Exact Chebyshev Equioscillation, Riesz–Bernstein–Schur–Markov Derivative Extremals, Squared-Fejér Jackson Moduli, and Positive-Fourier Chebyshev Lebesgue Envelopes

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Abstract

We present a unified, completely self-contained, non-asymptotic treatment of four cornerstone extremal and stability theorems in constructive approximation theory on $[-1,1]$ and the circle $\mathbb{R}/2\pi\mathbb{Z}$. First, via a multiplicity-credit root-counting argument on the $n+1$ Chebyshev extrema $\eta_k = \cos(k\pi/n)$, we prove Chebyshev's sharp monic minimax theorem $\|p\|_\infty \geq 2^{1-n}$ $(n \geq 1)$ with unique attainment at $\widetilde{T}_n = 2^{1-n}T_n$, together with the de la Vallée Poussin lower bound and exact monomial error $E_n(x^{n+1}) = 2^{-n}$ for all $n \geq 0$. Second, from a six-line Fejér-kernel derivation of the $2n$-node Riesz trigonometric derivative interpolation identity, we establish Bernstein's sharp trigonometric and interior algebraic bounds, Schur's boundary-collar comparison against $nU_{n-1} = T_n'$, and the full global Markov inequality $\|p'\|_\infty \leq n^2\|p\|_\infty$ for every $p \in \mathcal{P}_n$, with uniqueness $p = \pm\|p\|_\infty T_n$ certified via exact alternating Chebyshev–Lobatto derivative weights $\ell_0'(1) = (2n^2+1)/6$ and $\ell_k'(1) = (-1)^k 2/(c_k(1-\eta_k))$. Third, by exact Parseval evaluation of the squared-Fejér (Jackson) kernel J_m—obtaining normalization $c_m = m(2m^2+1)/3$ and exact second trigonometric moment $\frac{3}{2(2m^2+1)}$—we prove explicit non-asymptotic trigonometric and algebraic Jackson bounds $E_n(f) \leq (1+\pi\sqrt{3})\,\omega(f,1/n)$ and $E_n(f) \leq \frac{\pi\sqrt{3}}{n+1}L$ for L-Lipschitz functions for every $n \geq 1$ (strict whenever f is nonconstant, resp. $L > 0$). Fourth, via a positive-coefficient cosine expansion of the alternating Chebyshev interpolant $F_n(\theta)$ and a $\frac{\pi}{n}$-periodic cotangent majorant, we prove the exact Chebyshev-root Lebesgue constant identity $\Lambda_n(Z_n) = \lambda_n(1) = \frac{1}{n}\sum_{r=1}^{n} \cot\frac{(2r-1)\pi}{4n} = \frac{\mathbf{1}_{\{n \text{ odd}\}}}{n} + \frac{2}{n}\sum_{k=0}^{\lfloor n/2\rfloor-1} \csc\frac{(2k+1)\pi}{2n}$ together with the universal logarithmic sandwich $\frac{2}{\pi}\ln n + \frac{2}{\pi}\ln\frac{4}{\pi} < \Lambda_n(Z_n) \leq \frac{2}{\pi}\ln n + 1$ for all $n \geq 1$. All identities and envelopes are verified across 172 deterministic numerical checks.

1 Introduction and Relation to Prior Work

Constructive approximation theory rests on four tightly interlocking pillars linking minimax optimality, inverse derivative inequalities, direct modulus-of-continuity convergence rates, and polynomial interpolation stability: Chebyshev's equioscillation and monic minimax theorem [3], the Bernstein–Markov–Schur polynomial derivative inequalities [6, 1, 7, 8], Jackson's direct approximation theorems via positive trigonometric kernels [5], and the logarithmic Lebesgue constant of Chebyshev-node Lagrange interpolation [4, 2, 9]. While standard monographs treat these pillars through separate complex-analytic, functional-analytic, or asymptotic arguments, every non-asymptotic sharp constant admits a short, elementary, fully self-contained algebraic-trigonometric proof.

Throughout, let $\|f\|_\infty := \sup_{x\in[-1,1]} |f(x)|$ for $f \in C([-1,1])$, let $\mathcal{P}_n$ denote the real algebraic polynomials of degree at most n, let $E_n(f) := \inf_{q\in\mathcal{P}_n} \|f-q\|_\infty$, and let $C_{2\pi}$ denote continuous

2π-periodic real functions equipped with $\|g\|_\infty := \sup_{\theta\in\mathbb{R}} |g(\theta)|$ and best degree-n trigonometric approximation error $E_n^{\mathrm{trig}}(g)$. Here we establish four self-contained theorems valid for every integer $n \geq 1$: (i) Chebyshev monic minimax optimality with an airtight multiplicity-credit uniqueness proof and the de la Vallée Poussin lower bound (Theorem 2.1); (ii) the Fejér-kernel proof of Riesz's derivative identity, Bernstein's interior inequality, Schur's collar envelope, and Markov's global n^2 inequality with Chebyshev–Lobatto uniqueness (Theorem 3.1); (iii) exact Parseval normalization and second trigonometric moment of the squared-Fejér kernel yielding explicit non-asymptotic Jackson bounds (Theorem 4.1); and (iv) the positive-Fourier-coefficient proof of the exact Chebyshev-root Lebesgue constant, its dual cotangent–cosecant identity, and an elementary calculus proof of the universal logarithmic sandwich $\frac{2}{\pi}\ln n + \frac{2}{\pi}\ln\frac{4}{\pi} < \Lambda_n(Z_n) \leq \frac{2}{\pi}\ln n + 1$ (Theorem 5.1).

2 Chebyshev Equioscillation and Monic Minimax Optimality

For every integer $n \geq 0$ and $x \in [-1, 1]$, define the Chebyshev polynomial of the first kind $T_n(x) := \cos(n \arccos x)$ and, for $n \geq 0$, the second kind $U_n(\cos\theta) := \frac{\sin((n+1)\theta)}{\sin\theta}$ ($\theta \in (0, \pi)$, extended continuously to ± 1). For $n \geq 1$, let $\eta_k := \cos(k\pi/n)$ ($k = 0, \ldots, n$) denote the $n + 1$ Chebyshev–Lobatto extrema and $z_j := \cos\frac{(2j-1)\pi}{2n}$ ($j = 1, \ldots, n$) the n Chebyshev roots.

Theorem 2.1 (Chebyshev Minimax Monic Theorem and de la Vallée Poussin Lower Bound)**.** (a) (Exact Recurrence, Extrema, and Boundary Derivatives.) *$T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for all $n \geq 1$. For every $n \geq 1$, $T_n \in \mathcal{P}_n$ has leading coefficient 2^{n-1}, $\|T_n\|_\infty = 1$, $T_n(\eta_k) = (-1)^k$ ($0 \leq k \leq n$), $T_n(z_j) = 0$ ($1 \leq j \leq n$), $T_n'(x) = nU_{n-1}(x)$, and $T_n'(\pm 1) = (\pm 1)^{n-1}n^2$.*

(b) (Sharp Monic Minimax Theorem with Unique Attainment.) *For every $n \geq 1$ and every monic polynomial $p(x) = x^n + \sum_{j=0}^{n-1} c_j x^j$,*

$$\|p\|_\infty \geq 2^{1-n}, \tag{1}$$

with equality if and only if $p = \widetilde{T}_n := 2^{1-n}T_n$.

(c) (de la Vallée Poussin Lower Bound and Monomial Error.) *Let $f \in C([-1, 1])$, $p \in \mathcal{P}_n$ ($n \geq 0$), and $-1 \leq x_0 < x_1 < \cdots < x_{n+1} \leq 1$ be points on which $r := f - p$ strictly alternates in sign ($r(x_k)r(x_{k+1}) < 0$ for $0 \leq k \leq n$). Then*

$$E_n(f) \geq \min_{0\leq k\leq n+1} |f(x_k) - p(x_k)|. \tag{2}$$

In particular, for every $n \geq 0$, $E_n(x^{n+1}) = 2^{-n}$, uniquely attained by $p_n^*(x) = x^{n+1} - 2^{-n}T_{n+1}(x)$.

Proof. Part (a). Writing $x = \cos\theta$ ($\theta \in [0, \pi]$), the sum-to-product identity $\cos((n + 1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta)$ yields the three-term recurrence, which by induction shows $T_n \in \mathcal{P}_n$ has leading coefficient 2^{n-1} for all $n \geq 1$. Direct evaluation gives $T_n(\eta_k) = \cos(k\pi) = (-1)^k$, $\|T_n\|_\infty = 1$, and $T_n(z_j) = \cos((j - 1/2)\pi) = 0$. Differentiating $T_n(\cos\theta) = \cos(n\theta)$ gives $-\sin\theta\, T_n'(\cos\theta) = -n\sin(n\theta)$, i.e., $T_n'(x) = nU_{n-1}(x)$; taking $\theta \to 0^+$ gives $T_n'(1) = n^2$, and parity $T_n(-x) = (-1)^n T_n(x)$ gives $T_n'(-1) = (-1)^{n-1}n^2$.

Part (b). Suppose p is monic of degree n with $\|p\|_\infty \leq 2^{1-n}$ and $p \neq \widetilde{T}_n$, so $q := \widetilde{T}_n - p \in \mathcal{P}_{n-1} \setminus \{0\}$ obeys $(-1)^k q(\eta_k) = 2^{1-n} - (-1)^k p(\eta_k) \geq 0$ for all $k = 0, \ldots, n$. Let $m(x)$ denote the multiplicity of $x \in [-1, 1]$ as a root of q. At any interior extremum η_k ($1 \leq k \leq n - 1$) where $q(\eta_k) = 0$, we have $(-1)^k p(\eta_k) = 2^{1-n} = \|p\|_\infty$, so p attains an interior global extremum at η_k and hence $p'(\eta_k) = 0 = \widetilde{T}_n'(\eta_k)$, forcing $m(\eta_k) \geq 2$. Assign one unit of root-multiplicity credit to each of the n closed subintervals $I_j := [\eta_j, \eta_{j-1}]$ ($j = 1, \ldots, n$) as follows: if neither endpoint

of I_j is a zero of q, then $q(\eta_j)$ and $q(\eta_{j-1})$ have strictly opposite signs, so by the Intermediate Value Theorem q has at least one root strictly inside (η_j, η_{j-1}), credited to I_j; if an interior node η_k $(1 \le k \le n-1)$ has $q(\eta_k) = 0$, its multiplicity $m(\eta_k) \ge 2$ supplies one credit to I_k and one credit to I_{k+1}; and if $\eta_0 = 1$ (resp. $\eta_n = -1$) is a root, its multiplicity ≥ 1 supplies one credit to I_1 (resp. I_n). Every subinterval I_j receives at least 1 credit without over-allocating any root's multiplicity, so q has at least n roots in $[-1, 1]$ counted with multiplicity—contradicting $0 \ne q \in \mathcal{P}_{n-1}$ [3].

Part (c). Let $\mu := \min_{0 \le k \le n+1} |r(x_k)| > 0$. If some $q \in \mathcal{P}_n$ satisfied $\|f - q\|_\infty < \mu$, then $q - p = r - (f - q)$ would share the strict sign of $r(x_k)$ at all $n+2$ points $x_0 < \cdots < x_{n+1}$, yielding at least $n+1$ distinct zeros of $q - p \in \mathcal{P}_n$, whence $q \equiv p$, contradicting $\|f - q\|_\infty < \mu \le \|f - p\|_\infty$. For $f(x) = x^{n+1}$, every $q \in \mathcal{P}_n$ makes $x^{n+1} - q(x)$ a monic polynomial of degree $n+1$, so (1) yields $E_n(x^{n+1}) = 2^{-n}$ with unique minimizer $p_n^* = x^{n+1} - \widetilde{T}_{n+1}$. □

3 Riesz–Bernstein–Schur–Markov Derivative Extremals

Theorem 3.1 (Riesz Interpolation, Bernstein Interior Bound, and Global Schur–Markov Inequality)**.** *Let $n \ge 1$ be an integer.*

(a) (Exact Riesz–Fejér Derivative Identity and Sharp Bernstein Trigonometric Bound.) *For every degree-$\le n$ real trigonometric polynomial $S(\theta) = a_0 + \sum_{m=1}^{n}(a_m \cos(m\theta) + b_m \sin(m\theta))$ and every $\theta \in \mathbb{R}$, with $\phi_k := \frac{(2k-1)\pi}{2n}$ $(k = 1, \dots, 2n)$,*

$$S'(\theta) = \frac{1}{4n} \sum_{k=1}^{2n} (-1)^{k-1} \frac{S(\theta + \phi_k)}{\sin^2(\phi_k/2)}, \qquad \frac{1}{4n} \sum_{k=1}^{2n} \frac{1}{\sin^2(\phi_k/2)} = n. \tag{3}$$

Consequently, $\|S'\|_\infty \le n\|S\|_\infty$, with equality for nonzero S if and only if $S(\theta) = A\cos(n\theta) + B\sin(n\theta)$.

(b) (Bernstein Interior Bound, Schur Collar Envelope, and Global Markov Inequality.) *For every $p \in \mathcal{P}_n$,*

$$|p'(x)| \le \begin{cases} \dfrac{n}{\sqrt{1-x^2}} \|p\|_\infty \le n^2 \|p\|_\infty, & |x| \le \cos\dfrac{\pi}{2n}, \\ |T_n'(x)| \, \|p\|_\infty \le n^2 \|p\|_\infty, & \cos\dfrac{\pi}{2n} \le |x| \le 1. \end{cases} \tag{4}$$

In particular, $\|p'\|_\infty \le n^2 \|p\|_\infty$ on $[-1, 1]$. Moreover, the Chebyshev–Lobatto Lagrange basis $\{\ell_k\}_{k=0}^n$ at $\eta_k = \cos(k\pi/n)$ obeys

$$\ell_0'(1) = \frac{2n^2 + 1}{6} > 0, \qquad \ell_k'(1) = \frac{(-1)^k \cdot 2}{c_k(1 - \eta_k)} \quad (1 \le k \le n,\ c_n = 2,\ c_k = 1 \text{ for } 1 \le k \le n-1), \tag{5}$$

so $(-1)^k \ell_k'(1) > 0$ for all $k = 0, \dots, n$, and equality $\|p'\|_\infty = n^2 \|p\|_\infty$ holds if and only if $p = \pm\|p\|_\infty T_n$ (for nonzero p, attained exclusively at $x = \pm 1$ when $n \ge 2$, and everywhere on $[-1, 1]$ when $n = 1$).

Proof. Part (a). By translation invariance and linearity, it suffices to verify (3) at $\theta = 0$ for $e^{im\theta}$ $(-n \le m \le n)$. Recall the Fejér kernel identity $K_{n-1}(\phi) := \sum_{j=-(n-1)}^{n-1} (1 - |j|/n) e^{ij\phi} = \frac{1}{n}\big(\frac{\sin(n\phi/2)}{\sin(\phi/2)}\big)^2$. Since $n\phi_k/2 = \frac{(2k-1)\pi}{4}$, we have $\sin^2(n\phi_k/2) = \frac{1}{2}$ and $\sin(n\phi_k) = (-1)^{k-1}$ for every $k = 1, \dots, 2n$, whence $\frac{1}{4n\sin^2(\phi_k/2)} = \frac{1}{2} K_{n-1}(\phi_k)$. Using $(-1)^{k-1} = (e^{in\phi_k} - e^{-in\phi_k})/(2i)$,

$$\frac{1}{4n} \sum_{k=1}^{2n} (-1)^{k-1} \frac{e^{im\phi_k}}{\sin^2(\phi_k/2)} = \frac{1}{4i} \sum_{j=-(n-1)}^{n-1} \left(1 - \frac{|j|}{n}\right) \sum_{k=1}^{2n} \left(e^{i(n+j+m)\phi_k} - e^{i(-n+j+m)\phi_k}\right).$$

By geometric summation on $\phi_k = \frac{(2k-1)\pi}{2n}$, $\sum_{k=1}^{2n} e^{ir\phi_k} = 2n(-1)^q$ if $r = 2nq$ ($q \in \mathbb{Z}$) and 0 otherwise. For $m = 0$, $|\pm n + j| \in [1, 2n-1]$ is never a multiple of $2n$, giving $0 = i \cdot 0$. For $1 \le m \le n$, $n + j + m \in [2, 3n-1]$ hits $2n$ solely at $j = n - m$ (contributing $-2n$), while $-n + j + m \in [-2n+2, n-1]$ hits 0 solely at $j = n - m$ (contributing $+2n$), both with Fejér weight $1 - \frac{n-m}{n} = \frac{m}{n}$; subtracting gives $\frac{1}{4i} \cdot \frac{m}{n} \cdot (-4n) = im$. Conjugation gives the case $-n \le m \le -1$, proving the first identity in (3) [7]. Evaluating at $S(\theta) = \sin(n\theta)$ at $\theta = 0$ gives the weight sum n, and the triangle inequality immediately gives $\|S'\|_\infty \le n\|S\|_\infty$ [1]. If $|S'(\theta_0)| = n\|S\|_\infty > 0$, then every term $(-1)^{k-1}S(\theta_0 + \phi_k)$ must equal $\sigma\|S\|_\infty$ for a common sign $\sigma \in \{\pm 1\}$; each $\theta_0 + \phi_k$ is therefore a global extremum of S, so $S'(\theta_0 + \phi_k) = 0$ as well. Hence $R(\theta) := S(\theta) - \sigma\|S\|_\infty \sin(n(\theta - \theta_0))$ has double zeros at all $2n$ distinct points $\theta_0 + \phi_k \pmod{2\pi}$, meaning the degree-$\le 2n$ algebraic polynomial $z^n R(\theta)$ ($z = e^{i\theta}$) has at least $4n$ roots counted with multiplicity and must vanish identically.

Part (b). Setting $S(\theta) := p(\cos\theta)$ gives $S'(\theta) = -\sin\theta\, p'(\cos\theta)$, so Part (a) yields $|p'(x)| \le \frac{n}{\sqrt{1-x^2}}\|p\|_\infty$ for all $x \in (-1, 1)$. For $|x| \le z_1 := \cos\frac{\pi}{2n}$, concavity of sin on $[0, \pi/2]$ gives $\sqrt{1-x^2} \ge \sin\frac{\pi}{2n} \ge \frac{2}{\pi} \cdot \frac{\pi}{2n} = \frac{1}{n}$ (strictly $> \frac{1}{n}$ for $n \ge 2$), establishing the first line of (4).

For the boundary collar $x \in [z_1, 1]$, assume without loss of generality $\|p\|_\infty \le 1$ and set $q(x) := p'(x)/n \in \mathcal{P}_{n-1}$, so $|q(z_k)| \le 1/\sin\theta_k = |U_{n-1}(z_k)|$ at all n Chebyshev roots $z_k = \cos\theta_k$ ($\theta_k = \frac{(2k-1)\pi}{2n}$). Because the $n-1$ roots of U_{n-1} are $\cos(k\pi/n) < z_1$ (for $n \ge 2$, and $U_0 \equiv 1$ for $n = 1$), U_{n-1} is strictly positive and nondecreasing on $[z_1, 1]$ (strictly increasing on $[z_1, 1]$ for $n \ge 2$ by Rolle's theorem, as all $n-2$ roots of U'_{n-1} lie below $\cos(\pi/n) < z_1$), with $U_{n-1}(1) = n$. Suppose for contradiction that $|q(x_0)| > U_{n-1}(x_0)$ at some $x_0 \in (z_1, 1]$; replacing q by $\pm q$ so $q(x_0) > U_{n-1}(x_0) > 0$ and setting $\lambda := U_{n-1}(x_0)/q(x_0) \in (0, 1)$, the polynomial $h := U_{n-1} - \lambda q \in \mathcal{P}_{n-1}$ satisfies $(-1)^{k-1}h(z_k) \ge (1-\lambda)/\sin\theta_k > 0$ for all $k = 1, \dots, n$. Thus h has at least $n-1$ distinct roots in (z_n, z_1) plus an n-th distinct root at $x_0 > z_1$, forcing $h \equiv 0$, contradicting $h(z_1) > 0$ [8]. Hence $|p'(x)| \le nU_{n-1}(x) = T'_n(x) \le n^2$ on $[z_1, 1]$; replacing x by $-x$ covers $[-1, -z_1]$, proving (4) and Markov's bound $\|p'\|_\infty \le n^2\|p\|_\infty$ [6].

Finally, the Lagrange polynomials for $\{\eta_k\}_{k=0}^n$ are $\ell_k(x) = \frac{(-1)^{k+1}(1-x^2)T'_n(x)}{c_k n^2(x-\eta_k)}$ ($1 \le k \le n$) and $\ell_0(x) = \frac{(1+x)T'_n(x)}{2n^2}$. Differentiating at $x = 1$ immediately yields $\ell'_k(1) = \frac{(-1)^k 2}{c_k(1-\eta_k)}$ for $k \ge 1$. Differentiating the Chebyshev differential equation $(1-x^2)T''_n - xT'_n + n^2T_n = 0$ once and evaluating at $x = 1$ gives $T''_n(1) = \frac{n^2(n^2-1)}{3}$, whence $\ell'_0(1) = \frac{T'_n(1)+2T''_n(1)}{2n^2} = \frac{2n^2+1}{6} > 0$, proving (5). For the equality characterization, $p \equiv 0$ trivially satisfies $p = \pm\|p\|_\infty T_n$; assume therefore that $p \not\equiv 0$ ($\|p\|_\infty > 0$). For $n \ge 2$, both interior inequalities in (4) are strict on $(-1, 1)$ (since $\sin\frac{\pi}{2n} > \frac{1}{n}$ and $T'_n(x) < n^2$ on $[z_1, 1)$), so $\|p'\|_\infty = n^2\|p\|_\infty$ forces $\max(|p'(1)|, |p'(-1)|) = n^2\|p\|_\infty$, i.e., $|p'(1)| = n^2\|p\|_\infty$ or $|p'(-1)| = n^2\|p\|_\infty$. By (5), $\sum_{k=0}^n |\ell'_k(1)| = \sum_{k=0}^n (-1)^k \ell'_k(1) = T'_n(1) = n^2$, so $|p'(1)| = |\sum_{k=0}^n p(\eta_k)\ell'_k(1)| = n^2\|p\|_\infty$ holds iff $p(\eta_k) = \sigma(-1)^k\|p\|_\infty$ for a common sign $\sigma \in \{\pm 1\}$ and all $k = 0, \dots, n$, uniquely identifying $p = \sigma\|p\|_\infty T_n$ (and symmetrically if $|p'(-1)| = n^2\|p\|_\infty$, replacing $p(x)$ by $p(-x)$ gives the same unique family $p = \pm\|p\|_\infty T_n$, which then attains $|p'(\pm 1)| = n^2\|p\|_\infty$ at both endpoints). For $n = 1$, $p(x) = ax + b$ satisfies $\|p\|_\infty = |a| + |b|$, so $|p'(x)| = |a| = \|p\|_\infty$ iff $b = 0$, again giving $p = \pm\|p\|_\infty T_1$ (attained everywhere on $[-1, 1]$). □

4 Squared-Fejér Kernel Moments and Non-Asymptotic Jackson Bounds

For $g \in C_{2\pi}$ (or $f \in C([-1, 1])$) and $\delta > 0$, define the modulus of continuity $\omega(g, \delta) := \sup_{|t-s| \le \delta} |g(t) - g(s)|$. Note that for every $u \ge 0$ and $\delta > 0$, setting $k := \lfloor u/\delta \rfloor$ and subdividing an interval of length u into $k+1$ subintervals of length $\le \delta$ gives $\omega(g, u) \le (k+1)\,\omega(g, \delta) \le (1 + u/\delta)\,\omega(g, \delta)$.

Theorem 4.1 (Exact Squared-Fejér Moments and Explicit Non-Asymptotic Jackson Bounds)**.** (a) (Exact Normalization and Second Trigonometric Moment.) *For every integer $m \geq 1$, the nonnegative even degree-$(2m-2)$ trigonometric polynomial kernel*

$$J_m(t) := \frac{1}{c_m}\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^4, \qquad c_m := \frac{m(2m^2+1)}{3}, \tag{6}$$

satisfies exact unit mass, exact second trigonometric moment, and explicit first moment:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} J_m(t)\,dt = 1, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2\left(\frac{t}{2}\right) J_m(t)\,dt = \frac{3}{2(2m^2+1)}, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} |t|\, J_m(t)\,dt < \frac{\pi\sqrt{3}}{2m}. \tag{7}$$

(b) (Explicit Non-Asymptotic Trigonometric and Algebraic Jackson Bounds.) *For every integer $n \geq 1$, every $g \in C_{2\pi}$, and every $f \in C([-1,1])$,*

$$E_n^{\mathrm{trig}}(g) \leq (1+\pi\sqrt{3})\,\omega\left(g, \frac{1}{n}\right), \qquad E_n(f) \leq (1+\pi\sqrt{3})\,\omega\left(f, \frac{1}{n}\right), \tag{8}$$

with strict inequality whenever g (resp. f) is nonconstant. Moreover, if $g \in C_{2\pi}$ (resp. $f \in C([-1,1])$) is L-Lipschitz ($\omega(\cdot,u) \leq Lu$, $L \geq 0$), then

$$E_n^{\mathrm{trig}}(g) \leq \frac{\pi\sqrt{3}}{n+1}\,L, \qquad E_n(f) \leq \frac{\pi\sqrt{3}}{n+1}\,L, \tag{9}$$

with strict inequality whenever $L > 0$.

Proof. Part (a). Since $\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^2 = \sum_{j=-(m-1)}^{m-1}(m-|j|)e^{ijt}$, Parseval's identity gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^4 dt = \sum_{j=-(m-1)}^{m-1}(m-|j|)^2 = m^2 + 2\sum_{r=1}^{m-1} r^2 = \frac{m(2m^2+1)}{3} = c_m,$$

confirming unit mass. For the second trigonometric moment, factor $\sin(t/2)\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^2 = \sin(mt/2)\frac{\sin(mt/2)}{\sin(t/2)}$ and use $\frac{\sin(mt/2)}{\sin(t/2)} = \sum_{r=0}^{m-1} e^{i(r-(m-1)/2)t}$ together with $2i\sin(mt/2) = e^{imt/2} - e^{-imt/2}$:

$$2i\sin\left(\frac{mt}{2}\right)\frac{\sin(mt/2)}{\sin(t/2)} = \sum_{r=0}^{m-1} e^{i(r+1/2)t} - \sum_{s=-m}^{-1} e^{i(s+1/2)t}.$$

Because the $2m$ half-integer frequencies $r+1/2$ ($r=-m,\dots,m-1$) are mutually orthogonal on $[-\pi,\pi]$ (their pairwise differences are nonzero integers) and each has modulus 1, Parseval's identity yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sin^2\left(\frac{t}{2}\right)\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^4 dt = 2m \implies \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2\left(\frac{t}{2}\right) J_m(t)\,dt = \frac{m/2}{c_m} = \frac{3}{2(2m^2+1)}.$$

Since $|t| \leq \pi|\sin(t/2)|$ on $[-\pi,\pi]$ (by concavity of sin on $[0,\pi/2]$), Cauchy–Schwarz and $2(2m^2+1) > 4m^2$ yield

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|t|\,J_m(t)\,dt \leq \pi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2\left(\frac{t}{2}\right)J_m(t)\,dt\right)^{1/2} = \pi\sqrt{\frac{3}{2(2m^2+1)}} < \frac{\pi\sqrt{3}}{2m}.$$

Part (b). Fix $n \geq 1$ and set $m := \lfloor n/2 \rfloor + 1$, so $2m-2 \leq n$ and $2m \geq n+1 > n$. The convolution $(J_m * g)(\theta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta-t)\,J_m(t)\,dt$ is a trigonometric polynomial of degree $\leq 2m-2 \leq n$ [5]. By unit mass, nonnegativity of J_m, and (7),

$$|g(\theta) - (J_m * g)(\theta)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\theta)-g(\theta-t)|\,J_m(t)\,dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\omega(g,|t|)\,J_m(t)\,dt.$$

If g is L-Lipschitz ($L \geq 0$), $\omega(g,|t|) \leq L|t|$, giving $E_n^{\mathrm{trig}}(g) \leq \frac{\pi\sqrt{3}}{2m} L \leq \frac{\pi\sqrt{3}}{n+1} L$ (strict whenever $L > 0$ by (7)). For general $g \in C_{2\pi}$, applying $\omega(g,|t|) \leq (1+n|t|)\,\omega(g,1/n)$ yields $E_n^{\mathrm{trig}}(g) \leq (1+\frac{\pi\sqrt{3}\,n}{2m})\,\omega(g,1/n) \leq (1+\pi\sqrt{3})\,\omega(g,1/n)$ (with $1+\pi\sqrt{3} \approx 6.4414$, strict whenever g is nonconstant so $\omega(g,1/n) > 0$). For $f \in C([-1,1])$, define the even function $g(\theta) := f(\cos\theta) \in C_{2\pi}$. Since $|\cos\theta_1 - \cos\theta_2| \leq |\theta_1 - \theta_2|$, we have $\omega(g,u) \leq \omega(f,u)$. Because J_m and g are even, $J_m * g$ is an even trigonometric polynomial of degree $\leq n$, hence $(J_m * g)(\theta) = p(\cos\theta)$ for a unique $p \in \mathcal{P}_n$ with $\|f - p\|_\infty = \|g - J_m * g\|_\infty$, yielding the algebraic bounds in (8)–(9). □

5 Exact Chebyshev Lebesgue Constant and Logarithmic Envelope

For $n \geq 1$ and any n distinct nodes $X = \{x_1, \ldots, x_n\} \subset [-1,1]$, let $(\mathcal{I}_X f)(x) := \sum_{j=1}^n f(x_j)\,\ell_j(x) \in \mathcal{P}_{n-1}$ be the Lagrange interpolation operator, $\lambda_X(x) := \sum_{j=1}^n |\ell_j(x)|$ its Lebesgue function, and $\Lambda_n(X) := \|\mathcal{I}_X\|_{C\to C} = \max_{x\in[-1,1]} \lambda_X(x)$ its Lebesgue constant.

Theorem 5.1 (Positive-Fourier Chebyshev Lebesgue Constant and Universal Logarithmic Sandwich)**.** *Let $Z_n = \{z_j = \cos\theta_j\}_{j=1}^n$ with $\theta_j = \frac{(2j-1)\pi}{2n}$ ($n \geq 1$).*

(a) (Lebesgue Stability Inequality.) *For every $f \in C([-1,1])$ and every n-point node set $X \subset [-1,1]$,*

$$\|f - \mathcal{I}_X f\|_\infty \leq \big(1 + \Lambda_n(X)\big)\, E_{n-1}(f). \tag{10}$$

(b) (Exact Positive-Fourier Identity and Global Maximum at $x = \pm 1$.) *The alternating Chebyshev-root interpolant $F_n(\theta) := \sum_{r=1}^n (-1)^{r-1} \ell_r(\cos\theta)$ has strictly positive cosine coefficients:*

$$F_n(\theta) = \frac{\mathbf{1}_{\{n \text{ odd}\}}}{n} + \frac{2}{n} \sum_{\substack{1\leq j\leq n-1 \\ n-j \text{ odd}}} \frac{\cos(j\theta)}{\sin\Big(\frac{(n-j)\pi}{2n}\Big)}. \tag{11}$$

Consequently, $\Lambda_n(Z_n)$ is attained at $x = \pm 1$ and satisfies the exact dual identity

$$\Lambda_n(Z_n) = \lambda_{Z_n}(\pm 1) = \frac{1}{n} \sum_{r=1}^n \cot\Big(\frac{(2r-1)\pi}{4n}\Big) = \frac{\mathbf{1}_{\{n \text{ odd}\}}}{n} + \frac{2}{n} \sum_{k=0}^{\lfloor n/2\rfloor - 1} \frac{1}{\sin\Big(\frac{(2k+1)\pi}{2n}\Big)}. \tag{12}$$

(c) (Universal Non-Asymptotic Logarithmic Sandwich.) *For every integer $n \geq 1$,*

$$\frac{2}{\pi} \ln n + \frac{2}{\pi} \ln\Big(\frac{4}{\pi}\Big) < \Lambda_n(Z_n) \leq \frac{2}{\pi} \ln n + 1, \tag{13}$$

with equality on the right if and only if $n = 1$. Hence $\|f - \mathcal{I}_{Z_n} f\|_\infty \leq (2 + \frac{2}{\pi}\ln n) E_{n-1}(f)$.

Proof. Part (a). For any $q \in \mathcal{P}_{n-1}$, $\mathcal{I}_X q = q$, so $\|f - \mathcal{I}_X f\|_\infty \leq \|f - q\|_\infty + \|\mathcal{I}_X(f-q)\|_\infty \leq (1 + \Lambda_n(X))\|f - q\|_\infty$; taking the infimum over $q \in \mathcal{P}_{n-1}$ gives (10).

Part (b). By discrete orthogonality of $\{T_j\}_{j=0}^{n-1}$ on the Chebyshev roots Z_n (namely $\sum_{s=1}^n \cos(j\theta_s)\cos(m\theta_s) = \frac{n}{2}\delta_{j,m}$ for $1 \leq j, m \leq n-1$, n for $j = m = 0$, and 0 for $j \neq m$), each Lagrange basis function expands as $\ell_r(\cos\theta) = \frac{1}{n} + \frac{2}{n}\sum_{j=1}^{n-1} \cos(j\theta_r)\cos(j\theta)$. Multiplying by $(-1)^{r-1} = \sin(n\theta_r)$ and summing over $r = 1, \ldots, n$, note that for any integer m with $2n \nmid m$, multiplying $\sum_{r=1}^n \sin(m\theta_r)$ by $2\sin\frac{m\pi}{2n}$ telescopes to $1 - \cos(m\pi) = 1 - (-1)^m$, giving 0 if m is even and $1/\sin\frac{m\pi}{2n}$ if m is odd. Applying $\sin(n\theta_r)\cos(j\theta_r) = \frac{1}{2}(\sin((n+j)\theta_r) + \sin((n-j)\theta_r))$ for $1 \leq j \leq n-1$, both $n+j$ and $n-j$ share parity, and when $n-j$ is odd we have $\sin\frac{(n+j)\pi}{2n} = \sin\frac{(n-j)\pi}{2n} > 0$, yielding $\sum_{r=1}^n (-1)^{r-1}\cos(j\theta_r) = 1/\sin\frac{(n-j)\pi}{2n}$. Combined with $\frac{1}{n}\sum_{r=1}^n (-1)^{r-1} = \mathbf{1}_{\{n \text{ odd}\}}/n$, this proves (11). Since every cosine coefficient in (11) is positive, $F_n(\theta) \leq F_n(0)$ for all $\theta \in \mathbb{R}$.

On the other hand, from $T_n'(z_r) = n(-1)^{r-1}/\sin\theta_r$ we have $\ell_r(\cos\theta) = \frac{(-1)^{r-1}}{n}\frac{\cos(n\theta)\sin\theta_r}{\cos\theta-\cos\theta_r}$. Using the trigonometric identity $\frac{2\sin\theta_r}{\cos\theta-\cos\theta_r} = \cot\frac{\theta_r-\theta}{2} + \cot\frac{\theta_r+\theta}{2} = \cot\frac{\theta_r-\theta}{2} - \cot\frac{(2\pi-\theta_r)-\theta}{2}$ and setting $\phi_s := \frac{(2s-1)\pi}{2n}$ ($s = 1,\dots,2n$, so $\phi_{2n+1-r} = 2\pi - \theta_r$), the triangle inequality yields, for every $\theta \in [0,\pi] \setminus \{\theta_r\}$,

$$\lambda_{Z_n}(\cos\theta) = \frac{|\cos(n\theta)|}{2n}\sum_{r=1}^{n}\left|\cot\frac{\theta_r-\theta}{2} - \cot\frac{\phi_{2n+1-r}-\theta}{2}\right| \le \frac{|\cos(n\theta)|}{2n}\sum_{s=1}^{2n}\left|\cot\frac{\phi_s-\theta}{2}\right| =: \Phi_n(\theta).$$

The majorant Φ_n is even and $\frac{\pi}{n}$-periodic (shifting $\theta \mapsto \theta + \frac{\pi}{n}$ cyclically permutes $\{\phi_s\}$ modulo 2π while preserving $|\cos(n\theta)|$ and $|\cot(\cdot/2)|$), so every $\theta \in \mathbb{R}$ reduces to some $\tilde\theta \in [0, \frac{\pi}{2n}]$. On $[0, \frac{\pi}{2n})$, we have $\cos(n\tilde\theta) > 0$, $\cot\frac{\phi_r-\tilde\theta}{2} > 0$, and $|\cot\frac{\phi_{2n+1-r}-\tilde\theta}{2}| = \cot\frac{\phi_r+\tilde\theta}{2} > 0$ for $r = 1,\dots,n$, so $\Phi_n(\tilde\theta) = F_n(\tilde\theta) \le F_n(0)$ (and at $\tilde\theta = \frac{\pi}{2n}$, $\lim_{\theta\to\phi_1}\Phi_n(\theta) = 1 = F_n(\phi_1) \le F_n(0)$). At $\theta = 0$, $\lambda_{Z_n}(1) = \frac{1}{n}\sum_{r=1}^{n}\frac{\sin\theta_r}{1-\cos\theta_r} = \frac{1}{n}\sum_{r=1}^{n}\cot(\theta_r/2) = F_n(0)$ (and by symmetry $\lambda_{Z_n}(-1) = \lambda_{Z_n}(1)$), proving (12) [4, 2].

Part (c). Because $u \mapsto \cot u$ is strictly decreasing on $(0,\pi/2)$, comparing Riemann rectangles on $[\frac{(2r-1)\pi}{4n}, \frac{(2r+1)\pi}{4n}]$ ($r = 1,\dots,n-1$) and $[\frac{(2n-1)\pi}{4n}, \frac{\pi}{2}]$ (using $\frac{\pi}{2n}\cot u > \frac{\pi}{4n}\cot u > \int$ on the last half-step) gives

$$\frac{\pi}{2}\Lambda_n(Z_n) = \frac{\pi}{2n}\sum_{r=1}^{n}\cot\!\left(\frac{(2r-1)\pi}{4n}\right) > \int_{\pi/(4n)}^{\pi/2}\cot u\,du = \ln\!\left(\frac{1}{\sin(\pi/(4n))}\right) > \ln\!\left(\frac{4n}{\pi}\right),$$

since $\sin u < u$ for $u > 0$, establishing the strict lower bound in (13).

For the upper bound, $n = 1$ gives $\Lambda_1(Z_1) = \cot(\pi/4) = 1 = \frac{2}{\pi}\ln 1 + 1$. For $n \ge 2$, $\cot''(u) = 2\cot u\csc^2 u > 0$ makes $\cot u$ strictly convex on $(0,\pi/2)$, so the midpoint inequality on $[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}]$ for $r = 2,\dots,n$ yields

$$\frac{\pi}{2n}\sum_{r=2}^{n}\cot\!\left(\frac{(2r-1)\pi}{4n}\right) < \int_{\pi/(2n)}^{\pi/2}\cot u\,du = \ln\!\left(\frac{1}{\sin(\pi/(2n))}\right).$$

Setting $u := \frac{\pi}{2n} \in (0, \frac{\pi}{4}]$ for $n \ge 2$ and multiplying by $\frac{2}{\pi}$ gives

$$\Lambda_n(Z_n) < \frac{1}{n}\cot\!\left(\frac{u}{2}\right) + \frac{2}{\pi}\ln\!\left(\frac{1}{\sin u}\right) = \frac{2}{\pi}\ln n + \frac{2}{\pi}\left(f(u) + \ln\frac{2}{\pi}\right), \qquad f(u) := u\cot\!\left(\frac{u}{2}\right) - \ln\!\left(\frac{\sin u}{u}\right).$$

Using $\cot(u/2) - \cot u = 1/\sin u$ and putting $v := u/2 \in (0,\pi/4]$, direct differentiation yields

$$f'(u) = \frac{1}{u} + \frac{1}{\sin u} - \frac{u}{2\sin^2(u/2)} = \frac{\sin^2 v + v\tan v - 2v^2}{u\sin^2 v}.$$

On $(0,\pi/4]$, $\sin v \ge v - v^3/6$ (so $\sin^2 v \ge v^2 - v^4/3 + v^6/36$) and $\tan v \ge v + v^3/3$ (as $(\tan v - v - v^3/3)' = \tan^2 v - v^2 > 0$), whence $\sin^2 v + v\tan v - 2v^2 \ge v^6/36 > 0$. Thus f is strictly increasing on $(0,\pi/2]$, so for every $n \ge 2$, $f(\frac{\pi}{2n}) < f(\frac{\pi}{2}) = \frac{\pi}{2} - \ln\frac{2}{\pi}$, giving $\Lambda_n(Z_n) < \frac{2}{\pi}\ln n + 1$. □

6 Computational Verification and Numerical Illustrations

All four theorems were verified deterministically in IEEE-754 `float64` arithmetic via the MIT-licensed companion repository `lluiseriksson/constructive-approximation-extremals`. Across 172 structured checks—covering Chebyshev extrema and perturbed monic polynomials ($n \le 20$, seed `20261129`), Riesz derivative interpolation, Chebyshev–Lobatto weights, and random polynomials against the Schur–Markov envelope ($n \le 20$, seed `20261130`), exact squared-Fejér moments ($m \le 30$) and Jackson convergence on $|x|$ ($n \le 64$), and the dual cotangent–cosecant Lebesgue identity and logarithmic sandwich ($1 \le n \le 128$)—Table 1 and Figures 1–2 confirm zero violations.

Table 1: Deterministic verification across 172 checks (`pytest -q`: 4/4 passed in 5.41 s). All identities hold to machine precision (10^{-17}–10^{-12}) and all inequalities hold with zero violations.

Theorem	Checks	Max Identity Error	Tightest Observed Ratio / Value
Thm. 2.1 (Chebyshev Minimax)	60	9.99×10^{-16} ($T_n(\eta_k)=(-1)^k$)	$\min(\|p_{\text{monic}}\|_\infty/2^{1-n}) = 1.0227 > 1$
Thm. 3.1 (Riesz–Schur–Markov)	55	1.88×10^{-12} (Riesz), 5.68×10^{-13} (Lobatto)	$\max(\|p'\|/(\text{env}\|p\|_\infty)) = 0.9174 \le 1$
Thm. 4.1 (Squared-Fejér Jackson)	40	2.78×10^{-17} (exact $\sin^2(t/2)$ moment)	$\max(\|\|x\| - J_m * g\|_\infty/\frac{\pi\sqrt{3}}{n+1}) = 0.4789 < 1$
Thm. 5.1 (Chebyshev Lebesgue)	17	4.44×10^{-16} (cot–csc), 2.44×10^{-13} ($\max\lambda_n$)	$\Lambda_{128}(Z_{128}) = 4.0514 < 4.0889 = \frac{2}{\pi}\ln 128 + 1$

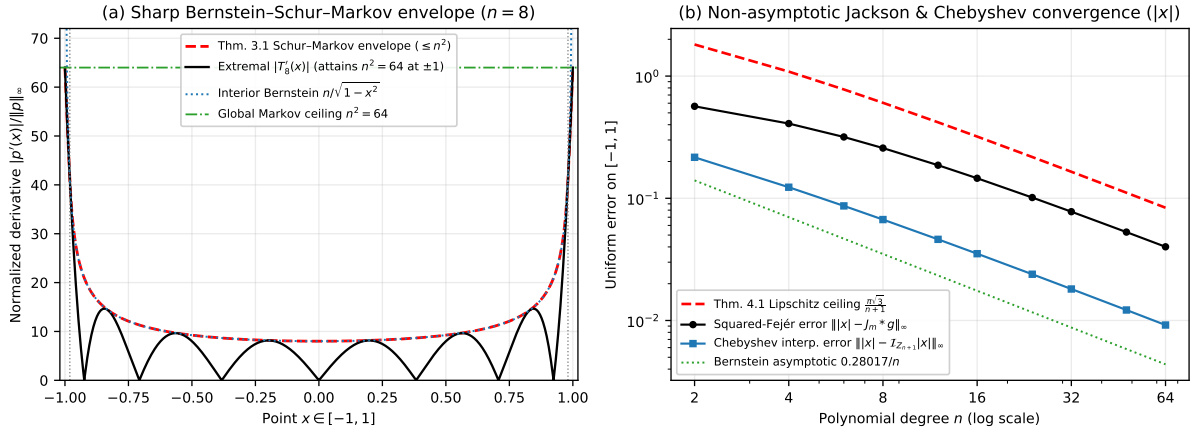

Figure 1: **(a)** Piecewise sharp Bernstein–Schur–Markov derivative envelope (4) for $n = 8$, matching $|T_8'(x)|$ in the boundary collars $|x| \ge \cos(\pi/16)$ and attaining $n^2 = 64$ uniquely at $x = \pm1$. **(b)** Non-asymptotic squared-Fejér Jackson convergence (9) and Chebyshev-node interpolation error on $f(x) = |x|$ ($2 \le n \le 64$).

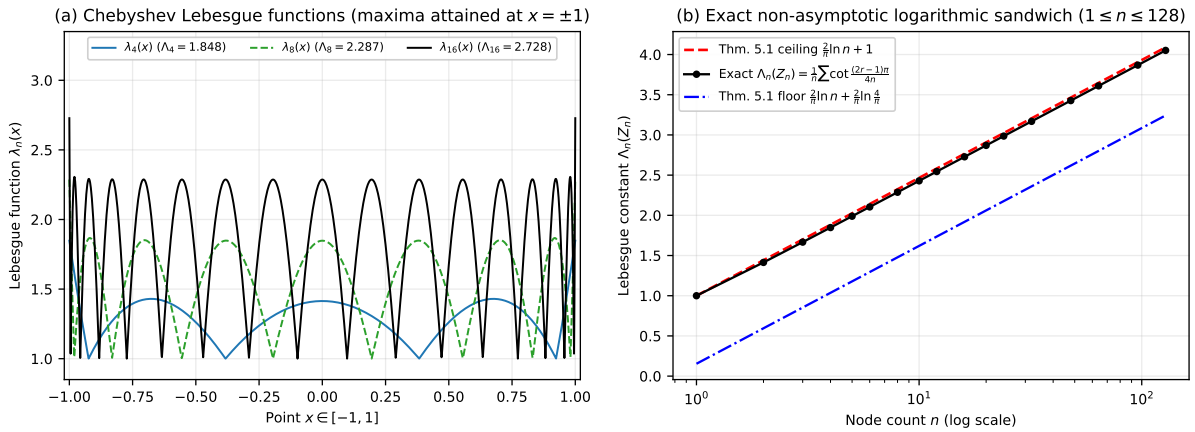

Figure 2: **(a)** Chebyshev-root Lebesgue functions $\lambda_n(x)$ for $n \in \{4, 8, 16\}$, attaining their global maxima $\Lambda_n(Z_n)$ at $x = \pm1$ (12). **(b)** Exact $\Lambda_n(Z_n)$ across $1 \le n \le 128$ tightly enclosed by the non-asymptotic logarithmic sandwich $\frac{2}{\pi}\ln n + \frac{2}{\pi}\ln\frac{4}{\pi} < \Lambda_n(Z_n) \le \frac{2}{\pi}\ln n + 1$ (13).

7 Scope and Limitations

Three structural boundaries merit explicit note:

1. *Sharp Jackson–Favard constants and higher smoothness.* Theorem 4.1 uses the elementary squared-Fejér kernel J_m to obtain explicit non-asymptotic constants ($1 + \pi\sqrt{3}$ and $\pi\sqrt{3}$); achieving the exact sharp Favard–Akhiezer–Krein constant $\frac{\pi}{2}$ for 1-Lipschitz functions or $O(n^{-r}\omega(f^{(r)}, 1/n))$ rates for C^r functions requires Favard's optimal spline/trigonometric multipliers or iterated Jackson convolutions [9].

2. *Higher-order derivatives (V. A. Markov) and L^p norms.* Theorem 3.1 treats first-order uniform (L^∞) derivatives; sharp bounds for k-th derivatives $\|p^{(k)}\|_\infty \le T_n^{(k)}(1)\|p\|_\infty$ (V. A. Markov) and L^p analogues require further extremal comparison arguments.

3. *Asymptotic Lebesgue constant precision.* Theorem 5.1 proves the clean non-asymptotic ceiling $\frac{2}{\pi}\ln n + 1$ valid for all $n \ge 1$; the exact asymptotic constant is $\frac{2}{\pi}(\gamma + \ln\frac{8}{\pi}) \approx 0.962523$ [2], which sits only 0.0375 below the universal ceiling 1.

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