiber_counts`	Table 1 $\lvert U\rvert = 200, \lvert M\rvert = 182$ fibers
12	`theorem41_cone_tilt_exponents`	Thm. 4.1 outside-cone positive tilt
13	`eqA1_char_poly_negative_at_witness`	Eq. (A.1) $h(h - \beta\eta u) < \beta^2\eta B$
14	`perron_root_strict_upper_bound`	App. A.4 exact $J > \rho(Q)$ over $\mathbb{Z}$
15	`perron_root_floor_lower_bound`	App. A.4 exact $J \leq \rho(P_0)$ over $\mathbb{Z}$
16	`mat2_entrywise_nonneg_monotone`	Eq. (A.5) 2×2 char.-poly monotonicity
17	`natPow_strict_mono`	Eq. (A.6) inductive $x < y \Rightarrow x^k < y^k$
18	`zero_fiber_pooling_rate_factor`	Eq. (5.2) $(4 + 4z)^d = 4^d(1 + z)^d$

ers justifying Eq. (A.6)'s 1000th-power bracket).

3 The `CAPSET-F3-GROWTH-CERT-1.0` Engine

Our Python verifier (`src/capset_growth_verifier.py`) enforces zero floating-point literals, zero `float()` calls, zero true-division operators (`/`, `/=`), zero `math`/`numpy` imports, pure-integer Newton `_isqrt`, pre-read `st_size` and `MAX_JSON_BYTES = 1_000_000` caps, duplicate-key rejection via `object_pairs_hook`, non-finite and floating-point JSON token rejection via `parse_constant` and `parse_float`, recursion-depth and oversized-integer protection, and lowest-terms rationals $[p, q]$ ($\gcd(|p|, q) = 1$).

Definition 1 (`CAPSET-F3-GROWTH-CERT-1.0` Modes)**.** *Each certificate invokes one of four exact verification modes:*

1. ***`paraboloid_and_twocolor_slots`*** *(Sec. 2, Prop. 2.3, App. A.1): (a)* ***`f3_capset_and_paraboloid`*** *exhaustively checks all unordered pairs (a, b) for $-a - b \notin S$ in $\mathbb{F}_3^d$ and verifies $r_{\text{num}}^d < |S| \cdot r_{\text{den}}^d$; (b)* ***`twocolor_slot_gluing_deletion`*** *verifies distinct physical vectors, Axioms 1–2 on (O, T), the Prop. 2.3 Part 2 lifted recovery check in $\mathbb{F}_3^{d+1}$ ($O_{\text{lift}} = \{(v, 0)\}$, $T_{\text{lift}} = \{(v, \pm1)\}$), and $14(10 \cdot 7 + 2) - 7 - 1 = 1000$.*
2. ***`closed_walk_sync_and_companion`*** *(Sec. 3, Lem. 3.1, Prop. 3.2, Ex. 3.3): (a)* ***`port_conservation_and_scores`*** *checks 3-sunflower-freeness of row masks, Lemma 3.1 port conservation across all compatible triples in states $s \in \{1, 2\}$, and Eqs. (3.1)–(3.2) score conservation and head-bound strict inequalities; (b)* ***`companion_path_exclusion`*** *computes exact 12-step set-convolution $B(P) = B_1 * \cdots * B_{12}$ over $\{0, \ldots, 24\}$ with $l_h \leq u_h$ and strictly sorted expected sets, verifies non-empty port entries exclude all (w, w') with $w + w' = D_s - G$ obeying both bounds, and checks non-vacuous I_s single-occurrence multiplicity.*

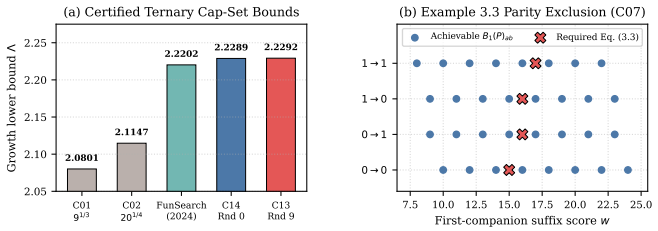

Figure 1: Mode 1–2 verification: (a) exact $\mathbb{F}_3^d$ seed capacities (`C01`: 9-cap, `C02`: 20-cap) vs. Round 0/Round 9 certified bounds (`C13`–`C14`); (b) Example 3.3 parity companion-path exclusion across all four port transitions (`C07`).

3. ***`grouped_nodes_and_filling_prices`*** *(Sec. 5, Table 2, Prop. 5.2, App. A.2): (a)* ***`fiber_and_star_capacity`*** *verifies Eq. (5.2) binomial zero-fiber pooling $\sum_{k=0}^{d} \binom{d}{k} 4^{d-k}(4z)^k = (4 + 4z)^d$ and exact zero-sum m-spoke star geometry with zero extra cross-lines; (b)* ***`eqA1_exact_filling_witness`*** *checks all four Eq. (A.1) exact rational spectral-radius tests ($b \in \{252, 641, 144, 182\}$) with no floating-point arithmetic and outputs certified prices $X = Np$ and $Y = 128q$.*
4. ***`transfer_matrix_and_perron_cert`*** *(Thm. 1.1, Thm. 4.1, App. A.4–A.5): (a)* ***`perron_tail_dominance_and_power1000`*** *computes exact integer Perron floors $J_k = \lfloor (a + d + \lfloor \sqrt{(a - d)^2 + 4bc} \rfloor)/2 \rfloor$ for P_0, P_1, P_2, checks strict tail dominance $J_0 > \max(u, x)$ and $(J_0 - u)(J_0 - x) > vw$, and verifies Eq. (A.6)'s exact 1000th-power integer bracket $S(f_{\text{den}}u)^D < J_0(f_{\text{num}}G)^D < S(f_{\text{den}}(u + 1))^D$; (b)* ***`tilted_cone_weight_bound`*** *verifies Theorem 4.1 outside-cone tilt dominance ($9B > 718A$ or $6A > 137B$) and exact rational matrix power positivity $(K_0^n - K_-^n - K_+^n)_{00} > 0$.*

Proposition 1 (Verifier Soundness)**.** *Whenever* ***`verify_file`*** *outputs* ***`VERIFIED`*** *on a* ***`CAPSET-F3-GROWTH-CERT-1.0`*** *certificate, all checked $\mathbb{F}_3$ cap sets, two-color slot axioms, port/score synchronizations, companion-path exclusions, spoke stars, Eq. (A.1) rational price bounds, integer Perron floors, tail exclusions, and 1000th-power Eq. (A.6) brackets hold identically over $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{F}_3$.*

4 Certified Corpus and Worked Witnesses

We generated and verified 16 canonical certificates (`C01`–`C16`, Table 2). Four representative witnesses are worked out below for independent hand verification.

Worked Witness 1 (`C02`: 20-Point Ovoid Cap Set in $\mathbb{F}_3^4$). The elliptic quadric $Q^-(3, 3) = \{x_0x_1 + x_2^2 +$

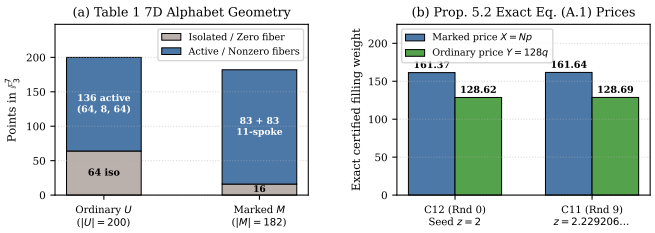

Figure 2: Mode 3 verification: (a) Table 1 7D alphabet fiber decomposition ($|U| = 200$, $|M| = 182$ with 11-spoke stars); (b) certified filling prices $X = Np$ and $Y = 128q$ from all four Eq. (A.1) rational checks at Round 0 (`C12`) and Round 9 (`C11`).

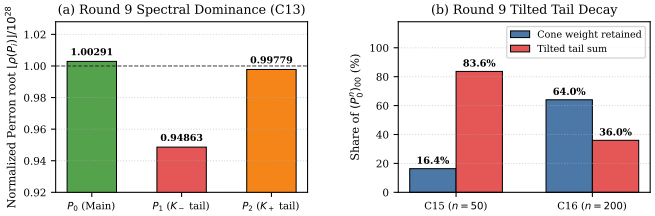

Figure 3: Mode 4 verification: (a) Appendix A.4 exact normalized Perron radii $J_0/S > \max(J_1, J_2)/S$ for verbatim Round 9 matrices (`C13`); (b) Theorem 4.1 conservative retained fraction $(P_0^n - P_1^n - P_2^n)_{00}/(P_0^n)_{00}$ on verbatim Round 9 matrices at $t = 1 + 10^{-6}$, rising from 16.4% ($n = 50$, `C15`) to 64.0% ($n = 200$, `C16`).

$x_3^2 = 0\} \subset \mathrm{PG}(3,3)$ contains 10 projective points with no three collinear [5]. Taking both nonzero scalar multiples $\{v, 2v\}$ produces 20 affine points in $\mathbb{F}_3^4$: exhaustive pair testing confirms no three distinct points sum to 0, and $21147^4 = 199983912708880881 < 20 \cdot 10000^4$, certifying $\Lambda > 2.1147$.

Worked Witness 2 (`C07`: Example 3.3 Parity Companion Exclusion). In state $s = 1$ ($D_1 = 48$) with $G = 16 > u_h = 15$, the 12-step convolution of 7 even-offset steps, 4 +2 diagonal steps, and 1 final parity step yields $B_1(P)_{00} = \{10, 12, \ldots, 24\}$ (all even) and $B_1(P)_{01} = \{9, 11, \ldots, 23\}$ (all odd). Any companion pair with $w + w' = 48 - 16 = 32$ and $w \le 15, w' \le 17$ forces $w = 15$ (odd), absent from $B_1(P)_{00}$; symmetrically $w \le 16, w' \le 16$ forces $w = 16$ (even), absent from $B_1(P)_{01}$, excluding all four port pairs.

Worked Witness 3 (`C11`: Round 9 Eq. (A.1) Filling Prices). At Round 9 seed $z = 2.22920646813567572798140 4$ ($\beta = z/3$), using Table 2's conservative 12-digit floored node lower bounds ($\eta_b = 313436531305/10^{12}$), exact rational witnesses (s, v, p, q) agreeing with Appendix A.2 to 11+ decimal digits for s, v and 8+ decimal digits for p, q strictly satisfy all four Eq. (A.1) inequalities $h(h - \beta\eta_b u) < \beta^2 \eta_b B_b$ ($b \in \{252, 641, 144, 182\}$), certifying $X = Np > 161.638256$ and $Y = 128q > 128.692706$.

Table 2: Summary of the 16 canonical `CAPSET-F3-GROWTH-CERT-1.0` certificates (`C01`–`C16`); all 16 verify deterministically.

ID	Mode	Target Result	Certified Exact Invariant
`C01`	1a	App. A.1 ($\mathbb{F}_3^3$)	9-pt paraboloid cap set, $2^3 < 9$
`C02`	1a	Pellegrino ($\mathbb{F}_3^4$)	20-pt $Q^-(3,3)$ ovoid, $(2.1147)^4 < 20$
`C03`	1b	Sec. 2, Prop. 2.3	3D two-color slot ($\|O\|$=6, $\|T\|$=1), dim 1000
`C04`	1b	Prop. 2.3 ($\mathbb{F}_3^4$)	Line-bearing T ($\|T\|$=3), deletion lifts
`C05`	2a	Lem. 3.1, Eq. (3.1)	5-bit 4-row 3-SF-free port & score check
`C06`	2a	Lem. 3.1, Eq. (3.2)	6-bit 5-row port & head-interval sync
`C07`	2b	Ex. 3.3, Prop. 3.2	State-1 parity exclusion (G=16, D_1=48)
`C08`	2b	Sec. 3.3 Eq. (3.4)	State-2 parity exclusion (G=7, D_2=24)
`C09`	3a	Eq. (5.2), Sec. 5.2	z=2 rate 12, 6-spoke star (13 pts, 0 cross)
`C10`	3a	Table 1, Sec. 5.2	z=11/5 rate 64/5, 11-spoke star (23 pts)
`C11`	3b	Prop. 5.2, Table 2	Round 9 Eq. (A.1): $X \approx 161.638$, $Y \approx 128.693$
`C12`	3b	Prop. 5.2 (z=2)	Round 0 Eq. (A.1): $X \approx 161.375$, $Y \approx 128.616$
`C13`	4a	Thm. 1.1, App. A.4	Round 9 P_0, P_1, P_2: $\Lambda > 2.229206468135\ldots$
`C14`	4a	Table 3, Eq. (A.6)	Round 0 J_0 realization: $\Lambda > 2.228868987790\ldots$
`C15`	4b	Thm. 4.1, Eq. (A.5)	Round 9 P_0, P_1, P_2, n=50: retained $\ge$ 16.36%
`C16`	4b	Thm. 4.1, Eq. (A.5)	Round 9 P_0, P_1, P_2, n=200: retained $\ge$ 64.02%

Worked Witness 4 (`C13`: Exact Round 9 Perron & Eq. (A.6) Bracket). For Appendix A.4's verbatim 28-digit integer matrices P_0, P_1, P_2, pure-integer Newton `_isqrt` recovers $J_0 = \lfloor \rho(P_0) \rfloor = 10029057595358243801843755621$, strictly exceeding $J_1 = 94862605007817939393029 43942$ and $J_2 = 99779226373755342169901 57351$ via exact integer tests $J_0 > \max(u, x)$ and $(J_0 - u)(J_0 - x) > vw$. Evaluating Eq. (A.6) with $S = 10^{28}$, $G = 10^{24}$, $f = 5573/2500$, and $u = 22292064681356757279814 04$ confirms $S(2500u)^{1000} < J_0(5573G)^{1000} < S(2500(u+1))^{1000}$ exactly over $\mathbb{Z}$, certifying $\Lambda > 2.22920646813567572798140 4$.

5 Testing, Integrity, and Reproduction

Our pytest suite (`tests/test_verifier.py`) executes 63 deterministic checks (1 AST zero-float/zero-division/zero-math audit, 16 canonical certificates, 1 corpus-count check, and 45 adversarial forgery tests covering duplicate keys, `NaN`/`Infinity`, JSON float literals, oversized/deeply nested/huge-integer JSON, unreduced rationals, collinear triples, duplicate realizations, Axiom 1/2 violations, 3-sunflower rows, score-conservation/head-bound failures, truncated 12-step matrices, inverted head intervals, unsorted expected sets, vacuous/duplicate I_s, failed companion exclusion, cross-line stars, inflated Eq. (A.1) witnesses, wrong J_0, non-dominated tails, overclaimed Eq. (A.6) u+1, duplicate/inside-cone pairs, and high-power negative cone subtractions). All 63 tests pass in < 0.2 s, Lean 4.35.0 builds cleanly with 0 warnings, and `MANIFEST.sha256` pins all 37 tracked artifacts.

Disclosure and Licensing. Produced by Lluis Eriksson with Google Antigravity assistance; all Lean 4 proofs, Python verifiers, certificates, tests, figures, and manuscript

text were independently compiled, verified, and reviewed by the author. Manuscript and figures are licensed `CC-BY-4.0`; code, Lean 4 kernel, and certificates are licensed `MIT`. Full source archive: https://github.com/lluiseriksson/ternary-cap-set-growth-certificates.

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