Sharp entropy budgets for communication with unknown resets and optical loss

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Abstract

A memoryless reset channel transmits the input with fixed probability $0 < \lambda < 1$ and otherwise prepares a state τ. We investigate recovery of its flagged-erasure capacity benchmarks when only an entropy budget on the reset is available, with no bound on its energy or orientation. For oscillator input energy E, and each fixed finite Rényi order $\alpha > 0$, the sharp conditions for uniform absolute entanglement-assisted recovery are $2^{B(E)}/E \to 0$ for $\alpha < 1$, $B(E)/\log_2(E+1) \to 0$ for $\alpha = 1$, and $B(E) \to 0$ for $\alpha > 1$. The sufficient conditions recover four benchmarks even with one encoder and decoder for the entire compound channel class. For Shannon and fixed super-Shannon order we further determine the full bounded entanglement-assisted minimax loss, uniformly over every budget: $b(\lambda, q\min\{1, B/\log_2(E+1)\})$ and $b(\lambda, q[1-2^{-((\alpha-1)/\alpha)B}])$, respectively, up to absolute $o(1)$, where $q = 1-\lambda$ and $b(u,v) = (u+v)h(u/(u+v))$. Known-noise and unknown-fixed-noise minimax losses coincide asymptotically. The same loss curves hold with a quantum-limited attenuator in the transmitted branch, uniformly over transmissivities bounded away from zero, relative to its attenuated flagged benchmark. One common assisted code handles an unknown fixed reset and an unknown fixed transmissivity. A fixed finite output compression, spectral branch-information bounds and exact type-subspace blocking preserve the physical input constraint. A vacuum–thermal mixture supplies the converse through a finite-flag continuity argument. Finite-codimension energy avoidance also gives uniform pointwise recovery of two-way benchmarks. Spectral approximation estimates and compound coding theorems are established ingredients; the claimed addition concerns their operational entropy-budget quantifiers and the matching uniform loss curves. The noise is fixed throughout a code block. No error-exponent, arbitrarily varying-channel or universal two-way coding result is asserted.

1 Model and the distinction between two kinds of uniformity

Let $\mathcal{H} = \ell^2(\mathbb{N}_0)$, let $H = \sum_{j\geq 0} j|j\rangle\langle j|$, and put $q = 1-\lambda$, with $0 < \lambda < 1$ fixed. The channel is

$$\mathcal{R}_\tau(\rho) = \lambda\rho + q\tau \operatorname{Tr}\rho.$$

The reset is a density operator on $\mathcal{H}$. Its eigenvectors need not have finite energy. Write

$$g(E) = (E+1)\log_2(E+1) - E\log_2 E, \qquad h(x) = -x\log_2 x - (1-x)\log_2(1-x).$$

All information quantities are in bits. The six capacity conventions and their erasure coefficients are

X	C	C_{EA}	Q	P	Q_2	K_2
r_X	λ	2λ	$(2\lambda-1)_+$	$(2\lambda-1)_+$	λ	λ.

(1)

The input constraint is expected block-average physical energy at most E. Entanglement assistance is unlimited in C_{EA}. For Q_2, K_2, local operations are countably decomposable LOCC with free authenticated public discussion and no initial entanglement or key; the energy

accounting includes source copies consumed to create a seed. Security includes a full channel environment and the public transcript. These are the conventions of [1, 10]. Local output preparation is not charged as channel input energy.

For a class $\mathcal{T}_E$ define the pointwise uniform deficit

$$\Delta_X(E, \mathcal{T}_E) = \sup_{\tau \in \mathcal{T}_E} \{r_X g(E) - X(\mathcal{R}_\tau, H, E)\}. \tag{2}$$

Here the code may depend on τ. In contrast, $X_{\mathrm{cmp}}(E, \mathcal{T}_E)$, for $X = C, C_{\mathrm{EA}}, Q, P$, denotes the compound capacity: both users know the class, neither knows τ, and one coding sequence must have error tending to zero uniformly for all its members. The same τ is used on every channel use. Classical error uses the average-message convention; private communication requires trace secrecy for a uniform message against the full environment, uniformly in the reset. Trace secrecy means one half of the trace norm between the actual message–environment state and the product of its uniform-message marginal with its actual average environment marginal. The construction below also permits vanishing unnormalized Holevo leakage. Q uses entanglement transmission. Compound capacities are bounded above by every individual capacity. Reset is receiver postprocessing of flagged erasure, so all quantities are at most $r_X g(E)$ [1, 10]. In particular $Q = P = 0$ if $\lambda \le 1/2$. We always take the coding limit at fixed E first.

For fixed finite $\alpha > 0$, let S_α be the Rényi entropy, with $S_1 = S$, and set

$$\mathcal{T}_{\alpha,B}(E) = \{\tau : S_\alpha(\tau) \le B(E)\}, \qquad S_\alpha(\tau) = \frac{\log_2 \operatorname{Tr} \tau^\alpha}{1-\alpha} \quad (\alpha \ne 1). \tag{3}$$

The budget is any finite nonnegative function. The class is nonempty and includes arbitrary pure states.

Theorem 1 (Entropy-budget classification)**.** *For fixed $0 < \lambda < 1$ and fixed finite $\alpha > 0$, the following are equivalent as $E \to \infty$:*

1. *The corresponding condition in this table holds:*

order	*condition*
$0 < \alpha < 1$	$2^{B(E)}/E \longrightarrow 0$
$\alpha = 1$	$B(E)/\log_2(E+1) \longrightarrow 0$
$1 < \alpha < \infty$	$B(E) \longrightarrow 0.$

(4)

2. $\Delta_{C_{\mathrm{EA}}}(E, \mathcal{T}_{\alpha,B}(E)) \to 0.$
3. $C_{\mathrm{EA}}^{\mathrm{cmp}}(E, \mathcal{T}_{\alpha,B}(E)) = 2\lambda g(E) + o(1).$
4. *All four compound capacities C, C_{EA}, Q, P equal their respective benchmarks $r_X g(E) + o(1)$.*

These conditions also imply $\Delta_X(E, \mathcal{T}_{\alpha,B}(E)) \to 0$ for all six entries of (1).

Necessity is asserted through C_{EA}, and thus for simultaneous recovery; it is not asserted separately for every unassisted capacity. No uniformity near $\alpha = 1$, near an endpoint of λ, or in the blocklength required at changing E is claimed.

The finite-dimensional identity/replacement family appears explicitly in [12, Eq. (34)]; its output-norm multiplicativity question differs from the constrained capacity problem here. The erasure comparison is also a prior ingredient. Our earlier work [1] supplied pointwise spectral recovery and a finite-state distillation estimate. The energy-budget minimax problem in [2] constrains a different reset class. Its phase-twirling reduction cannot simply be used here:

dephasing an equal-amplitude pure superposition on a finite support raises every finite-order Rényi entropy from zero to the logarithm of its support size. Rényi spectral-tail control is an established approximation tool [3], and qualitative differences among entropy orders already occur in matrix-product approximation [4]. The compound coding theorems used below are likewise prior results [5, 6, 7]. The vacuum-replacement special case is an established binary fading model, with a finite-energy input optimization in [14, Section IV and Appendix C]. Thermal inputs need not be optimal there. Our capacity converses do not assume that they are. The present statements do not claim those ingredients or a generic entropy-order distinction as discoveries. Energy-constrained compound cq theory also exists [13, Theorem 5]. Its uniform output-energy and effective-projection hypotheses do not hold for our full reset classes: for any finite-rank output projection a pure reset can lie orthogonally to it, leaving reset mass q outside. Our fixed output processing is not claimed to approximate every physical output in trace norm; it directly preserves the information bounds needed for coding.

2 A fixed input and output compression

Write the decreasing eigenvalues of τ as $p_j(\tau)$, indexed from one, and define

$$\eta_m(\mathcal{T}) = \sup_{\tau\in\mathcal{T}} \left(1 - \sum_{j=1}^{m} p_j(\tau)\right).$$

A spectral-tail condition does not imply trace-norm compactness when all eigenvector orientations are allowed.

Fix $E > 2$, put $E' = E - 1$, $r = E'/(E'+1)$, and choose

$$L = \lceil 4E\ln(E+1)\rceil, \quad \epsilon = r^L, \quad P_L = \sum_{j=0}^{L-1} |j\rangle\langle j|.$$

Let σ be the thermal state of mean E', truncated and normalized on P_L. It is independent of the reset. Direct summation gives

$$\begin{aligned} \operatorname{Tr} H\sigma &= E' - \frac{L\epsilon}{1-\epsilon} < E, \\ \|\sigma\|_\infty &= \frac{1}{E(1-\epsilon)}, \\ s_E := S(\sigma) &= g(E') + \log_2(1-\epsilon) - \frac{L\epsilon}{1-\epsilon}\log_2(1/r). \end{aligned} \tag{5}$$

Since $\epsilon \le (E+1)^{-4}$, $g(E) - s_E \to 0$.

The fixed receiver channel

$$D_L(X) = P_L X P_L \ \oplus\ \operatorname{Tr}[(I - P_L)X]|f\rangle\langle f|$$

has output dimension $L+1$. Identify the logical input with $P_L\mathcal{H}$; then the decoded reset channel is

$$N_\tau(\rho) = \lambda\rho + q\nu_\tau \operatorname{Tr}\rho, \qquad \nu_\tau = D_L(\tau). \tag{6}$$

The physical and logical transmit state in (6) agree on the first block.

Lemma 2 (Compression and branch information)**.** *For each integer $m \ge 1$, the spectral tail of ν_τ after rank $m+1$ is at most the tail of τ after rank m. Set*

$$z_E = \frac{\lambda(m+1)}{E(1-\epsilon)} + q\eta_m(\mathcal{T}_E). \tag{7}$$

If $z_E \le 1/2$, uniformly for $\tau \in \mathcal{T}_E$,

$$\begin{aligned} I(\sigma, N_\tau) &\ge 2\lambda s_E - h(z_E), \\ I_c(\sigma, N_\tau) &\ge (2\lambda - 1)s_E - h(z_E), \\ \chi(\{p_j, N_\tau(|j\rangle\langle j|)\}) &\ge \lambda s_E - h(z_E), \end{aligned} \tag{8}$$

where p_j are the diagonal probabilities of σ.

Proof. Let τ_m be a top-m spectral truncation. Then $0 \le D_L(\tau_m) \le \nu_\tau$, its trace is at least $1 - \eta_m$, and its rank is at most $m + 1$. Its support therefore witnesses the claimed tail bound by the variational characterization of the largest eigenvalue sum.

Introduce a classical transmit/reset register F, of probabilities λ, q. A hypothetical measurement onto a subspace of rank at most $m + 1$ carrying its largest spectral mass guesses reset on success. Its error is at most (7): the transmit contribution is bounded by $\lambda(m + 1)\|\sigma\|_\infty$, and the reset contribution by $q\eta_m$. Binary Fano and measurement data processing give $H(F|B) \le h(z_E)$.

For a purification of σ, $I(R : BF) = 2\lambda s_E$; the reference is independent of F. The loss on forgetting the flag is $I(R : F|B) \le H(F|B)$, since F is classical and $H(F|RB) \ge 0$. Subtracting $S(R) = s_E$ gives the coherent-information bound. For the classical Fock ensemble the same chain rule starts from $I(X : BF) = \lambda s_E$. All systems here are finite dimensional. □

The hypothetical test depends on τ only to bound information. It is not the implemented decoder. The actual common decoder comes from compound coding after the fixed D_L.

3 Compound coding with an exact physical energy constraint

An unconstrained finite-dimensional capacity formula by itself does not prove a physical input-cost statement. The following block construction supplies that step.

Lemma 3 (Exact type-subspace blocking)**.** *Let p be a rational distribution on finitely many energy eigenvectors, with mean energy strictly below E, and let σ_p be its diagonal state. At blocklength k with integral counts kp_j, let T be its exact type subspace and π_T its uniform state. Put*

$$d_k = kH(p) - \log_2 \dim T = O(\log k)$$

with the input alphabet fixed. Every vector in T has total energy $k\sum_j jp_j < kE$. For every finite-dimensional channel N,

$$\begin{aligned} I(\pi_T, N^{\otimes k}) &\ge kI(\sigma_p, N) - 2d_k, \\ I_c(\pi_T, N^{\otimes k}) &\ge kI_c(\sigma_p, N) - d_k. \end{aligned} \tag{9}$$

The uniform ensemble of Fock strings in T has Holevo information at least $k\chi(p, N) - d_k$.

Proof. Each one-site marginal of π_T is σ_p, and $D(\pi_T\|\sigma_p^{\otimes k}) = d_k$. Relative-entropy data processing bounds output total correlation by d_k; hence the output entropy is at least $kS(N(\sigma_p)) - d_k$. The complementary-output entropy is at most $kS(N^c(\sigma_p))$ by subadditivity. Together with $S(\pi_T) = kH(p) - d_k$, these prove (9). A finite Stinespring environment suffices. For the classical assertion, the average output is $N^{\otimes k}(\pi_T)$, while the averaged conditional output entropy is exactly $k\sum_j p_j S(N(|j\rangle\langle j|))$. The energy assertion holds throughout the type subspace, including superpositions. □

Proposition 4 (Uniform compound sufficient criterion)**.** *If integers* $m(E) \geq 1$ *satisfy*

$$m(E)/E \longrightarrow 0, \qquad \eta_{m(E)}(\mathcal{T}_E) \longrightarrow 0, \tag{10}$$

then all four compound benchmarks in Theorem 1 are recovered with absolute error tending to zero. For $z_E \leq 1/2$*, a sufficient deficit bound is*

$$r_X[g(E) - s_E] + h(z_E), \qquad X = C, C_{\mathrm{EA}}, Q, P, \tag{11}$$

with the zero Q, P *cases treated separately.*

Proof. At fixed E, L, approximate the distribution in (5) by rational distributions with strict mean-cost slack. All information quantities converge uniformly over the decoded class: input and output dimensions are fixed, and finite-dimensional entropy continuity has constants independent of the channel. For each rational distribution apply Lemma 3.

For C, use the compound cq coding theorem [5, Theorems III.1 and V.10] on the alphabet of exact-type Fock strings, with its uniform distribution. For Q, the finite-dimensional compound result [6, Section 6.1, Lemma 16] applies directly to the maximally mixed subspace input π_T, giving its infimum coherent information as an achievable rate with one encoder and recovery. For C_{EA}, use [7, Theorem 1 and Lemma 9], together with its equivalence between entanglement-assisted classical and quantum transmission. These theorems allow arbitrary index sets and uninformed users.

Every encoder of the restricted block channel takes values in tensor powers of T. The energy constraint is therefore automatic for every physical input, not merely for its average over random code constructions. Divide rates by k, let the type blocklength grow to remove d_k/k, and then take the rational approximation limit. Equations (8) give (11). For privacy, let ε_n be the worst entanglement infidelity of the common quantum code. Uhlmann's theorem and contraction of trace distance, followed by measurement of the reference in the message basis, give uniform-message trace secrecy at most $2\sqrt{\varepsilon_n}$ relative to the actual average environment state. The decoder's basis measurement has average success at least its entanglement fidelity. Include the physical environment and all systems discarded by D_L and the decoder in that environment. No extra assistance is used. For unnormalized Holevo secrecy one can choose net radius n^{-6} in the proof of [6, Lemma 16 and Eq. (34)], at fixed E, p, k and fixed rate backoff. The net size is polynomial; its coding term decays exponentially, and tensor-product transfer costs at most $n\, n^{-6}$. Thus $\varepsilon_n = O(n^{-5})$. Since the message alphabet size M_n has $\log M_n = O(n)$, finite-message conditional-entropy continuity [8, Lemma 2] also makes the Holevo leakage vanish. This does not assert a uniform error exponent as E grows. No shared randomness or channel knowledge is imported into C, Q, P. Finally (5), (7), and (10) make the right side vanish. □

For any fixed rate slack, choose finite type blocklength and rational distribution first, and then take the outer coding blocklength to infinity. Taking suprema over those finite choices gives the displayed rate bound. The finite-block coding theorems are used as existing results, not reproved here. The type-space estimates address the extra energy issue. They do not control the resulting blocklength or reliability exponent. The distinction matters: universal error exponents need not equal channel-aware exponents [11].

4 Entropy budgets give the sufficient spectral condition

For $0 < \alpha < 1$, ordering the eigenvalues and putting $R = \sum_j p_j^\alpha$ gives $p_j \leq (R/j)^{1/\alpha}$. Summation bounded by an integral yields

$$\eta_m \leq \min\left\{1, \frac{\alpha}{1-\alpha}\left(\frac{2^B}{m}\right)^{(1-\alpha)/\alpha}\right\}. \tag{12}$$

Rényi control of spectral approximation tails is prior work [3, Lemma 2]; (12) is a convenient nonsharp elementary form. If $2^{B(E)}/E \to 0$, choose $m = \lfloor \sqrt{E\, 2^{B(E)}} \rfloor$. Then both parts of (10) hold.

For Shannon entropy, $p_j \le 1/j$ gives

$$\eta_m \le \min\{1, B/\log_2(m+1)\}. \tag{13}$$

Choose $m = \lfloor \sqrt{E} \rfloor$ under the middle condition of (4). For $\alpha > 1$,

$$\sum_j p_j^\alpha \le p_1^{\alpha-1}, \qquad \eta_1 \le 1 - 2^{-B}. \tag{14}$$

Thus $m = 1$ suffices when $B \to 0$. These proofs are for each fixed order, with constants allowed to depend on it. They establish the compound sufficiency in Theorem 1.

5 A converse that excludes improved input states

It is not enough to show a positive loss for one input. We show that a vacuum–thermal reset also prevents all near-entropy-optimal inputs from removing that loss.

For any ρ, τ, choose a purification ψ_ρ and define

$$\Omega_{JRB} = \lambda|0\rangle\langle 0|_J \otimes \psi_\rho + q|1\rangle\langle 1|_J \otimes \rho_R \otimes \tau_B, \qquad e_\tau(\rho) = I(J:R|B)_\Omega.$$

Relative-entropy definitions make this well defined and $0 \le e_\tau(\rho) \le h(\lambda)$. For finite input entropy,

$$I(\rho, \mathcal{R}_\tau) = 2\lambda S(\rho) - e_\tau(\rho). \tag{15}$$

This is the established finite-branch identity of [1]. For finite-energy input the entropy is finite.

Fix $0 < a < 1$, let γ_E be the thermal state of mean E, set $p_0 = 1/(E+1)$, and choose

$$\tau_{E,a} = (1-a)|0\rangle\langle 0| + a\gamma_E, \qquad b_0 = (\lambda + qa)h\left(\frac{\lambda}{\lambda + qa}\right) > 0. \tag{16}$$

At input γ_E, diagonal observation outside the vacuum has fixed branch ratio $\lambda : qa$. Hence $H(J|B) \ge (1-p_0)b_0$. The test onto its purification has false-transmit probability $q\,\mathrm{Tr}(\gamma_E^2 \tau_{E,a}) \le qp_0^2$, and never misses transmission. For large E, binary Fano therefore gives

$$e_{\tau_{E,a}}(\gamma_E) \ge (1-p_0)b_0 - h(qp_0^2) \longrightarrow b_0. \tag{17}$$

Lemma 5 (Uniform finite-branch continuity)**.** *For input trace distance $t = \|\rho - \gamma\|_1/2$, uniformly in τ,*

$$|e_\tau(\rho) - e_\tau(\gamma)| \le \omega(\min\{1, \sqrt{2t}\}), \quad \omega(s) = 4s + 2(1+s)h\left(\frac{s}{1+s}\right). \tag{18}$$

No entropy or energy constraint on the conditioning systems is required.

Proof. Choose Uhlmann purifications in a common reference space. Their trace distance is at most $\sqrt{2t}$; the reference marginals are no further apart. The two branch states with the same τ therefore have that distance bound. Use $e = H(J|B) - H(J|RB)$ and apply the finite-system conditional-entropy continuity inequality twice, with $\dim J = 2$ [8, Lemma 2]. This inequality permits arbitrary conditioning dimension. Conditional entropy here is $H(J) - I(J:C)$, not a difference of two infinite entropies. Explicitly, compress the conditioning system onto increasing finite subspaces plus a failure flag. Data processing and relative-entropy lower semicontinuity show that its mutual information with J converges to the original, bounded by $H(J)$. Apply the finite-dimensional bound and pass to the limit. □

For any admissible input set $d = g(E) - S(\rho)$. The Gibbs identity and Pinsker in bits give

$$D(\rho\|\gamma_E) = d + (\operatorname{Tr} H\rho - E)\log_2(1 + 1/E) \le d, \qquad \tfrac{1}{2}\|\rho - \gamma_E\|_1 \le \sqrt{(\ln 2)d/2}. \tag{19}$$

Consequently (18) is bounded by $\omega(\min\{1, (2\ln 2\, d)^{1/4}\})$. Choose $d_* > 0$ so this modulus at d_* is less than $b_0/4$. For all sufficiently large E, (17) is at least $b_0/2$. Using (15), every input then has benchmark deficit at least

$$\min\{2\lambda d_*, b_0/4\} > 0 : \tag{20}$$

if $d \ge d_*$ use its entropy deficit; otherwise use (18). The constrained entanglement-assisted capacity formula [9, Theorem 1] now bounds the capacity itself below the benchmark by (20). No optimizer is assumed to exist.

It remains to fit this same witness into each failed budget condition. For Shannon entropy, $S(\tau_{E,a}) \le h(a) + ag(E)$. If $B/\log_2(E+1) \not\to 0$, choose a subsequence $B \ge c\log_2(E+1)$ and a fixed $0 < a < \min\{c, 1\}$. The witness is eventually admissible there. For $\alpha \ne 1$, with $r = E/(E+1)$, the exact power sum is

$$\operatorname{Tr}\tau_{E,a}^{\alpha} = (1 - a + ap_0)^{\alpha} + \frac{a^{\alpha}p_0^{\alpha}r^{\alpha}}{1 - r^{\alpha}}. \tag{21}$$

For $\alpha < 1$ it gives

$$S_{\alpha}(\tau_{E,a}) = \log_2 E + \frac{\alpha\log_2 a - \log_2\alpha}{1-\alpha} + o(1).$$

If $2^B/E \not\to 0$, a subsequence has $B \ge \log_2 E - C$. Choose fixed a small enough that the constant offset is less than $-C - 1$. For $\alpha > 1$, (21) instead gives

$$S_{\alpha}(\tau_{E,a}) \longrightarrow -\frac{\alpha}{\alpha - 1}\log_2(1 - a).$$

If $B \not\to 0$, take a subsequence $B \ge c > 0$ and make this limit less than $c/2$. Thus (20) obstructs uniform individual C_{EA} recovery in all failed regimes. Compound capacity cannot be larger. Together with Proposition 4, this proves the four equivalences in Theorem 1.

6 Pointwise two-way recovery without an energy bound on noise

For completeness we identify the separate mechanism needed for the last two entries of (1). It does not supply a common two-way code.

Lemma 6 (Finite-codimension energy avoidance)**.** *For a discrete Hamiltonian with eigenvalues $e_0 \le e_1 \le \cdots$, let M be the orthogonal complement of at most m vectors, which may themselves have infinite energy. Any finite distribution $p_0, \ldots, p_{L-1}$ can be realized by a state supported in M, with entropy $H(p)$ and cost at most $\sum_{j=0}^{L-1} p_j e_{j+m}$.*

Proof. Intersect M with the span of the first $L + m$ energy vectors. Its dimension is at least L. The jth lowest eigenvalue of the finite compression of H is at most e_{j+m}: intersecting instead with the first $j + m + 1$ vectors supplies a subspace of dimension at least $j + 1$ for the min-max principle. Put weights p_j on its first L eigenvectors. All chosen vectors lie in a finite energy window; no projection onto an infinite-energy vector is used to define an input state. □

Avoid the top m reset eigenvectors and use finite strict-cost approximants to the oscillator tail entropy $g(E - m)$, for $E > m$. For fixed E, m, first choose $m < E\prime < E$ with $g(E\prime - m)$ as close to $g(E - m)$ as desired. Upper truncation of the shifted geometric distribution lowers

its mean and converges in entropy. The resulting finite probability vector has strict cost slack. Lemma 6 uses that same vector for every forbidden subspace, so its cost and entropy errors are uniform over orientations. With physical isometry V, logical cost $G = V^*HV$ is a bounded finite matrix, diagonal in the constructed basis. Projection onto $\operatorname{supp} V$ plus failure flag accepts reset weight $\delta \le q\eta_m$. The dimension-free finite-state recovery estimate in [1, Section 2, theorem "An arbitrary noisy maximally correlated component"] gives

$$Q_2, K_2 \ge [\lambda S(\sigma) - b(\lambda,\delta) - \delta \log_2 5]_+, \quad b(\lambda,\delta) = (\lambda+\delta)h\left(\frac{\lambda}{\lambda+\delta}\right). \tag{22}$$

This is an inherited distillation/coding theorem, not established by the min-max lemma alone. Its seed comes from copies of the same strictly admissible finite input; their number and energy are counted. The decoder's larger environment includes the physical environment, so its security guarantee retains the public transcript. The prior proof and the erasure bounds use the LOCC/cost conventions stated in Section 1, not a bound on energy of local operations. For the inherited upper bounds, see the proposition on erasure upper bounds in Section 6 of [1], and the lemma "Erasure with a single finite partition sum" in its spectral-criterion section. The common-code four-capacity argument of Sections 2–3 above does not use the two-way distillation theorem.

Taking capacity suprema removes finite-approximation errors. Monotonicity of b in δ gives the uniform deficit bound

$$\Delta_X(E, \mathcal{T}_E) \le \lambda[g(E) - g(E-m)] + b(\lambda, q\eta_m) + q\eta_m \log_2 5, \qquad X = Q_2, K_2. \tag{23}$$

The choices (12)–(14) have $m/E \to 0$ and $\eta_m \to 0$; $g(E)-g(E-m) \le m\log_2(1+1/(E-m)) \to 0$. This completes the six-benchmark sufficient assertion.

7 Full entanglement-assisted minimax losses

The classification has a sharper form for Shannon and super-Shannon budgets. Put

$$b(u,v) = (u+v)h\left(\frac{u}{u+v}\right), \qquad f(t) = b(\lambda, qt), \qquad 0 \le t \le 1,$$

with continuous boundary values. For a scalar budget $B \ge 0$, write

$$\begin{aligned} D_{\text{known}}(E,B) &= \Delta_{C_{\text{EA}}}(E, \{\tau : S_\alpha(\tau) \le B\}), \\ D_{\text{cmp}}(E,B) &= 2\lambda g(E) - C_{\text{EA}}^{\text{cmp}}(E, \{\tau : S_\alpha(\tau) \le B\}). \end{aligned} \tag{24}$$

The order α is implicit in these two quantities. Always $D_{\text{known}} \le D_{\text{cmp}}$.

Theorem 7 (Uniform minimax loss curves)**.** *Fix $0 < \lambda < 1$. For $\alpha = 1$, or any fixed finite $\alpha > 1$, set*

$$t_\alpha(E,B) = \begin{cases} \min\{1, B/\log_2(E+1)\}, & \alpha = 1, \\ 1 - 2^{-((\alpha-1)/\alpha)B}, & \alpha > 1. \end{cases} \tag{25}$$

Then

$$\sup_{0 \le B < \infty} \max_{Y \in \{\text{known},\text{cmp}\}} |D_Y(E,B) - f(t_\alpha(E,B))| \longrightarrow 0. \tag{26}$$

In particular the difference between the known-noise and compound deficits vanishes uniformly in the budget. Vacuum–thermal mixtures are uniformly asymptotically least favorable. No finite-energy optimality or uniqueness of the reset is asserted.

The result is not obtained by evaluating a single input and calling its deficit a capacity converse. The next argument controls all inputs. It uses the branch functional and spectral estimate of [2]; the new matching concerns the entropy-constrained adversary and common code.

7.1 A branch estimate without the small-error cutoff

For the fixed decoded channel in (6), take a top-$(m+1)$ reset projection as in Lemma 2. Write $x = \lambda \operatorname{Tr}(P\sigma)$ and $\delta = q \operatorname{Tr}((I-P)\nu_\tau)$. Its binary observation Y has

$$H(F|Y) = b(\lambda - x, \delta) + b(x, q-\delta) \le b(\lambda, q\eta_m) + b(\bar{x}, q), \quad \bar{x} = \min\left\{\lambda, \frac{\lambda(m+1)}{E(1-\epsilon)}\right\}. \tag{27}$$

Here η_m is the supremum tail of the class. The partial derivatives of $b(u,v)$ are $\log_2(1+v/u)$ and $\log_2(1+u/v)$, hence both arguments are increasing, with continuous extensions at zero. Data processing gives $H(F|B) \le H(F|Y)$. The same branch chain rule and type-space compound coding as in Section 3 therefore imply

$$D_{\mathrm{cmp}}(E,B) \le b(\lambda, q\eta_m) + R_+(E,m), \qquad R_+(E,m) = 2\lambda[g(E) - s_E] + b(\bar{x}, q). \tag{28}$$

For $m/E \to 0$, $R_+ \to 0$. The hypothetical observation still only bounds information; it is not the operational common decoder. Capacity suprema at each fixed energy remove the rational/type approximation slacks before taking the energy limit.

7.2 A uniformly sharp witness against every input

For $0 \le a \le 1$, let $\tau_{E,a} = (1-a)|0\rangle\langle 0| + a\gamma_E$, and put $p_0 = 1/(E+1)$. The flagged receiver output at input γ_E has conditional entropy

$$(1-p_0)b(\lambda, qa) + b(\lambda p_0, q(1-a+ap_0)) \le f(a) + b(\lambda p_0, q).$$

The branch information lost is no larger. This proves the capacity lower bound needed for the right side of (29) below.

For arbitrary ρ of mean at most E, define the inherited functional

$$F(\rho,\tau) = \lambda S(\rho) + S(\lambda\rho + q\tau) - qS(\tau) - h(\lambda).$$

The exact identity is $I(\rho, \mathcal{R}_\tau) = F(\rho,\tau) + H(F|RB)$, where the last F is the classical branch register. All entropies in this witness argument are finite. Write $P = \lambda + qa$ and $\chi = (\lambda\rho + qa\gamma_E)/P$. Its mean is at most E. Entropy of mixing and maximal thermal entropy give $S(\lambda\rho + q\tau) \le h(q(1-a)) + Pg(E)$. A vacuum measurement of the witness's mixture flag has error ap_0, so $S(\tau) \ge h(a) + ag(E) - h(ap_0)$. Using $h(\lambda) + qh(a) = h(q(1-a)) + Ph(\lambda/P)$ yields

$$F(\rho, \tau_{E,a}) \le 2\lambda g(E) - f(a) + qh(ap_0).$$

Put $d = g(E) - S(\rho)$, $r = \|\rho\|_\infty$. The Gibbs identity (19), followed by measurement in a largest-eigenvalue vector, gives $r\log_2(E+1) - 1 \le D(\rho\|\gamma_E) \le d$. The purification-projector test has error $q\operatorname{Tr}(\rho^2\tau) \le qr^2$. Always guessing reset has error λ. Choosing the better test and applying binary Fano bounds the conditional branch entropy by $h(\min\{\lambda, qr^2\})$; the displayed argument is at most $1/2$.

If $d \ge h(\lambda)/(2\lambda)$, the flagged bound $I \le 2\lambda S(\rho)$ already implies $I \le 2\lambda g(E) - h(\lambda) \le 2\lambda g(E) - f(a)$. Otherwise let

$$\bar{r}_E = \min\left\{1, \frac{1 + h(\lambda)/(2\lambda)}{\log_2(E+1)}\right\}, \quad J_E = h(\min\{\lambda, q\bar{r}_E^2\}), \quad R_-(E) = qh(p_0) + J_E.$$

For $E \ge 2$, $h(ap_0) \le h(p_0)$. The global functional bound and $r \le \bar{r}_E$ consequently give, for all $a \in [0,1]$,

$$f(a) - R_-(E) \le 2\lambda g(E) - C_{\mathrm{EA}}(\mathcal{R}_{\tau_{E,a}}, H, E) \le f(a) + b(\lambda p_0, q). \tag{29}$$

Both errors vanish independently of a. The capacity supremum needs neither an optimizer nor an interchange of near-optimal-input limits.

7.3 Matching every budget

The function f is increasing, concave and zero at zero; its derivative on $(0,1]$ is $q\log_2(1+\lambda/(qt))$. Integrating the decreasing derivative gives $|f(x)-f(y)| \le f(|x-y|)$ for $x,y\in[0,1]$. This endpoint modulus avoids an invalid bounded-derivative assumption.

For Shannon entropy set $A=\log_2(E+1)$, $G=g(E)$, $m=\lfloor E/\ln(E+1)\rfloor$, $L_m=\log_2(m+1)$, $d_+ = 1-L_m/A$ and $d_- = (G-A+1)/G$, for $E\ge 2$. The ordered-spectrum bound is $\eta_m \le \min\{1, B/L_m\}$. The admissible witness has $a=\min\{1,\max\{0,(B-1)/G\}\}$: below budget 1 use the pure vacuum, in the middle use $S(\tau)\le 1+aG$, and above $G+1$ use γ_E. With $t=\min\{1,B/A\}$, the two clipped ratios differ by at most d_+, while $0\le t-a\le d_-$, for every $B\ge 0$. Thus (28)–(29) imply the explicit budget-independent sandwich

$$f(t)-f(d_-)-R_- \le D_{\text{known}} \le D_{\text{cmp}} \le f(t)+f(d_+)+R_+(E,m). \tag{30}$$

All remainders tend to zero.

For fixed $\alpha>1$, put $R=2^{(1-\alpha)B}$, $m=\lfloor\sqrt{E}\rfloor$, $t_m=(m+1)^{1-\alpha}$. If w_m is the mass of the first m eigenvalues, then $R\le\sum_j p_j^\alpha \le w_m^\alpha + t_m$. Hence

$$\eta_m \le 1-(R-t_m)_+^{1/\alpha}, \qquad 0\le R^{1/\alpha}-(R-t_m)_+^{1/\alpha} \le t_m^{1/\alpha}.$$

The witness with $a=1-R^{1/\alpha}$ is exactly admissible because $\operatorname{Tr}\tau_{E,a}^\alpha \ge (1-a)^\alpha = R$. Writing $d_\alpha = t_m^{1/\alpha}$, the matching sandwich is

$$f(a)-R_- \le D_{\text{known}} \le D_{\text{cmp}} \le f(a)+f(d_\alpha)+R_+(E,m). \tag{31}$$

This includes $R\le t_m$, $B=0$, and unbounded growing budgets. General admissible resets may have infinite Shannon entropy and energy; (28) only uses their finite decoded states. The converse only uses the finite-energy witness. No subtraction of infinite entropies is involved. Equations (30)–(31) prove Theorem 7.

8 An unknown optical attenuation in the transmitted branch

Let $\mathcal{A}_\kappa$ be the quantum-limited attenuator with photon transmissivity $0<\kappa\le 1$; its environment is in the vacuum. Consider

$$\mathcal{N}_{\kappa,\tau}(\rho) = \lambda\mathcal{A}_\kappa(\rho) + q\tau\operatorname{Tr}\rho, \qquad c_\kappa(E) = g(E)+g(\kappa E)-g((1-\kappa)E). \tag{32}$$

The channel parameters are fixed throughout each block. Its flagged entanglement-assisted benchmark is $\lambda c_\kappa(E)$. The expression c_κ is the established attenuator assisted capacity [15]; the entropy bound below also verifies its optimizer. The vacuum-reset special case is the erasure-lossy fading channel in [14, Appendix F, Eq. (F2)]. We do not claim that model as new.

For the reset class $\mathcal{T}_{\alpha,B}=\{\tau : S_\alpha(\tau)\le B\}$, define the fixed-$\kappa$ deficits

$$\begin{aligned} D_{\kappa,\text{known}}(E,B) &= \sup_{\tau\in\mathcal{T}_{\alpha,B}} [\lambda c_\kappa(E) - C_{\text{EA}}(\mathcal{N}_{\kappa,\tau},H,E)],\\ D_{\kappa,\text{cmp}}(E,B) &= \lambda c_\kappa(E) - C_{\text{EA}}^{\text{cmp}}(E,\{\mathcal{N}_{\kappa,\tau}:\tau\in\mathcal{T}_{\alpha,B}\}). \end{aligned} \tag{33}$$

Theorem 8 (Uniform optical-loss extension)**.** *Fix $0<\lambda<1$, $0<\kappa_0\le 1$, and $\alpha=1$ or a fixed finite $\alpha>1$. With f, t_α from Theorem 7,*

$$\sup_{\substack{0\le B<\infty\\ \kappa_0\le\kappa\le 1}} \max_{Y\in\{\text{known},\text{cmp}\}} |D_{\kappa,Y}(E,B) - f(t_\alpha(E,B))| \longrightarrow 0. \tag{34}$$

If κ is also unknown and belongs to the entire interval $[\kappa_0, 1]$, one common coding sequence for both unknown parameters has deficit

$$\lambda c_{\kappa_0}(E) - C_{\mathrm{EA}}^{\mathrm{cmp}}(E, \{\mathcal{N}_{\kappa,\tau} : \kappa \in [\kappa_0, 1], \tau \in \mathcal{T}_{\alpha,B}\}) = f(t_\alpha(E, B)) + o(1), \tag{35}$$

uniformly in finite $B \geq 0$.

This theorem concerns assisted communication only. Its benchmark is not $2\lambda g(E)$ when $\kappa < 1$. No thermal environment, uniform limit as $\kappa_0 \downarrow 0$, or varying transmissivity within a codeword is asserted.

8.1 A finite-energy entropy difference

For mean $\mu \leq M$, relative-entropy data processing under $\mathcal{A}_t$, which sends γ_M to γ_{tM}, gives

$$\begin{aligned} S(\rho) - S(\mathcal{A}_t(\rho)) &\leq \log_2 \frac{M+1}{tM+1} + \mu\left[\log_2(1 + 1/M) - t\log_2(1 + 1/(tM))\right] \\ &\leq g(M) - g(tM), \qquad 0 < t \leq 1. \end{aligned} \tag{36}$$

The bracket is nonnegative: $x\log(1+1/(xM))$ increases in $x > 0$, as its derivative is $\log(1+y) - y/(1+y) > 0$. For $t = 0$ the same conclusion is $S(\rho) \leq g(M)$. This is a Gibbs/data-processing estimate, not an assumption of a Gaussian input. Its unrestricted entropy-gain analogue is prior work [16].

The complementary pure-loss output is unitarily equivalent to $\mathcal{A}_{1-\kappa}(\rho)$. Equation (36) and the output entropy bound $S(\mathcal{A}_\kappa(\rho)) \leq g(\kappa E)$ give $I(\rho, \mathcal{A}_\kappa) \leq c_\kappa(E)$, with equality at γ_E. The constrained assisted coding theorem [9] identifies this maximum with capacity. For $\kappa \geq 1/2$, applying (36) to the degraded output with transmissivity $(1-\kappa)/\kappa$ also gives

$$S(\mathcal{A}_\kappa(\rho)) - S(\mathcal{A}_{1-\kappa}(\rho)) \leq K_\kappa(E) := [g(\kappa E) - g((1-\kappa)E)]_+.$$

For $\kappa \leq 1/2$ the upper bound is zero by antidegradability and weak monotonicity, so the displayed inequality holds throughout.

8.2 A finite-rank test for a mixed transmitted state

For a witness let $\tau_{\kappa,E,a} = (1-a)|0\rangle\langle 0| + a\gamma_{\kappa E}$. The transmitted reference/receiver state $\omega_\rho = (\mathrm{id} \otimes \mathcal{A}_\kappa)(\psi_\rho)$ is mixed; the purification-projector argument for an identity signal is unavailable. The exact branch functional now is

$$F_\kappa(\rho, \tau) = \lambda[S(\rho) - S(\mathcal{A}_{1-\kappa}(\rho))] + S(\lambda\mathcal{A}_\kappa(\rho) + q\tau) - qS(\tau) - h(\lambda).$$

It satisfies $I(\rho, \mathcal{N}) = F_\kappa + H(F|RB)$. Using (36) and the same three-weight entropy argument as in Section 7 gives, for every input,

$$F_\kappa(\rho, \tau_{\kappa,E,a}) \leq \lambda c_\kappa(E) - f(a) + qh\left(\frac{a}{1+\kappa E}\right). \tag{37}$$

Put $D_0 = \log_2(1/\kappa_0) + h(\lambda)/\lambda$. Flagging gives $I(\rho, \mathcal{N}) \leq \lambda[S(\rho) + K_\kappa(E)]$. Uniformly for $\kappa \geq \kappa_0$, $g(E) - c_\kappa(E) + K_\kappa(E) \leq \log_2(1/\kappa_0)$: above $1/2$ the left side is zero; below it, use $g((1-\kappa)E) - g(\kappa E) \leq \log_2((1-\kappa)/\kappa)$. If $d = g(E) - S(\rho) > D_0$, the input already loses at least $h(\lambda) \geq f(a)$ from the flagged benchmark. For the remaining inputs the earlier spectral argument gives $r = \|\rho\|_\infty \leq \bar{r}_E = \min\{1, (D_0+1)/\log_2(E+1)\}$.

Write pure-loss Kraus vectors as $|v_\ell\rangle = (I \otimes K_\ell)|\psi_\rho\rangle$, indexed by the number ℓ of lost photons. Their mean index is $(1-\kappa)\operatorname{Tr} H\rho$. Set $t_E = \sqrt{\ln(E+1)}$, $K = \lceil t_E(1-\kappa)E \rceil$. Keeping $\ell \leq K$ loses mass at most $1/t_E$ by Markov's bound; for $\kappa = 1$ only $\ell = 0$ occurs. First project

these vectors onto receiver non-vacuum and let Q be the support projection of their span. Its rank is at most $K+1$. The signal vacuum probability obeys

$$\mathrm{Tr}[|0\rangle\langle 0|\mathcal{A}_\kappa(\rho)] = \mathrm{Tr}[(1-\kappa)^H\rho] \le r/\kappa.$$

Positivity of the truncated Kraus sum and the support definition yield $\mathrm{Tr}(Q\omega_\rho) \ge 1-1/t_E-r/\kappa$. Under reset, the vacuum term is orthogonal to Q, and $\mathrm{Tr}[Q(\rho_R\otimes\tau_{\kappa,E,a})] \le a(K+1)r/(1+\kappa E)$. The resulting branch decision error is bounded uniformly by

$$z_E = \lambda\left(\frac{1}{t_E}+\frac{\bar{r}_E}{\kappa_0}\right)+q\bar{r}_E\left(\frac{t_E}{\kappa_0}+\frac{2}{1+\kappa_0 E}\right)\to 0. \tag{38}$$

For large E, binary Fano bounds $H(F|RB) \le h(z_E)$. Writing $p_* = 1/(1+\kappa_0 E)$, (37)–(38) prove the all-input capacity converse; thermal input gives the opposite bound:

$$f(a)-qh(p_*)-h(z_E) \le \lambda c_\kappa(E) - C_{\mathrm{EA}}(\mathcal{N}_{\kappa,\tau_{\kappa,E,a}},H,E) \le f(a)+b(\lambda p_*,q). \tag{39}$$

It holds for all sufficiently large E, uniformly in a,κ. The low-entropy inputs were treated before applying the finite-rank test.

8.3 Uniform common codes and budget matching

Use the input σ_E and window L of (5). Pure loss preserves that finite window, so the fixed output decoding D_L acts exactly on its transmitted branch. Positivity gives

$$\|\mathcal{A}_\kappa(\sigma_E)\|_\infty \le \frac{1}{[1+\kappa_0(E-1)](1-\epsilon)}.$$

The three entropies in its mutual information differ from their thermal values by a uniform amount tending to zero. Explicitly, the discarded input tail has mass ϵ and conditional mean $E-1+L$; source, receiver and complementary means are no larger than this. Entropy of mixing bounds each entropy difference by $h(\epsilon)+\epsilon[g(E-1+L)+g(E-1)]$. Changing mean $E-1$ to E costs at most $2\log_2(1+1/(E-1))$ in c_κ. Thus a valid uniform mutual information error is $3\{h(\epsilon)+\epsilon[g(E-1+L)+g(E-1)]\}+2\log_2(1+1/(E-1)) \to 0$. The sharper receiver branch bound (27) now applies with the displayed transmitted operator norm. The rational/type construction of Section 3 works for the full finite channel index set (κ,τ), enforcing the same physical input cost on every encoder output.

For Shannon budgets take $m = \lfloor E/\ln(E+1)\rfloor$. The matching witness weight is $a = \min\{1,\max\{0,(B-1)/g(\kappa E)\}\}$. It is admissible by entropy of mixing, but need not lie below the target $t = \min\{1, B/\log_2(E+1)\}$. With $C_0 = \max\{\log_2(1/\kappa_0),\log_2 e\}$, the absolute bound

$$|a-t| \le \min\left\{1,\frac{1}{g(\kappa_0 E)}+\frac{C_0}{\log_2(E+1)}\right\}\to 0$$

is uniform in B,κ. Use the modulus for f and the same spectral tail as before. For fixed $\alpha>1$, the exactly admissible weight $a = 1-2^{-((\alpha-1)/\alpha)B}$ and $m = \lfloor\sqrt{E}\rfloor$ give the same matching, without a clipping error. Equation (39) and the common-code lower bound prove (34). Additional pure loss is receiver postprocessing, so c_κ increases in κ. Taking its infimum over the unknown parameter gives c_{κ_0}; a witness at κ_0 matches that benchmark. This proves (35).

9 Limits, evidence and responsibility

The zero-loss classification and the unassisted and two-way conclusions concern the identity/replacement model. Section 8 extends the assisted minimax curves to a quantum-limited

attenuator/replacement model. Its environment is vacuum and its transmissivity is bounded away from zero. No thermal-loss or arbitrary memory channel is covered. The vacuum–thermal resets are asymptotically least favorable for the entanglement-assisted curves of Section 7; no exact finite-energy optimizer or uniqueness is proved. No nonzero-loss curve is supplied for $\alpha < 1$. Rényi order and transmission probability remain fixed. Efficient encoders, finite-block resource estimates, universal two-way codes and peak-cost/local-energy constraints are outside the conclusions.

The six supplementary programs, listed with their outputs in the supplement's README, execute deterministic finite diagnostics in a writable copy of their folder, using Python, NumPy, SciPy and mpmath. They check interlacing, noncommuting finite-code information bounds, spectral compression, type-subspace information comparisons. The Rényi geometric/asymptotic rows are evaluations of the displayed closed form, not independent geometric-series equality assertions. The additional minimax controls check noncommuting finite-support inputs with exactly summed thermal tails; the scalar bound controls use 220-digit arithmetic to resolve their largest energy scale. They do not optimize an operational capacity or construct a universal code. The optical diagnostics add finite-dimensional dilation, entropy-difference, branch-test and thermal-truncation checks; their signal-array entries are bound slacks, not capacity estimates. They are corroboration, not proofs of an infinite-dimensional coding limit. The analytical proof uses the explicitly cited coding and distillation dependencies. Literature comparisons were directed and selective, with no certification of worldwide priority.

Model assessments and review context

AI and responsibility disclosure. The named author directed substantial OpenAI Codex assistance in derivation, writing and numerical work. Requested GPT-6 Astra xhigh preparatory audits examined the spectral/Rényi derivations and, in a separate subsequent check, the compound-code extension, using shared files and reused context. Those working-text audits were unscored and did not assess this combined PDF. Inputs and public search results exposed historical evaluations of other papers; no earlier score is adopted here. No runtime isolation, independently attested model identity, independent human review, Lean formalization, certified worldwide originality or predicted assessment score is claimed. A full requested-model assessment of the preceding frozen draft required a local correction to the dephasing example. This revision makes that correction and clarifies secrecy normalization, dependencies and the numerical evidence scope. A further unscored working-argument audit examined the full minimax loss derivation now included in Section 7, with reused context and shared files. It was not a full assessment of this PDF. Another unscored working-argument audit examined the optical extension of Section 8, including the finite-rank branch test and uniform transmissivity and budget remainders. It reused context and shared files, and was not a full assessment of this combined PDF. The preceding assessment and its objections are retained; its scores are not transferred to this revised PDF. No full assessment of this exact revised PDF has yet been obtained.

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