

A sharp polynomial Lyapunov threshold for a stable cubic family

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Abstract

We determine exactly when the globally asymptotically stable cubic family introduced by Liu and Fitzsimmons admits polynomial Lyapunov certificates. In their normalization, with damping $a > 0$ and angular forcing $b \in \mathbb{R}$, the criterion is

$$|b|^{2/3} < 1 + a^{2/3}.$$

The same strict inequality characterizes positive homogeneous polynomial certificates with weak nonpositive Lie derivative, strict polynomial certificates, and local real-analytic certificates. Equality admits none of them. Below the threshold we construct a certificate as a finite product of positive quadratic forms. The proof pairs positive harmonic functions with an explicit analytic square-root test and constructs a complementary limiting harmonic pair at the boundary. Poisson smoothing and equal-weight quantiles then produce a polynomial with a strict margin. For fixed b , an explicit sufficient degree is $O_b(\delta^{-3})$, where δ is the damping excess above the threshold, while the minimum degree diverges; no matching degree rate is asserted. The threshold persists for every single even angular harmonic and nonzero angular speed. Classical binary-form factorization gives the corresponding sum-of-squares consequence without increasing the Lyapunov degree. The family, its global stability, the original counterexample to a polynomial converse theorem and the nonexistence of weak polynomial certificates for $|b| \geq 2(a + 1)$ are prior results of Liu and Fitzsimmons. Our classification concerns certificates, not a change in dynamical stability.

1 The family and the exact certificate region

For $a > 0$ and $b \in \mathbb{R}$, consider

$$\begin{aligned}\dot{x} &= -a(x^2 + y^2)x + b(x^2 - y^2)x - (x^2 + y^2)y, \\ \dot{y} &= -a(x^2 + y^2)y + b(x^2 - y^2)y + (x^2 + y^2)x.\end{aligned}\tag{1}$$

In polar coordinates $x = r \cos \theta, y = r \sin \theta$, this is

$$\dot{r} = r^3(-a + b \cos 2\theta), \quad \dot{\theta} = r^2.\tag{2}$$

Liu and Fitzsimmons introduced this family [1, Sec. IV-D]. They prove, for every $a > 0$ and $b \in \mathbb{R}$, global asymptotic stability of the origin through the Lyapunov function $r^2 e^{-b \sin 2\theta}$, and, for $a > 0$ and $|b| \geq 2(a + 1)$, that no positive definite homogeneous polynomial has nonpositive Lie derivative [1, Theorem IV.3]. The bound behind the latter is $|b|/2 < a + 1$, obtained from the second Fourier coefficient of the normalized logarithmic derivative together with $|g_2| < 1$ [1, Lemma III.3]. For the single field $(a, b) = (1, 5)$ they also exclude local real-analytic Lyapunov functions by a leading-term argument [1, Theorem II.2(4), Sec. III-D]; the family version [1, Theorem IV.3] is stated for homogeneous polynomials only, and the extension of the analytic statement to the region $|b| \geq 2(a + 1)$ is a routine consequence of their argument (see Section 6), not a statement we find there. Their work already refutes the homogeneous polynomial Lyapunov

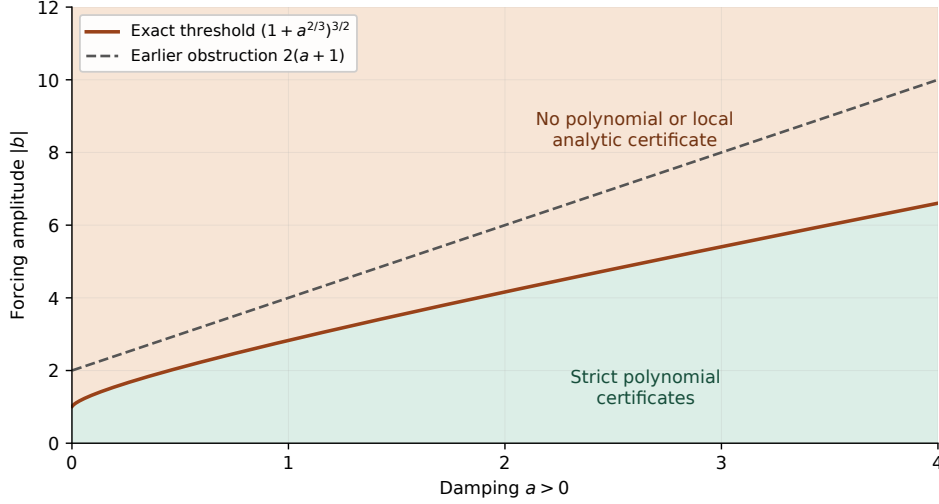


Figure 1: Certificate regions for the stable cubic family. All displayed parameters with $a > 0$ give globally asymptotically stable dynamics. The solid curve is excluded from the polynomial and local analytic certificate region. The dashed line marks the sufficient nonexistence condition for polynomial certificates in [1, Theorem IV.3]; it is not a stability boundary.

converse. We determine the exact polynomial and local analytic certificate region of this family; it lies strictly inside $|b| < 2(a + 1)$ (Remark 2). The test functions below pair the whole sequence of Fourier coefficients at once, not only the second one.

For a vector field f , write $L_f V = \nabla V \cdot f$. A *weak certificate* means $V(0) = 0$, $V > 0$ away from the origin, and $L_f V \leq 0$. A *strict certificate* has $L_f V < 0$ away from the origin. For a local analytic certificate these conditions are required on a punctured neighborhood of zero. All homogeneous certificates below have positive degree.

Theorem 1. For (1), the following are equivalent:

1. $|b| < (1 + a^{2/3})^{3/2}$.
2. There exists a positive definite homogeneous polynomial weak certificate.
3. There exists a strict homogeneous polynomial certificate that is a finite product of positive definite real quadratic forms.
4. There exists a local real-analytic weak certificate.

Thus equality at the threshold is excluded even for weak certificates. Every system in (1) is nevertheless globally asymptotically stable.

The stability assertion is recalled from [1, Sec. IV-D], not new. For orientation, its nonpolynomial certificate is

$$H(r, \theta) = r^2 \exp(-b \sin 2\theta), \quad L_f H = -2ar^2 H.$$

It is positive, coercive, continuously differentiable at the origin, and smooth away from it. The new assertion in Theorem 1 is the exact algebraic and analytic threshold.

Unbounded required polynomial degree in fixed dimension and field degree was already established by Ahmadi and Parrilo [2, Theorem 4.3]. Ahmadi and El Khadir [3, Proposition 6.1] also compare polynomial and rational certificate complexity (a degree-five family). Our degree result concerns approach to the explicit boundary above. Compactness, positive harmonic representations, and the binary-form sum-of-squares principle are classical tools, not new general converse theorems.

Remark 2. For $a > 0$ one has $1 + a < (1 + a^{2/3})^{3/2} \leq \sqrt{2}(1 + a) < 2(1 + a)$. Indeed, for $x, y > 0$ and $p = 2/3$, the p -power sum satisfies $x + y \leq (x^p + y^p)^{1/p} \leq 2^{1/p-1}(x + y)$; take $x = 1, y = a$. So the new threshold is strictly below the earlier sufficient nonexistence bound, and the polynomial and analytic obstruction of [1] covers only part of the excluded region.

2 Polynomial certificates as positive harmonic data

Use normalized circle averages $\langle v \rangle = (2\pi)^{-1} \int_0^{2\pi} v(\phi) d\phi$. The circular Hilbert transform $\mathcal{H}$ is normalized by

$$\mathcal{H} \cos(k\phi) = \sin(k\phi), \quad \mathcal{H} \sin(k\phi) = -\cos(k\phi), \quad \mathcal{H} 1 = 0, \quad k \geq 1.$$

On smooth real functions it is skew-adjoint for this average. The formulas used below can equally be verified termwise from Fourier series.

Lemma 3. *Let P be a positive definite real homogeneous polynomial of degree $2N$, $N \geq 1$, and set $p(\theta) = P(\cos \theta, \sin \theta)$. There exist $\alpha_1, \dots, \alpha_N \in \mathbb{D}$ such that*

$$h(z) = \frac{1}{N} \sum_{j=1}^N \frac{1 + \alpha_j z}{1 - \alpha_j z}, \quad \frac{p'(\theta)}{2Np(\theta)} = \operatorname{Im} h(e^{2i\theta}).$$

Here $h(0) = 1$, and $u = \operatorname{Re} h$ has a strictly positive minimum on the closed disk. The weak certificate condition is equivalent to

$$w(\phi) := a - b \cos \phi - \mathcal{H}u(\phi) \geq 0. \quad (3)$$

Proof. The strict Fejér–Riesz factorization writes

$$p(\theta) = |C|^2 \prod_{j=1}^L |1 - \alpha_j e^{2i\theta}|^2, \quad L \leq N, \quad |\alpha_j| < 1.$$

This is the factorization of [1, Lemma III.2 and proof of Lemma III.3]; for completeness here is a direct proof. The even trigonometric polynomial p has only the frequencies $2j$, $|j| \leq N$, so $q(\psi) = p(\psi/2) = \sum_{|j| \leq L} c_j e^{ij\psi}$ is a positive trigonometric polynomial of actual degree $L \leq N$, with $c_{-j} = \bar{c}_j$ and $c_L \neq 0$. If $L = 0$, p is constant and nothing is to be proved. Otherwise $A(z) = z^L \sum_j c_j z^j$ is a polynomial of degree $2L$ with $A(0) \neq 0$, $A(e^{i\psi}) = e^{iL\psi} q(\psi) \neq 0$, and $A(z) = z^{2L} \overline{A(1/\bar{z})}$. Hence its roots are not on the unit circle and come in pairs $\zeta, 1/\bar{\zeta}$ with equal multiplicity. Let $\zeta_1, \dots, \zeta_L$ be the roots in $\mathbb{D}$ (none is zero). For $|z| = 1$ one has $|z - \zeta| = |1 - \bar{\zeta}z|$ and $|z - 1/\bar{\zeta}| = |1 - \bar{\zeta}z|/|\zeta|$, so $q(\psi) = |A(e^{i\psi})| = K \prod_l |1 - \alpha_l e^{i\psi}|^2$ with $\alpha_l = \bar{\zeta}_l$, $K = |c_L| / \prod_l |\zeta_l| > 0$. Put $\psi = 2\theta$ and $|C|^2 = K$. Pad the list with zero parameters to length N . Each kernel has real part

$$\operatorname{Re} \frac{1 + \alpha z}{1 - \alpha z} = \frac{1 - |\alpha z|^2}{|1 - \alpha z|^2} > 0 \quad (|z| \leq 1).$$

Differentiating the logarithm of a factor gives twice the imaginary part of that kernel, proving the angular identity. The imaginary part of h is $\mathcal{H}u$ because $h(0) = 1$. Finally,

$$L_f P = 2Nr^{2N+2} p(\theta) \left[-a + b \cos 2\theta + \frac{p'(\theta)}{2Np(\theta)} \right],$$

which proves (3). □

Conversely, an equal-weight average of these kernels yields a polynomial of degree $2N$. Explicitly, put

$$Q_\alpha(x, y) = (1 + |\alpha|^2)(x^2 + y^2) - 2\operatorname{Re} \alpha(x^2 - y^2) + 4\operatorname{Im} \alpha xy. \quad (4)$$

Its eigenvalues are $(1 - |\alpha|)^2, (1 + |\alpha|)^2$. The product $P = \prod_j Q_{\alpha_j}$ is positive definite and satisfies the angular identity in Lemma 3. A strict margin in (3) therefore gives a strict polynomial certificate.

3 The strict obstruction

Rotation by $\pi/2$ changes the sign of b , so assume $b \geq 0$. For $c > 0$, define

$$\begin{aligned} s &= \sqrt{1 + c^2}, & \eta &= \frac{c + i}{s}, \\ M_c(z) &= c^2 - 1 + 2ic \frac{1 + \eta z}{1 - \eta z}, & f_c(z) &= \sqrt{M_c(z)}. \end{aligned} \quad (5)$$

The Cayley transform has positive real part on $\mathbb{D}$; hence M_c takes values in the upper half-plane. We choose the analytic square root in the first quadrant. In particular,

$$f_c(0) = c + i, \quad f'_c(0) = \frac{2ic}{s}. \quad (6)$$

For $0 < \sigma < 1$, write $f_c(\sigma e^{i\phi}) = F_\sigma(\phi) + iG_\sigma(\phi)$. Both functions are positive and smooth. Their means and first coefficient give

$$\langle F_\sigma \rangle = c, \quad \langle G_\sigma \rangle = 1, \quad \langle G_\sigma \cos \phi \rangle = \frac{\sigma c}{s}, \quad F_\sigma + \mathcal{H}G_\sigma = c. \quad (7)$$

If u, w come from a weak polynomial certificate, then $\langle u \rangle = 1$, $u \geq \varepsilon > 0$, and $w \geq 0$. Using (3) and skew-adjointness,

$$\begin{aligned} \langle uF_\sigma + wG_\sigma \rangle &= \langle u(F_\sigma + \mathcal{H}G_\sigma) \rangle + a \langle G_\sigma \rangle - b \langle G_\sigma \cos \phi \rangle \\ &= c + a - \frac{b\sigma c}{s} \geq \varepsilon c. \end{aligned} \quad (8)$$

Every function in this computation is smooth for $\sigma < 1$; no singular boundary pairing is assumed. Letting $\sigma \uparrow 1$ yields

$$b \leq \frac{(a + c)s}{c} - \varepsilon s < \frac{(a + c)\sqrt{1 + c^2}}{c}. \quad (9)$$

The derivative of the last expression is

$$\frac{c^3 - a}{c^2 \sqrt{1 + c^2}}.$$

Its minimum occurs at $c = a^{1/3}$, with value $(1 + a^{2/3})^{3/2}$. This proves the strict necessary inequality, including nonexistence at equality. The positive minimum of u , rather than mere nonnegativity, is what excludes the boundary.

4 An explicit limiting pair on the boundary

The converse needs more than a necessary moment inequality. We construct positive harmonic data at the critical parameters, then approximate them while retaining a positive margin.

Fix $c > 0$, with s, η, f_c as above, and put

$$a_0 = c^3, \quad b_0 = s^3.$$

Define

$$q_c(z) = 1 - c^4 + \frac{b_0}{2} [(c - i)z + (c + i)z^{-1}], \quad (10)$$

$$h_c(z) = ia_0 - \frac{ib_0}{2}(z + z^{-1}) + i \frac{q_c(z)}{f_c(z)}. \quad (11)$$

Lemma 4. *The function h_c is analytic in $\mathbb{D}$ (it extends analytically across zero), continuous on $\overline{\mathbb{D}}$, and satisfies $h_c(0) = 1$. Its boundary data obey*

$$u_c = \operatorname{Re} h_c \geq 0, \quad w_c = a_0 - b_0 \cos \phi - \operatorname{Im} h_c(e^{i\phi}) \geq 0,$$

and $u_c w_c \equiv 0$: w_c vanishes on one closed arc of the circle and u_c on the complementary closed arc. Moreover $\operatorname{Re} h_c > 0$ in $\mathbb{D}$.

Proof. Step 1: algebra. Since $c + i = s\eta$ and $c - i = s\bar{\eta}$,

$$M_c(z) = \frac{(c + i)^2 - \eta z(c - i)^2}{1 - \eta z} = s^2 \frac{\eta^2 - \bar{\eta}z}{1 - \eta z}, \quad (12)$$

because $\eta\bar{\eta}^2 = \bar{\eta}$. Using $\eta^2 + \bar{\eta}^2 = 2(c^2 - 1)/s^2$ and $1 - c^4 = s^2(1 - c^2)$, expansion of the right-hand side shows

$$q_c(z) = -\frac{s^2}{2} M_c(z)(1 - \eta z)(1 - \bar{\eta}/z) = -\frac{s^4}{2} (\eta^2 - \bar{\eta}z)(1 - \bar{\eta}/z). \quad (13)$$

So M_c is a Möbius map with its only pole at $z = \bar{\eta}$ and its only zero at $z = \eta^3$ (since $\eta^2/\bar{\eta} = \eta^3$). Both lie on the unit circle and are distinct, because $\eta^4 \neq 1$ for $c > 0$.

Step 2: branch and interior. M_c is analytic near $\overline{\mathbb{D}} \setminus \{\bar{\eta}\}$, and in $\mathbb{D}$ $\operatorname{Im} M_c = 2c \operatorname{Re} \frac{1+\eta z}{1-\eta z} > 0$. The square root with values in the closed first quadrant is continuous on the closed upper half-plane and analytic on the open one. Hence f_c is analytic and zero-free in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}} \setminus \{\bar{\eta}\}$. By (6), $f_c(0) = c + i$ and $f'_c(0) = M'_c(0)/(2f_c(0)) = 2ic/s$. In q_c/f_c only the term in z^{-1} is singular at 0. Its coefficient is $\frac{b_0}{2} \frac{c+i}{f_c(0)} = \frac{b_0}{2}$, which cancels the term $-\frac{ib_0}{2} z^{-1}$ of (11). The constant coefficient of h_c is

$$ia_0 + \frac{i(1 - c^4)}{f_c(0)} - \frac{ib_0}{2} \frac{f'_c(0)}{f_c(0)} = ic^3 + (1 - c^2)(1 + ic) + c(c - i) = 1,$$

using $1/(c + i) = (c - i)/s^2$. Hence h_c is analytic in $\mathbb{D}$ and $h_c(0) = 1$.

Step 3: the circle away from the exceptional points. Write $\eta = e^{i\beta}$, $z = e^{i\phi}$, $\psi = \phi + \beta$. Then

$$M_c(e^{i\phi}) = c^2 - 1 - 2c \cot(\psi/2),$$

which is real and strictly increasing for $\psi \in (0, 2\pi)$, from $-\infty$ to $+\infty$. So $M_c < 0$ on one open arc and $M_c > 0$ on the complementary open arc, the zero η^3 and the pole $\bar{\eta}$ being the endpoints. The boundary root is $f_c = \sqrt{M_c} > 0$ where $M_c > 0$, and $f_c = iG$ with $G = \sqrt{-M_c} > 0$ where $M_c < 0$. On the circle $1 - \bar{\eta}/z = \overline{1 - \eta z}$, so by (13) $q_c = -\frac{s^2}{2} M_c |1 - \eta z|^2$ is real: negative where $M_c > 0$ and positive where $M_c < 0$. The terms $ia_0 - ib_0 \cos \phi$ of (11) are purely imaginary there. Where $M_c > 0$, iq_c/f_c is purely imaginary, so $u_c = 0$ and $w_c = -q_c/f_c > 0$. Where $M_c < 0$, $iq_c/f_c = q_c/G$ is real, so $u_c = q_c/G > 0$, $\operatorname{Im} h_c = a_0 - b_0 \cos \phi$ and $w_c = 0$.

Step 4: the two exceptional points. For $z \in \mathbb{D} \setminus \{0, \bar{\eta}\}$ we have $|f_c|^2 = |M_c|$ and, by (12), $|M_c| = s^2 |z - \eta^3|/|z - \bar{\eta}|$ (use $|\eta^2 - \bar{\eta}z| = |z - \eta^3|$ and $|1 - \eta z| = |z - \bar{\eta}|$). With $|1 - \bar{\eta}/z| = |z - \bar{\eta}|/|z|$, (13) gives

$$\left| \frac{q_c(z)}{f_c(z)} \right| = \frac{s^2}{2} |M_c|^{1/2} |1 - \eta z| |1 - \bar{\eta}/z| = \frac{s^3}{2} \frac{|z - \eta^3|^{1/2} |z - \bar{\eta}|^{3/2}}{|z|}.$$

This is bounded and tends to 0 as $z \rightarrow \eta^3$ or $z \rightarrow \bar{\eta}$ within $\bar{\mathbb{D}}$, with orders $1/2$ and $3/2$ respectively. Since q_c/f_c is continuous elsewhere on $\bar{\mathbb{D}} \setminus \{0\}$, h_c extends continuously to the whole closed disk, with $h_c = ia_0 - ib_0 \cos \phi$ at both exceptional points. There $u_c = 0$ and $w_c = 0$, so the inequalities of Step 3 hold at the endpoints of both arcs.

Step 5: the interior. $\operatorname{Re} h_c$ is harmonic in $\mathbb{D}$, continuous on $\bar{\mathbb{D}}$ and ≥ 0 on the circle. By the minimum principle $\operatorname{Re} h_c \geq 0$ in $\mathbb{D}$, and strictly positive because $\operatorname{Re} h_c(0) = 1$. $\square$

This limiting pair is not a polynomial certificate at (a_0, b_0) . Its real part vanishes on an arc, unlike the strictly positive boundary real part in Lemma 3.

5 Constructing a finite polynomial certificate

By the rotation by $\pi/2$ assume $b \geq 0$, and put

$$a_*(b) = (\max\{b^{2/3} - 1, 0\})^{3/2}, \quad (14)$$

so that $b < (1 + a^{2/3})^{3/2}$ is equivalent to $a > a_*(b)$. We first fix a *critical function* h_* for each $b \geq 0$: an analytic function on $\mathbb{D}$, continuous on $\bar{\mathbb{D}}$, with $h_*(0) = 1$, $\operatorname{Re} h_* \geq 0$ on the circle and

$$\operatorname{Im} h_*(e^{i\phi}) + b \cos \phi \leq a_*(b) \quad (\phi \in \mathbb{R}). \quad (15)$$

- If $b > 1$, take $c = \sqrt{b^{2/3} - 1}$, so that $b_0 = s^3 = b$ and $a_0 = c^3 = a_*(b)$, and $h_* = h_c$. Then (15) is the inequality $w_c \geq 0$ of Lemma 4.
- If $0 \leq b \leq 1$, take $h_*(z) = 1 - ibz$. On the circle $\operatorname{Re} h_* = 1 + b \sin \phi \geq 0$ and $\operatorname{Im} h_* + b \cos \phi = 0 = a_*(b)$. This treats $b = 0$ and $b = 1$ without the singular endpoint $c = 0$ of the critical formula.

In both cases the measure $d\mu(t) = \operatorname{Re} h_*(e^{it}) dt/2\pi$ has a continuous density, hence no atoms, and total mass $\operatorname{Re} h_*(0) = 1$. Poisson's formula for $\operatorname{Re} h_*$ and its harmonic conjugate, together with $h_*(0) = 1$, give for $0 < \sigma < 1$

$$h_*(\sigma z) = \int_0^{2\pi} K_\sigma(t, z) d\mu(t), \quad K_\sigma(t, z) = \frac{1 + \sigma e^{-it} z}{1 - \sigma e^{-it} z}, \quad |z| \leq 1; \quad (16)$$

the imaginary constant is zero because $h_*(0) = 1$.

Poisson smoothing. $\operatorname{Im} h_*(\sigma e^{i\phi})$ and $b\sigma \cos \phi$ are the Poisson integrals at radius σ of $\operatorname{Im} h_*(e^{i\phi})$ and $b \cos \phi$. Integrating (15) against the positive Poisson kernel gives $\operatorname{Im} h_*(\sigma e^{i\phi}) + b\sigma \cos \phi \leq a_*(b)$, hence

$$\operatorname{Im} h_*(\sigma e^{i\phi}) + b \cos \phi \leq a_*(b) + b(1 - \sigma). \quad (17)$$

Discretization. Approximating μ by equal-weight empirical measures, with the kernel continuous jointly in t and $|z| \leq 1$, gives uniform approximation on the closed disk. Thus for $\sigma < 1$ with $a_*(b) + b(1 - \sigma) < a$, and then for sufficiently large N ,

$$h_N(z) = \frac{1}{N} \sum_{j=1}^N \frac{1 + \alpha_j z}{1 - \alpha_j z}, \quad \alpha_j = \sigma e^{-it_j},$$

retains a strict margin in (17). The product of the quadratic forms (4) is then a strict certificate. The next proposition gives an explicit choice of a sufficient N , valid for every $b \geq 0$.

Proposition 5 (A sufficient degree bound). *Let $b \geq 0$, $a_* = a_*(b)$ as in (14), and $\delta = a - a_* > 0$. Set*

$$\tau = \min\{1/2, \delta/(4b)\} \quad (\tau = 1/2 \text{ if } b = 0), \quad \sigma = 1 - \tau, \quad N = \max\left\{1, \left\lceil \frac{16\pi\sigma}{\delta\tau^2} \right\rceil\right\}.$$

There is a strict polynomial certificate of degree $2N$ whose normalized angular margin $a - b \cos 2\theta - p'/(2Np)$ is at least $\delta/2$. If $0 < \delta \leq 2b$, one may use

$$N \leq \left\lceil \frac{256\pi b^2}{\delta^3} \right\rceil,$$

and if $\delta > 2b$ (in particular for $b = 0$), one may use $N \leq \max\{1, \lceil 32\pi/\delta \rceil\}$. This is a sufficient bound, not the minimum degree.

Proof. Use h_* and μ as above. Partition $[0, 2\pi]$ into N successive quantile intervals I_j , each of μ -mass $1/N$ (possible since μ has no atoms), and choose $t_j \in I_j$. Intervals may cross zero-density gaps; their lengths still sum to 2π . For the kernel $K_\sigma(t, z)$ in (16), with $w = \sigma e^{-it}z$, $\partial_t K_\sigma = -2iw/(1-w)^2$, so

$$\sup_{t, |z| \leq 1} |\partial_t K_\sigma(t, z)| \leq \frac{2\sigma}{(1-\sigma)^2} = \frac{2\sigma}{\tau^2}.$$

The integral-minus-empirical error is therefore at most

$$\sum_j \int_{I_j} \frac{2\sigma}{\tau^2} |I_j| d\mu = \frac{2\sigma}{\tau^2} \cdot \frac{1}{N} \sum_j |I_j| = \frac{4\pi\sigma}{N\tau^2} \leq \frac{\delta}{4}.$$

The smoothing loss in (17) is $b\tau \leq \delta/4$, so $\text{Im } h_N + b \cos \phi \leq a_* + \delta/2$, a margin at least $\delta/2$ in a . If $\delta \leq 2b$, then $\tau = \delta/(4b) \leq 1/2$ and $\sigma \leq 1$ give $N \leq \lceil 16\pi \cdot 16b^2/\delta^3 \rceil$. If $\delta > 2b$, then $\tau = \sigma = 1/2$ and $N = \max\{1, \lceil 32\pi/\delta \rceil\}$. $\square$

Locating the quantiles with rigorous numerical enclosures is an additional computational task. Proposition 5 is an analytic construction and degree bound, not a claim that the floating-point experiment below implements a certified algorithm.

6 Local analytic certificates and degree divergence

We complete the equivalences of Theorem 1. Let V be a local real-analytic weak certificate, with Taylor expansion $V = P_m + P_{m+1} + \dots$ at the origin, where P_j is homogeneous of degree j and $P_m \neq 0$. Along each ray, $V(tx) = t^m P_m(x) + O(t^{m+1}) > 0$ gives $P_m \geq 0$. Since f is homogeneous of degree three, $L_f V(tx) = t^{m+2} L_f P_m(x) + O(t^{m+3}) \leq 0$, so $L_f P_m \leq 0$ (the polynomial $L_f P_m$ may well vanish identically; only the inequality is used). With $P_m(r, \theta) = r^m p_m(\theta)$ and (2), this reads

$$p'_m(\theta) + m(-a + b \cos 2\theta)p_m(\theta) \leq 0. \quad (18)$$

Lemma 6. *If $p \geq 0$ is a continuous, 2π -periodic, piecewise C^1 function, not identically zero, satisfying $p' + m g p \leq 0$ for a continuous periodic g , then $p > 0$ everywhere.*

Proof. Let $\rho(\theta) = \exp \int_0^\theta m g ds > 0$. Then $(\rho p)' \leq 0$. If $p(\theta_0) = 0$, the function ρp is nonnegative, nonincreasing on $[\theta_0, \theta_0 + 2\pi]$ and zero at θ_0 , hence zero throughout, and by periodicity $p \equiv 0$. $\square$

Hence P_m is positive definite. Its degree m is even, since $P_m(-x) = (-1)^m P_m(x)$ and $P_m \geq 0$, $P_m \neq 0$. So P_m is a positive definite homogeneous weak polynomial certificate, and the strict obstruction of Section 3 applies to it. This is the leading-term mechanism already used in [1, Sec. III-D] for the single field $(a, b) = (1, 5)$; the argument is unchanged for every $a > 0$, $b \in \mathbb{R}$, so combining it with [1, Theorem IV.3] also gives local analytic nonexistence for $|b| \geq 2(a + 1)$. Conversely, the constructed polynomial is analytic. Together with the preceding sections this proves Theorem 1.

Proposition 7. *For fixed $b > 1$, the minimum degree among positive homogeneous polynomial weak certificates tends to infinity as $a \downarrow a_*(b) = (b^{2/3} - 1)^{3/2}$.*

Proof. Suppose not. Then there are N and $a_n \downarrow a_*$ and positive definite homogeneous weak certificates P_n of degree at most $2N$ at parameters (a_n, b) ; passing to a subsequence, all have the same degree $2N' \leq 2N$. Let $p_n(\theta) = P_n(\cos \theta, \sin \theta)$ and normalize $\max p_n = 1$ (scaling P_n). The map from binary forms of degree $2N'$ to trigonometric polynomials with frequencies $0, \pm 2, \dots, \pm 2N'$ is a linear isomorphism of spaces of dimension $2N' + 1$. The coefficients of P_n are bounded, because $\max |p_n| = 1$ and all norms on this finite-dimensional space are equivalent. A subsequence therefore converges, with all derivatives, to $p = P(\cos \theta, \sin \theta)$ for a binary form P of degree $2N'$ (possibly of lower Fourier degree, which does not matter), with $p \geq 0$ and $\max p = 1$. Passing to the limit in $p'_n + 2N'(-a_n + b \cos 2\theta)p_n \leq 0$ gives the same inequality for p at $a = a_*$. By Lemma 6, $p > 0$, so P is a positive definite homogeneous polynomial with $L_f P \leq 0$ at the excluded boundary $b^{2/3} = 1 + a_*^{2/3}$. This contradicts Theorem 1. $\square$

This compactness method has antecedents in [2, Theorem 4.3]. Propositions 5 and 7 do not give matching upper and lower degree rates.

7 Single harmonics, angular speed and sum of squares

The same extremal argument applies to a family in every odd degree. Let $k \geq 1$, $a > 0$, $b \in \mathbb{R}$, and $\omega \neq 0$, and consider

$$\dot{r} = r^{2k+1}(-a + b \cos 2k\theta), \quad \dot{\theta} = \omega r^{2k}. \quad (19)$$

It is a polynomial vector field of degree $2k + 1$:

$$f(x, y) = [-a(x^2 + y^2)^k + b \operatorname{Re}(x + iy)^{2k}](x, y) + \omega(x^2 + y^2)^k(-y, x).$$

Corollary 8. *For (19), homogeneous polynomial weak, strict polynomial, and local real-analytic weak certificates exist exactly when*

$$|b|^{2/3} < a^{2/3} + |\omega|^{2/3}. \quad (20)$$

All these fields are globally asymptotically stable. The degree bound of Proposition 5, applied to $a/|\omega|$ and $|b|/|\omega|$, becomes $2kN$ for these fields.

Proof. Reductions. Multiplying the field by a positive constant does not change the certificates, since $L_{\lambda f} V = \lambda L_f V$; this replaces (a, b, ω) by $(a, b, \omega)/|\omega|$. The reflection $(x, y) \mapsto (x, -y)$ is a linear change of variables sending $\theta \mapsto -\theta$; it leaves $\cos 2k\theta$ and r unchanged and reverses the sign of $\theta' = \omega r^{2k}$, so it maps (19) with ω to (19) with $-\omega$, and polynomial or analytic certificates to certificates of the same kind and degree. The rotation by $\pi/(2k)$ maps $\cos 2k\theta$ to $-\cos 2k\theta$ and keeps θ' , so it replaces b by $-b$. Hence we may assume $\omega = 1$, $b \geq 0$; (20) becomes $b^{2/3} < 1 + a^{2/3}$ (with a, b replaced by $a/|\omega|, |b|/|\omega|$).

The Lie derivative. For a homogeneous $P = r^m p(\theta)$ and $\omega = 1$,

$$L_f P = \dot{r} \partial_r P + \dot{\theta} \partial_\theta P = r^{m+2k} [m(-a + b \cos 2k\theta)p + p'].$$

Let $m = 2N$ and P be positive definite. By Lemma 3, $p'/(2Np) = \operatorname{Im} h(e^{2i\theta})$; with $\phi = 2\theta$, $L_f P \leq 0$ is exactly

$$w(\phi) := a - b \cos(k\phi) - \mathcal{H}u(\phi) \geq 0, \quad u = \operatorname{Re} h, \quad \langle u \rangle = 1, \quad u \geq \varepsilon > 0. \quad (21)$$

Necessity (any k , any candidate polynomial). Use the test $f_c(\sigma z^k)$ in place of $f_c(\sigma z)$, $z = e^{i\phi}$: write $f_c(\sigma e^{ik\phi}) = F_\sigma(\phi) + iG_\sigma(\phi)$. This is the boundary value of the function $z \mapsto f_c(\sigma z^k)$, analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, so $G_\sigma = \mathcal{H}F_\sigma + \langle G_\sigma \rangle$ and $F_\sigma + \mathcal{H}G_\sigma = \langle F_\sigma \rangle$, with $\langle F_\sigma \rangle = \operatorname{Re} f_c(0) = c$ and $\langle G_\sigma \rangle = \operatorname{Im} f_c(0) = 1$. Because $F_\sigma + iG_\sigma$ takes values in the open first quadrant, $F_\sigma, G_\sigma > 0$.

Finally, expanding $f_c(\zeta) = \sum_{n \geq 0} \varphi_n \zeta^n$, only $n = 1$ contributes to $\langle G_\sigma \cos k\phi \rangle$, namely $\text{Im}(\sigma\varphi_1)/2$, and $\varphi_1 = f'_c(0) = 2ic/s$:

$$\langle G_\sigma \cos k\phi \rangle = \frac{\sigma c}{s}, \quad F_\sigma + \mathcal{H}G_\sigma = c.$$

(These are the identities (7) for $k = 1$; they were also checked numerically for $k = 1, 2, 3$, see Section 8.) The pairing (8) then holds verbatim with $\cos \phi$ replaced by $\cos k\phi$: by skew-adjointness of $\mathcal{H}$, $\langle uF_\sigma + wG_\sigma \rangle = \langle u(F_\sigma + \mathcal{H}G_\sigma) \rangle + a \langle G_\sigma \rangle - b \langle G_\sigma \cos k\phi \rangle = c + a - b\sigma c/s \geq \varepsilon c$. As in Section 3 this gives $b < (a+c)\sqrt{1+c^2}/c$ for every $c > 0$, hence $b < (1+a^{2/3})^{3/2}$. No k -fold angular symmetry of the candidate is assumed.

Sufficiency. Let $b \geq 0$, $a > a_*(b)$, and let $\alpha_1, \dots, \alpha_N$ be the nodes of Proposition 5, with h_N the corresponding strict-margin function: $\text{Im } h_N(e^{i\psi}) + b \cos \psi \leq a - \delta/2$ for all ψ . Put

$$P(x, y) = \prod_{j=1}^N |(x - iy)^k - \alpha_j(x + iy)^k|^2 = \prod_{j=1}^N \left[(1 + |\alpha_j|^2)(x^2 + y^2)^k - 2 \text{Re}(\alpha_j(x + iy)^{2k}) \right].$$

Each factor is a positive definite binary form of degree $2k$ (on the circle it equals $|1 - \alpha_j e^{2ik\theta}|^2 \geq (1 - |\alpha_j|)^2$), and P has degree $2kN$. Differentiating the logarithm of each factor, with $\zeta = e^{2ik\theta}$, gives $\frac{d}{d\theta} \log |1 - \alpha\zeta|^2 = 4k \text{Im} \frac{\alpha\zeta}{1 - \alpha\zeta} = 2k \text{Im} \frac{1 + \alpha\zeta}{1 - \alpha\zeta}$, hence

$$\frac{p'(\theta)}{2kNp(\theta)} = \text{Im } h_N(e^{2ik\theta}), \quad \frac{L_f P}{2kN r^{2kN+2k} p} \leq -\frac{\delta}{2} < 0$$

on the circle by the formula for $L_f P$ above (with $m = 2kN$, $\psi = 2k\theta$). Hence P is a strict polynomial certificate of degree $2kN$. The reductions above restore arbitrary $\omega \neq 0$ and $b \in \mathbb{R}$ without changing the degree. The reflection and rotation reductions, the product formula and the sign of $L_f P$ were also checked numerically in Cartesian coordinates for several (k, a, b, ω) , including $k = 2, 3$, $\omega < 0$ and $b < 0$ (Section 8).

Local analytic certificates. The argument of Section 6 carries over: for a local analytic weak certificate with leading term P_m one has $P_m \geq 0$ and $L_f P_m = r^{m+2k}[p'_m + m(-a + b \cos 2k\theta)p_m] \leq 0$ ($\omega = 1$). Lemma 6 gives that P_m is positive definite, hence of even degree, and the necessity argument above applies to it.

Stability. For $\omega \neq 0$ let $H = r^2 \exp[-\frac{b}{k\omega} \sin 2k\theta]$. Then $\partial_r H = 2H/r$ and $\partial_\theta H = -\frac{2b}{\omega} \cos(2k\theta)H$, so

$$L_f H = r^{2k+1}(-a + b \cos 2k\theta) \frac{2H}{r} - \frac{2b}{\omega} \cos(2k\theta) \omega r^{2k} H = -2ar^{2k} H.$$

Its bounded positive angular factor gives $e^{-\beta r^2} \leq H \leq e^{\beta r^2}$, $\beta = |b|/(k|\omega|)$, and C^1 regularity at zero (as for $k = 1$, by [1, Lemma III.1]). Decrease of H bounds all trajectories and prevents finite-time escape; with $L_f H < 0$ away from the origin, Lyapunov's direct method gives global asymptotic stability. $\square$

For $k = 1$, write $b \cos 2\theta + d \sin 2\theta = \rho \cos(2\theta - \gamma)$ with $\rho = \sqrt{b^2 + d^2}$; the rotation of coordinates by $\gamma/2$ (which commutes with $\theta' = \omega r^2$) turns it into $\rho \cos 2\theta$. The criterion becomes

$$(b^2 + d^2)^{1/3} < a^{2/3} + |\omega|^{2/3}.$$

No result for $\omega = 0$ or a general sum of different angular harmonics follows from this argument.

Corollary 9 (Classical binary-form SOS consequence). *The strict certificate region in Theorem 1 and Corollary 8 is also the region admitting a homogeneous certificate P such that P and $-L_f P$ are positive definite sums of squares. The same sufficient Lyapunov degree bounds apply.*

Proof. Any positive definite real binary form U of degree $2m$ is a sum of two squares of real forms of degree m . Indeed $U(x, 1)$ has positive leading coefficient and no real roots. Splitting its roots into nonreal conjugate pairs gives $U(x, 1) = F(x)\overline{F(x)}$. Homogenizing F yields $U = (\operatorname{Re} F)^2 + (\operatorname{Im} F)^2$. Apply this classical fact separately to the strict constructed P and $-L_f P$. The reverse implication is immediate. $\square$

Ahmadi and Parrilo [2, Theorem 5.3] show that any homogeneous polynomial vector field with a strict homogeneous polynomial Lyapunov function (V and $-\dot{V}$ positive definite) admits a homogeneous SOS one, *of possibly higher degree*. In the planar case treated here the classical two-square fact above keeps the degree; no new general SOS theorem is claimed. For any such field the coefficients of $L_f P$ depend linearly on those of P . Both P and $-L_f P$ are strictly positive on the compact unit circle, hence (by homogeneity) on $\mathbb{R}^2 \setminus \{0\}$, and this persists under sufficiently small changes of the coefficient vector of P . A rational coefficient vector can be chosen in that neighborhood, so a rational-coefficient strict polynomial certificate exists as well. This does not assert a two-square decomposition over the rationals.

8 Verification and limitations

The proofs above use explicit analytic identities and inequalities, not a numerical optimization result. The following computations are floating-point or high-precision *sanity checks*, not part of any proof, and none is interval-certified.

Reproducible checks in the supplement. The script `check_v2.py` (Python with numpy and mpmath, output `check_v2.output.json`) evaluates, with 40-digit arithmetic, Lemma 4 for $c \in \{1/8, 1/2, 1, 5/2, 8\}$: $h_c(0) = 1$, $\operatorname{Re} h_c \geq 0$ on a polar grid of the closed disk (the smallest value is of rounding size, below 10^{-38} in modulus), $w_c \geq 0$ and $u_c w_c = 0$ on the circle, identity (13), and the vanishing orders 0.5 and 1.5 of q_c/f_c at η^3 and $\bar{\eta}$ (measured slopes 0.500000001, 1.500000002). It also builds the products of Corollary 8 with $N = 1024$ factors and $\sigma = 0.98$, directly in Cartesian coordinates, for $(k, a, b, \omega) \in \{(1, 1.2, 2, 1), (2, 1.2, -2, -1), (3, 2.4, 4, 2), (2, 0.6, 0.8, 1), (2, 0.5, 0, 1)\}$; the largest sampled value of $L_f P/(2kNP)$ on the circle (20001 angles) is negative in each case (for example -0.688 for the first two, matching $a - \max(\operatorname{Im} h_N + b \cos)$). Finally it confirms the identities $\langle F_\sigma \rangle = c$, $\langle G_\sigma \rangle = 1$, $\langle G_\sigma \cos k\phi \rangle = \sigma c/s$, $F_\sigma + \mathcal{H}G_\sigma = c$ for $k = 1, 2, 3$ to about 10^{-13} .

An anecdotal example from version 1. One finite product of 2048 quadratic factors was evaluated at 8192 angular samples for $a = 1.2$, $b = 2\sqrt{2}$, $\sigma = 0.98$ (script `check_phase30.py`, output `phase-checks-30.json`); the sampled normalized margin was about 0.1545. It is illustrative only: it is not a certified positivity check, not a minimum-degree result, and nothing in the paper depends on it. A further exploratory Toeplitz-relaxation program of version 1 is dropped from the text; its script and stored output remain in the supplement, with their solver-status caveats, and no conclusion here relies on them.

Review status. The manuscript was written with AI assistance. Version 1 received separate automated reviews by Claude Sonnet 5.5 and Claude Haiku 5.5; version 2 responds to them. These are not human, blind or independent peer reviews (the reviewer and the drafting assistant belong to the same model family), and not formal verification. No Lean formalization of this argument exists; the Lean development mentioned in [1] concerns their Theorem II.2 only. A rigorously enclosed implementation of the quantile selection is not supplied; the quantitative construction is analytic.

Sources. The comparison with Liu–Fitzsimmons uses the complete text of arXiv:2607.16171v1 (the only version found on 21 October 2026). The cited theorem statements of Ahmadi–Parrilo (Theorems 4.3 and 5.3) and of Ahmadi–El Khadir (Proposition 6.1) were checked against their texts, but those works were not read in full. Targeted searches did not locate this boundary; worldwide priority is unconfirmed, particularly for the associated harmonic extremal problem. Minimum-degree asymptotics and general multiharmonic classification remain open here.

References

- [1] J. Liu and M. Fitzsimmons, *A globally asymptotically stable planar homogeneous polynomial vector field with no polynomial Lyapunov function*, arXiv:2607.16171v1, 17 July 2026. <https://arxiv.org/abs/2607.16171v1>.
- [2] A. A. Ahmadi and P. A. Parrilo, *Stability of polynomial differential equations: complexity and converse Lyapunov questions*, arXiv:1308.6833. Theorem numbers (4.3, 5.3) as in arXiv v1 and the author-hosted manuscript: https://web.mit.edu/~a_a_a/Public/Publications/Selected_Papers/dTAC.pdf.
- [3] A. A. Ahmadi and B. El Khadir, *On algebraic proofs of stability for homogeneous vector fields*, IEEE Trans. Automat. Control **65** (2020), 325–332. Proposition 6.1 and Lemma 6.2 as numbered in the 2018 preprint (cited as Proposition VI.1 in [1]): <https://optimization-online.org/wp-content/uploads/2018/03/6502.pdf>.