

A sharp polynomial Lyapunov threshold for a stable cubic family

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18 October 2026

Abstract

We determine exactly when the globally asymptotically stable cubic family introduced by Liu and Fitzsimmons admits polynomial Lyapunov certificates. In their normalization, with damping $a > 0$ and angular forcing $b \in \mathbb{R}$, the criterion is

$$|b|^{2/3} < 1 + a^{2/3}.$$

The same strict inequality characterizes positive homogeneous polynomial certificates with weak nonpositive Lie derivative, strict polynomial certificates, and local real-analytic certificates. Equality admits none of them. Below the threshold we construct a certificate as a finite product of positive quadratic forms. The proof pairs positive harmonic functions with an explicit analytic square-root test and constructs a complementary limiting harmonic pair at the boundary. Poisson smoothing and equal-weight quantiles then produce a polynomial with a strict margin. At a fixed nonzero boundary point, an explicit sufficient degree is $O_b(\delta^{-3})$, where δ is the damping excess, while the minimum degree diverges; no matching degree rate is asserted. The threshold persists for every single even angular harmonic and nonzero angular speed. Classical binary-form factorization gives the corresponding sum-of-squares consequence without increasing the Lyapunov degree. The family, its global stability and the original counterexample to a polynomial converse theorem are prior results. Our classification concerns certificates, not a change in dynamical stability.

1 The family and the exact certificate region

For $a > 0$ and $b \in \mathbb{R}$, consider

$$\begin{aligned}\dot{x} &= -a(x^2 + y^2)x + b(x^2 - y^2)x - (x^2 + y^2)y, \\ \dot{y} &= -a(x^2 + y^2)y + b(x^2 - y^2)y + (x^2 + y^2)x.\end{aligned}\tag{1}$$

In polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, this is

$$\dot{r} = r^3(-a + b \cos 2\theta), \quad \dot{\theta} = r^2.\tag{2}$$

Liu and Fitzsimmons introduced this family and established its global asymptotic stability, together with nonexistence of polynomial and local analytic certificates in the region $|b| \geq 2(a+1)$ [1]. Their work already refutes the homogeneous polynomial Lyapunov converse. We determine the exact certificate region of their family.

For a vector field f , write $L_f V = \nabla V \cdot f$. A *weak certificate* means $V(0) = 0$, $V > 0$ away from the origin, and $L_f V \leq 0$. A *strict certificate* has $L_f V < 0$ away from the origin. For a local analytic certificate these conditions are required on a punctured neighborhood of zero. All homogeneous certificates below have positive degree.

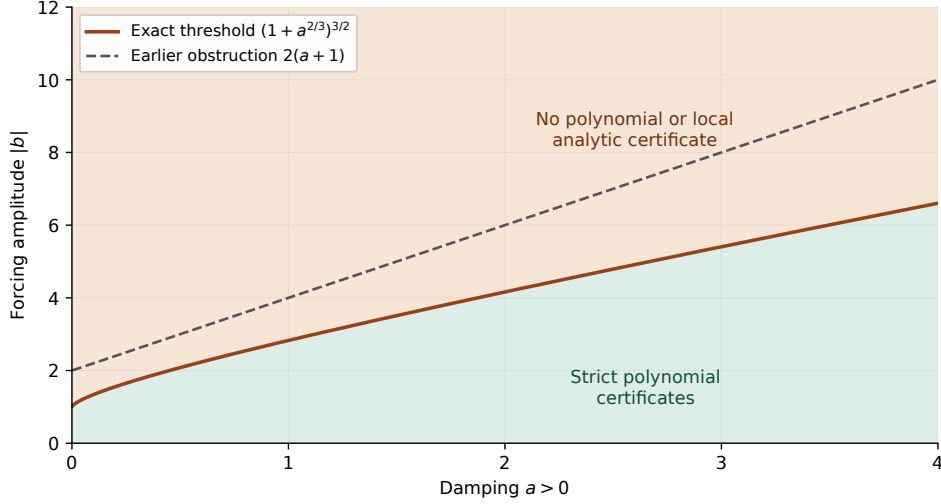


Figure 1: Certificate regions for the stable cubic family. All displayed parameters with $a > 0$ give globally asymptotically stable dynamics. The solid curve is excluded from the polynomial and local analytic certificate region. The dashed line marks the sufficient nonexistence condition in [1]; it is not a stability boundary.

Theorem 1. *For (1), the following are equivalent:*

1. $|b| < (1 + a^{2/3})^{3/2}$.
2. *There exists a positive definite homogeneous polynomial weak certificate.*
3. *There exists a strict homogeneous polynomial certificate that is a finite product of positive definite real quadratic forms.*
4. *There exists a local real-analytic weak certificate.*

Thus equality at the threshold is excluded even for weak certificates. Every system in (1) is nevertheless globally asymptotically stable.

The stability assertion is recalled from [1], not new. For orientation, its nonpolynomial certificate is

$$H(r, \theta) = r^2 \exp(-b \sin 2\theta), \quad L_f H = -2ar^2 H.$$

It is positive, coercive, continuously differentiable at the origin, and smooth away from it. The new assertion in Theorem 1 is the exact algebraic and analytic threshold.

Unbounded required polynomial degree in fixed dimension and field degree was already established by Ahmadi and Parrilo [2, Theorem 4.3]. Ahmadi and El Khadir [3, Proposition 6.1] also compare polynomial and rational certificate complexity. Our degree result concerns approach to the explicit boundary above. Compactness, positive harmonic representations, and the binary-form sum-of-squares principle are classical tools, not new general converse theorems.

2 Polynomial certificates as positive harmonic data

Use normalized circle averages $\langle v \rangle = (2\pi)^{-1} \int_0^{2\pi} v(\phi) d\phi$. The circular Hilbert transform $\mathcal{H}$ is normalized by

$$\mathcal{H} \cos(k\phi) = \sin(k\phi), \quad \mathcal{H} \sin(k\phi) = -\cos(k\phi), \quad \mathcal{H} 1 = 0, \quad k \geq 1.$$

On smooth real functions it is skew-adjoint for this average. The formulas used below can equally be verified termwise from Fourier series.

Lemma 2. *Let P be a positive definite real homogeneous polynomial of degree $2N$, $N \geq 1$, and set $p(\theta) = P(\cos \theta, \sin \theta)$. There exist $\alpha_1, \dots, \alpha_N \in \mathbb{D}$ such that*

$$h(z) = \frac{1}{N} \sum_{j=1}^N \frac{1 + \alpha_j z}{1 - \alpha_j z}, \quad \frac{p'(\theta)}{2Np(\theta)} = \operatorname{Im} h(e^{2i\theta}).$$

Here $h(0) = 1$, and $u = \operatorname{Re} h$ has a strictly positive minimum on the closed disk. The weak certificate condition is equivalent to

$$w(\phi) := a - b \cos \phi - \mathcal{H}u(\phi) \geq 0. \quad (3)$$

Proof. The strict Fejér–Riesz factorization, in the form used in [1, Lemmas III.2–III.3], writes

$$p(\theta) = |C|^2 \prod_{j=1}^L |1 - \alpha_j e^{2i\theta}|^2, \quad L \leq N, \quad |\alpha_j| < 1.$$

Pad the list with zero parameters to length N . Each kernel has real part

$$\operatorname{Re} \frac{1 + \alpha z}{1 - \alpha z} = \frac{1 - |\alpha z|^2}{|1 - \alpha z|^2} > 0 \quad (|z| \leq 1).$$

Differentiating the logarithm of a factor gives twice the imaginary part of that kernel, proving the angular identity. The imaginary part of h is $\mathcal{H}u$ because $h(0) = 1$. Finally,

$$L_f P = 2Nr^{2N+2}p(\theta) \left[-a + b \cos 2\theta + \frac{p'(\theta)}{2Np(\theta)} \right],$$

which proves (3). □

Conversely, an equal-weight average of these kernels yields a polynomial of degree $2N$. Explicitly, put

$$Q_\alpha(x, y) = (1 + |\alpha|^2)(x^2 + y^2) - 2 \operatorname{Re} \alpha (x^2 - y^2) + 4 \operatorname{Im} \alpha xy. \quad (4)$$

Its eigenvalues are $(1 - |\alpha|)^2, (1 + |\alpha|)^2$. The product $P = \prod_j Q_{\alpha_j}$ is positive definite and satisfies the angular identity in Lemma 2. A strict margin in (3) therefore gives a strict polynomial certificate.

3 The strict obstruction

Rotation by $\pi/2$ changes the sign of b , so assume $b \geq 0$. For $c > 0$, define

$$s = \sqrt{1 + c^2}, \quad \eta = \frac{c + i}{s}, \quad (5)$$

$$M_c(z) = c^2 - 1 + 2ic \frac{1 + \eta z}{1 - \eta z}, \quad f_c(z) = \sqrt{M_c(z)}.$$

The Cayley transform has positive real part on $\mathbb{D}$; hence M_c takes values in the upper half-plane. We choose the analytic square root in the first quadrant. In particular,

$$f_c(0) = c + i, \quad f'_c(0) = \frac{2ic}{s}. \quad (6)$$

For $0 < \sigma < 1$, write $f_c(\sigma e^{i\phi}) = F_\sigma(\phi) + iG_\sigma(\phi)$. Both functions are positive and smooth. Their means and first coefficient give

$$\langle F_\sigma \rangle = c, \quad \langle G_\sigma \rangle = 1, \quad \langle G_\sigma \cos \phi \rangle = \frac{\sigma c}{s}, \quad F_\sigma + \mathcal{H}G_\sigma = c. \quad (7)$$

If u, w come from a weak polynomial certificate, then $\langle u \rangle = 1$, $u \geq \varepsilon > 0$, and $w \geq 0$. Using (3) and skew-adjointness,

$$\begin{aligned}\langle uF_\sigma + wG_\sigma \rangle &= \langle u(F_\sigma + \mathcal{H}G_\sigma) \rangle + a \langle G_\sigma \rangle - b \langle G_\sigma \cos \phi \rangle \\ &= c + a - \frac{b\sigma c}{s} \geq \varepsilon c.\end{aligned}\tag{8}$$

Every function in this computation is smooth for $\sigma < 1$; no singular boundary pairing is assumed. Letting $\sigma \uparrow 1$ yields

$$b \leq \frac{(a+c)s}{c} - \varepsilon s < \frac{(a+c)\sqrt{1+c^2}}{c}.\tag{9}$$

The derivative of the last expression is

$$\frac{c^3 - a}{c^2\sqrt{1+c^2}}.$$

Its minimum occurs at $c = a^{1/3}$, with value $(1 + a^{2/3})^{3/2}$. This proves the strict necessary inequality, including nonexistence at equality. The positive minimum of u , rather than mere nonnegativity, is what excludes the boundary.

4 An explicit limiting pair on the boundary

The converse needs more than a necessary moment inequality. We construct positive harmonic data at the critical parameters, then approximate them while retaining a positive margin.

Fix $c > 0$, with s, η, f_c as above, and put

$$a_0 = c^3, \quad b_0 = s^3.$$

Define

$$q_c(z) = 1 - c^4 + \frac{b_0}{2}[(c-i)z + (c+i)z^{-1}],\tag{10}$$

$$h_c(z) = ia_0 - \frac{ib_0}{2}(z + z^{-1}) + i\frac{q_c(z)}{f_c(z)}.\tag{11}$$

Lemma 3. *The function h_c extends analytically across zero, continuously to $\overline{\mathbb{D}}$, and satisfies $h_c(0) = 1$. Its boundary data obey*

$$u_c = \operatorname{Re} h_c \geq 0, \quad w_c = a_0 - b_0 \cos \phi - \operatorname{Im} h_c(e^{i\phi}) \geq 0.$$

Their positivity sets are disjoint arcs.

Proof. The coefficient of z^{-1} in (11) cancels because $f_c(0) = c + i$. Its constant coefficient is

$$ia_0 + \frac{i(1-c^4)}{f_c(0)} - \frac{ib_0}{2} \frac{f'_c(0)}{f_c(0)} = 1,$$

by (6). No other interior singularity occurs, since f_c has no zero in $\mathbb{D}$.

Two exact rational identities determine the boundary behavior:

$$M_c(z) = s^2 \frac{\eta^2 - \bar{\eta}z}{1 - \eta z},\tag{12}$$

$$q_c(z) = -\frac{s^2}{2} M_c(z)(1 - \eta z)(1 - \bar{\eta}/z).\tag{13}$$

On the circle, M_c and q_c are real away from the exceptional points, and

$$q_c(z) = -\frac{s^2}{2}M_c(z)|1 - \eta z|^2.$$

On the arc where $M_c > 0$, the chosen boundary square root is real and positive; $q_c < 0$. Then $u_c = 0$ and $w_c = -q_c/f_c \geq 0$. On the complementary arc $M_c < 0$, write $f_c = iG$, $G > 0$. There $q_c > 0$, so $u_c = q_c/G \geq 0$ and $w_c = 0$.

For completeness, M_c has a simple boundary zero at $z = \eta^3$ and a simple boundary pole at $z = \bar{\eta}$. They are distinct for every finite $c > 0$. The Laurent polynomial q_c vanishes at both points. Thus q_c/f_c is respectively of orders $1/2$ and $3/2$, and extends continuously with value zero. The boundary inequalities therefore extend to both exceptional points. The minimum principle gives $\operatorname{Re} h_c \geq 0$ on $\bar{\mathbb{D}}$, strictly positive in the interior because $h_c(0) = 1$. $\square$

This limiting pair is not a polynomial certificate at (a_0, b_0) . Its real part vanishes on an arc, unlike the strictly positive boundary real part in Lemma 2.

5 Constructing a finite polynomial certificate

Suppose first $b > 1$ and $b < (1 + a^{2/3})^{3/2}$. Choose

$$c = \sqrt{b^{2/3} - 1}, \quad a_0 = (b^{2/3} - 1)^{3/2} < a, \quad b_0 = b. \quad (14)$$

Poisson smoothing of Lemma 3 gives, for $h_\sigma(z) = h_c(\sigma z)$,

$$\operatorname{Im} h_\sigma(e^{i\phi}) + b\sigma \cos \phi \leq a_0.$$

Consequently

$$\operatorname{Im} h_\sigma(e^{i\phi}) + b \cos \phi \leq a_0 + b(1 - \sigma) < a \quad (15)$$

when $\sigma < 1$ is sufficiently close to one.

For $0 \leq b \leq 1$, instead use $h(z) = 1 - ibz$. Its boundary real part is nonnegative, $h(0) = 1$, and $\operatorname{Im} h(e^{i\phi}) + b \cos \phi = 0$. Replacing z by σz gives a strict margin whenever $b(1 - \sigma) < a$. This treats $b = 0$ and $b = 1$ without using the singular endpoint $c = 0$ in the critical formula.

In both cases the unsmoothed h is analytic, continuous on the closed disk, has nonnegative real part and value 1 at zero. Poisson's formula and its harmonic conjugate imply

$$h(\sigma z) = \int_0^{2\pi} \frac{1 + \sigma e^{-it}z}{1 - \sigma e^{-it}z} d\mu(t), \quad d\mu(t) = \operatorname{Re} h(e^{it}) \frac{dt}{2\pi}. \quad (16)$$

The measure is a probability measure. The imaginary constant is zero because $h(0) = 1$. For fixed $\sigma < 1$ the kernel is continuous jointly in t and $|z| \leq 1$. Approximate μ by equal-weight empirical measures; uniform continuity makes the resulting approximation uniform on the closed disk. Thus, for sufficiently large N ,

$$h_N(z) = \frac{1}{N} \sum_{j=1}^N \frac{1 + \alpha_j z}{1 - \alpha_j z}, \quad \alpha_j = \sigma e^{-it_j},$$

retains the strict margin in (15). The product of the quadratic forms (4) is then a strict certificate. The next proposition gives an explicit choice of a sufficient N .

Proposition 4 (A sufficient degree bound). *Let $b > 1$, a_0 be as in (14), and $\delta = a - a_0 > 0$. Set*

$$\tau = \min\{1/2, \delta/(4b)\}, \quad \sigma = 1 - \tau, \quad N = \max\left\{1, \left\lceil \frac{16\pi\sigma}{\delta\tau^2} \right\rceil\right\}.$$

There is a strict polynomial certificate of degree $2N$ whose normalized angular margin $a - b \cos 2\theta - p'/(2Np)$ is at least $\delta/2$. For $0 < \delta \leq 2b$, one may use

$$N \leq \left\lceil \frac{256\pi b^2}{\delta^3} \right\rceil.$$

This is a sufficient bound, not the minimum degree.

Proof. The critical measure in (16) is nonatomic. Partition $[0, 2\pi]$ into N successive quantile intervals I_j , each of mass $1/N$, and choose $t_j \in I_j$. Intervals may cross zero-density gaps; their lengths still sum to 2π . For the kernel $K_\sigma(t, z)$ in (16),

$$\sup_{t, |z| \leq 1} |\partial_t K_\sigma(t, z)| \leq \frac{2\sigma}{(1 - \sigma)^2}.$$

The integral-minus-empirical error is therefore at most

$$\sum_j \int_{I_j} \frac{2\sigma}{\tau^2} |I_j| d\mu = \frac{4\pi\sigma}{N\tau^2} \leq \frac{\delta}{4}.$$

The smoothing loss is $b\tau \leq \delta/4$, leaving margin at least $\delta/2$. When $\delta \leq 2b$, insert $\tau = \delta/(4b)$ and $\sigma \leq 1$ to obtain the final bound. $\square$

Locating the quantiles with rigorous numerical enclosures is an additional computational task. Proposition 4 is an analytic construction and degree bound, not a claim that the floating-point experiment below implements a certified algorithm.

6 Local analytic certificates and degree divergence

We complete the equivalences of Theorem 1. Let V be a local real-analytic weak certificate, and let P_m be its first nonzero homogeneous Taylor term. Positivity of V implies $P_m \geq 0$, and comparison of the first Lie-derivative terms gives $L_f P_m \leq 0$. On the circle its profile satisfies

$$p'_m(\theta) + m(-a + b \cos 2\theta)p_m(\theta) \leq 0. \quad (17)$$

If $p_m(\theta_0) = 0$, multiply by the positive integrating factor of (17). The resulting function is nonnegative and nonincreasing from zero over one full turn. It must vanish throughout the turn, contradicting $P_m \neq 0$. Hence P_m is positive definite. Its degree is even, and the strict obstruction applies. This is the leading-term mechanism already used in [1]. Conversely, the constructed polynomial is analytic. Together with the preceding sections this proves Theorem 1.

Proposition 5. *For fixed $b > 1$, the minimum degree among positive homogeneous polynomial weak certificates tends to infinity as $a \downarrow a_0 = (b^{2/3} - 1)^{3/2}$.*

Proof. Otherwise select a sequence of certificates with bounded degrees, and a subsequence of one fixed even degree $2N$. Normalize their circle restrictions to maximum one. Their Fourier coefficients are bounded, so a subsequence converges with its derivative to a nonnegative trigonometric polynomial p of maximum one. The weak inequality passes to the limit at $a = a_0$. The integrating-factor argument above forces $p > 0$. It is the circle restriction of a homogeneous polynomial of degree $2N$, even if its Fourier degree drops. This would give a weak certificate at the excluded boundary, a contradiction. $\square$

This compactness method has antecedents in [2, Theorem 4.3]. Propositions 4 and 5 do not give matching upper and lower degree rates.

7 Single harmonics, angular speed and sum of squares

The same extremal argument applies to a family in every odd degree. Let $k \geq 1$, $a > 0$, $b \in \mathbb{R}$, and $\omega \neq 0$, and consider

$$\dot{r} = r^{2k+1}(-a + b \cos 2k\theta), \quad \dot{\theta} = \omega r^{2k}. \quad (18)$$

It is a polynomial vector field of degree $2k + 1$:

$$f(x, y) = [-a(x^2 + y^2)^k + b \operatorname{Re}(x + iy)^{2k}](x, y) + \omega(x^2 + y^2)^k(-y, x).$$

Corollary 6. *For (18), homogeneous polynomial weak, strict polynomial, and local real-analytic weak certificates exist exactly when*

$$|b|^{2/3} < a^{2/3} + |\omega|^{2/3}. \quad (19)$$

All these fields are globally asymptotically stable. For $|b| > |\omega|$, the degree bound of Proposition 4, applied to $a/|\omega|, |b|/|\omega|$, becomes $2kN$.

Proof. Positive time rescaling, reflection and rotation reduce to $\omega = 1$, $b \geq 0$. For an arbitrary certificate of degree $2N$, Lemma 2 gives $w = a - b \cos(k\phi) - \mathcal{H}u \geq 0$. Use the test $f_c(\sigma z^k)$ in place of $f_c(\sigma z)$. Its means are still $c, 1$, its real and imaginary parts are positive, and

$$\langle G_\sigma \cos(k\phi) \rangle = \frac{\sigma c}{s}, \quad F_\sigma + \mathcal{H}G_\sigma = c.$$

The pairing (8) and its strict conclusion are unchanged. This necessity does not assume that the candidate polynomial has k -fold angular symmetry.

For sufficiency take L nodes from the cubic construction and set

$$P(r, \theta) = r^{2kL} \prod_{j=1}^L |1 - \alpha_j e^{2ik\theta}|^2.$$

Each factor is the positive binary form

$$|(x - iy)^k - \alpha_j(x + iy)^k|^2 = (1 + |\alpha_j|^2)(x^2 + y^2)^k - 2 \operatorname{Re}[\alpha_j(x + iy)^{2k}].$$

The product has degree $2kL$ and satisfies

$$\frac{p'(\theta)}{2kLp(\theta)} = \operatorname{Im} h_L(e^{2ik\theta}).$$

Thus the strict margin persists. The local analytic argument is unchanged because the angular speed does not vanish away from zero. Finally,

$$H = r^2 \exp \left[-\frac{b}{k\omega} \sin 2k\theta \right], \quad L_f H = -2ar^{2k} H.$$

Its positive bounded angular factor gives coercivity and C^1 regularity at zero. Decrease of H bounds all trajectories and prevents finite-time escape; the displayed derivative and comparison with r^2 imply global asymptotic stability. $\square$

For $k = 1$, replacing $b \cos 2\theta$ by $b \cos 2\theta + d \sin 2\theta$ merely rotates its phase. The criterion becomes

$$(b^2 + d^2)^{1/3} < a^{2/3} + |\omega|^{2/3}.$$

No result for $\omega = 0$ or a general sum of different angular harmonics follows from this argument.

Corollary 7 (Classical binary-form SOS consequence). *The strict certificate region in Theorem 1 and Corollary 6 is also the region admitting a homogeneous certificate P such that P and $-L_f P$ are positive definite sums of squares. The same sufficient Lyapunov degree bounds apply.*

Proof. Any positive definite real binary form U of degree $2m$ is a sum of two squares of real forms of degree m . Indeed $U(x, 1)$ has positive leading coefficient and no real roots. Splitting its roots into nonreal conjugate pairs gives $U(x, 1) = F(x)\overline{F(x)}$. Homogenizing F yields $U = (\operatorname{Re} F)^2 + (\operatorname{Im} F)^2$. Apply this classical fact separately to the strict constructed P and $-L_f P$. The reverse implication is immediate. $\square$

This binary SOS principle and broader converse results are discussed in [2, Section 5 and Theorem 5.3]; no new general SOS theorem is claimed. If the vector field has rational coefficients, a sufficiently small rational perturbation of the coefficient vector of P preserves both strict inequalities on the compact unit circle. Thus a rational-coefficient polynomial certificate exists as well. This does not assert a two-square decomposition over the rationals.

8 Verification and limitations

The proof uses explicit analytic identities and inequalities, not a numerical optimization result. Supplementary author computations check the rational factorization of the critical pair, the value $h_c(0) = 1$, and five boundary parameter instances. One finite product of 2048 quadratic factors was evaluated at 8192 angular samples for $a = 1.2$, $b = 2\sqrt{2}$, $\sigma = 0.98$. The sampled normalized margin was about 0.1545. This is a floating-point diagnostic candidate, not a certified global positivity check or a minimum-degree result.

A separate diagnostic solves 18 necessary finite Toeplitz relaxations at three damping values and six truncation orders. The stored field named `degree` in that exploratory program means moment truncation order, not Lyapunov polynomial degree. Nine solver results are marked `optimal_inaccurate`; these labels are preserved. No solver status is used as a proof of feasibility, nonexistence or sharpness.

The same involved model reviewer separately checked the proof dossier and additional symbolic identities, finite Hilbert pairings and high-precision branch-point profiles. This is not blind or human peer review, an independent additional vote, or formal verification. No Lean build of the earlier source or this argument was performed. The quantitative construction is analytic; a rigorously enclosed implementation of its quantile selection is not supplied.

The comparison uses the complete mathematical text of Liu–Fitzsimmons v1 and the cited degree-complexity and SOS portions of the earlier works, not complete readings of the latter. Targeted searches did not locate this boundary; worldwide priority is unconfirmed, particularly for the associated harmonic extremal problem. Minimum-degree asymptotics and general multiharmonic classification remain open here. AI tools assisted this work.

References

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