AIRR.SCIENCE · ARR-2026-7Q0QKC6WC58YMAN6 v1 [Mathematical Physics] 29 Nov 2026

EXACT FOUR-PLAQUETTE TUBE FLOOR, REFLECTION-POSITIVE VARIATIONAL UPPER BOUNDS, AND CERTIFIED TWO-SIDED ENCLOSURES FOR THE GLUEBALL MASS GAP, STRING TENSION, AND MASS-TO-TENSION RATIO m_G/σ_{fund} IN d-DIMENSIONAL $\mathrm{SU}(N)$ WILSON LATTICE YANG–MILLS THEORY

LLUIS ERIKSSON

Abstract. We establish explicit, non-asymptotic, two-sided constructive enclosures simultaneously bounding the Osterwalder–Schrader glueball mass gap $m_G(d,N,\beta)$, the fundamental Wilson-loop string tension $\sigma_{\text{fund}}(d,N,\beta)$, and the dimensionless spectral ratios m_G/σ_{fund} and $m_G/\sqrt{\sigma_{\text{fund}}}$ in d-dimensional $\mathrm{SU}(N)$ Wilson lattice Yang–Mills theory on $\mathbb{Z}^d$ ($d \geq 3$, $N \geq 2$). Three main theorems anchor the analysis:

(1) *Universal four-plaquette tube floor and full Osterwalder–Schrader mass-gap lower bound:* Combining our $2(d-1)$-chromatic Hölder polymer bound, prefix-free surface entropy $C_{\text{surf}}(d) = (4d-6)^{4d-6}/(4d-7)^{4d-7}$, and bipartite girth-4 slab projection ($\delta(H_t) \geq 2 \implies |E(H_t)| \geq 4$ in every intermediate temporal slab $[t,t+1]$), we prove that every connected polymer bridge across temporal distance T consumes at least $4T$ plaquettes. Via tilted Kotecký–Preiss cluster pinning, this yields the explicit full-spectrum lower bound $m_G(d,N,\beta) = \inf(\sigma(H_{\text{OS}}) \setminus \{0\}) \geq 4\tau_{\max} = 4(-\ln(C_{\text{surf}}(d)\kappa_d(N,\beta)) + \ln\rho_* - a_0)$ on the entire Osterwalder–Schrader Hilbert space $\mathcal{H}_{\text{OS}} \ominus \mathbb{C}\Omega$.

(2) *All-coupling reflection-positive variational upper bounds, exact 3-cube Haar gluing, and ten-plaquette non-cube floor:* Using strict positive-definiteness of the temporal-gauge transfer matrix and two-direction Jensen convexity on every finite periodic torus $\mathbb{T}^d_L$ ($L \geq 2$, $1 \leq R,T \leq L-1$), we prove the exact all-coupling Wilson-loop lower bound $d_\rho^{-1}\langle\chi_\rho(W(\gamma_{R,T}))\rangle_{\mathbb{T}^d_L,\beta} \geq (d_\rho^{-1}\langle\chi_\rho(U_p)\rangle_{\mathbb{T}^d_L,\beta})^{RT} > 0$, implying $\sigma_\rho(d,N,\beta) \leq -\ln(d_\rho^{-1}\langle\chi_\rho(U_p)\rangle_\beta)$ for every subsequential infinite-volume limit with $\langle\chi_\rho(U_p)\rangle_\beta > 0$—and unconditionally for $\rho = \text{fund}$ at every $\beta > 0$, where we prove the uniform lower bound $N^{-1}\langle\mathrm{Re\,Tr}\,U_p\rangle_\beta \geq (2d(d-1)\beta)^{-1}\ln Z_{\mathrm{SU}(N)}(\beta/N) > 0$ via vertex-disjoint plaquette Gibbs entropy. Simultaneously, for any reflection-positive, translation-invariant state with unique vacuum Ω and $\mathrm{Cov}_\beta(O_{p_0},O_{p_1}) > 0$ ($O_p = \mathrm{Re\,Tr}\,U_p$), site-containing reflection positivity yields $m_G(d,N,\beta) \leq -\ln(\mathrm{Cov}_\beta(O_{p_0},O_{p_1})/\mathrm{Var}_\beta(O_{p_0}))$. Evaluating the unit 3-cube Haar integral explicitly (d_λ^{-4} cancelling d_λ^4 identically) yields $z''_{\text{cube}}(0,0) \geq c_N\,\mathrm{Var}_{\text{1-plaq}}(\mathrm{Re\,Tr}\,U)^2 r_{\text{fund}}(\beta)^4$ ($c_2 = 1$, $c_N = 2$ for $N \geq 3$), while an exact four-case combinatorial theorem proves that *every* non-cube cluster connecting p_0 to p_1 has total size at least 10, suppressing the Cauchy cluster remainder to $O(\kappa_d^8) \ll z''_{\text{cube}}(0,0) = \Theta(\beta^4)$.

(3) *Certified two-sided enclosures and universal $\beta \to 0^+$ limit $m_G/\sigma_{\text{fund}} \to 4$:* Combining Theorems 2.2 and 3.2 yields explicit two-sided brackets on σ_{fund}, m_G, m_G/σ_{fund}, and $m_G/\sqrt{\sigma_{\text{fund}}}$, proving $\lim_{\beta\to0^+} m_G/\sigma_{\text{fund}} = 4$ and $m_G/\sqrt{\sigma_{\text{fund}}} = 4\sqrt{\ln(c_N N^2/\beta)} + O\big(\ln(1/\beta)^{-1/2}\big)$ for every $d \geq 3$ and $N \geq 2$.

Analytical proofs are given in Sections 2–4; the combinatorial tube-floor, bipartite slab-dichotomy, ten-plaquette incidence-pair lemmas, and all 22 two-sided quantitative certificates across $d \in \{3,4,5\}$ and $\mathrm{SU}(2),\mathrm{SU}(3),\mathrm{SU}(4)$ are mechanically verified in a standalone zero-import, zero-`sorry`, zero-`Classical.choice` Lean 4.36.0 kernel backed by 120-digit interval arithmetic.

1. Introduction and Honest Scope

Two fundamental non-perturbative spectral quantities characterize pure $\mathrm{SU}(N)$ lattice Yang–Mills theory on $\mathbb{Z}^d$ ($d \geq 3$) [16, 12, 13]: the physical glueball mass gap $m_G(d,N,\beta) := \inf(\sigma(H_{\text{OS}}) \setminus$

Date: November 29, 2026.

Independent Researcher, Östersund, Sweden | editor@airr.science | CC-BY-4.0 / MIT | ARR-2026-7Q0QKC6WC58YMAN6

$\{0\}$) of the Osterwalder–Schrader Hamiltonian on the full physical Hilbert space $\mathcal{H}_{\mathrm{OS}}$, and the fundamental Wilson-loop string tension $\sigma_{\mathrm{fund}}(d,N,\beta) := \liminf_{R,T\to\infty}(-(RT)^{-1}\ln|\langle \mathrm{Tr}\, W(\gamma_{R,T})\rangle_\beta|)$. Their dimensionless ratios $m_G/\sigma_{\mathrm{fund}}$ and $m_G/\sqrt{\sigma_{\mathrm{fund}}}$ are central diagnostics in both strong-coupling lattice expansions [11, 14] and large-scale Monte Carlo spectroscopy [15, 10, 1].

At strong coupling $\beta \ll 1$, formal character expansions predict $\sigma_{\mathrm{fund}} \sim -\ln r_{\mathrm{fund}}(\beta)$ and $m_G \sim -4\ln r_{\mathrm{fund}}(\beta)$ (from the four-plaquette cross-section of the minimal closed glueball tube), suggesting the universal strong-coupling limit $m_G/\sigma_{\mathrm{fund}} \to 4$ as $\beta \to 0^+$ [11, 14, 13]. Yet rigorously proving *simultaneous, non-asymptotic, finite-β, two-sided* enclosures

$$\sigma^- \le \sigma_{\mathrm{fund}}(d,N,\beta) \le \sigma^+, \qquad m_G^- \le m_G(d,N,\beta) \le m_G^+, \qquad \frac{m_G^-}{\sigma^+} \le \frac{m_G}{\sigma_{\mathrm{fund}}} \le \frac{m_G^+}{\sigma^-},$$

with explicit positive numbers and zero hidden $O(\cdot)$ constants, requires solving four distinct technical problems simultaneously:

- proving a *full-spectrum* lower bound $m_G \ge 4\tau_{\max}$ on the *entire* Osterwalder–Schrader Hilbert space $\mathcal{H}_{\mathrm{OS}} \ominus \mathbb{C}\Omega$ (not merely for single-plaquette correlators), tracking the exact factor 4 from the minimal four-plaquette tube cross-section;
- proving a non-perturbative *upper* bound on the string tension σ_{fund} (i.e., a strict *lower* bound on large $R\times T$ Wilson loops uniformly in R,T) and a non-perturbative *upper* bound on the glueball mass gap m_G (via a variational Rayleigh quotient);
- evaluating the leading $T=1$ connected plaquette–plaquette covariance from the unit 3-cube Y_{cube} with exact group-theoretic cancellation of d_λ^4; and
- bounding the *entire* infinite series of non-cube cluster corrections strictly below the $O(\beta^4)$ 3-cube signal—which fails under the naive 6- or 8-plaquette floor at moderate β, and succeeds sharply once we prove the exact *ten-plaquette non-cube cluster floor* ($\sum_{Y\in\mathcal{C}}|Y| \ge 10$ for every non-cube cluster connecting p_0 to p_1).

The present paper—the sixth installment of our constructive lattice Yang–Mills series following [7, 6, 5, 4, 3]—resolves all four problems in closed form.

Honest Scope and Explicit Disclaimers. To state the exact mathematical boundary of our results clearly:

- *All-coupling upper bounds versus certified strong-coupling enclosures:* The finite-torus Wilson-loop lower bound (17) holds on every periodic torus $\mathbb{T}_L^d$ ($L \ge 2$) for every $\beta > 0$ and every irrep ρ. In any subsequential infinite-volume limit along even tori, this yields $\sigma_{\mathrm{fund}} \le -\ln(N^{-1}\langle \mathrm{Re}\,\mathrm{Tr}\, U_p\rangle_\beta)$ unconditionally for all $\beta > 0$, and $\sigma_\rho \le -\ln(d_\rho^{-1}\langle\chi_\rho(U_p)\rangle_\beta)$ whenever $\langle\chi_\rho(U_p)\rangle_\beta > 0$. Similarly, the variational bound $m_G \le -\ln(\mathrm{Cov}_\beta(O_{p_0},O_{p_1})/\mathrm{Var}_\beta(O_{p_0}))$ holds in any reflection-positive, translation-invariant state with unique vacuum Ω and $\mathrm{Cov}_\beta(O_{p_0},O_{p_1}) > 0$. By contrast, the explicit two-sided numerical enclosures in Theorem 4.1 and Table 1 apply at fixed lattice spacing $a=1$ in the convergent strong-coupling Kotecký–Preiss regime ($\beta \in [10^{-10}, 10^{-4}]$). This paper does *not* construct the 4D continuum limit $a \to 0$ (which requires weak coupling $\beta \to \infty$) and makes no claim toward the Clay Millennium Yang–Mills Existence and Mass Gap problem.
- *Separation of analytical and mechanized proofs:* Measure-theoretic Haar integration, $2(d-1)$-chromatic Hölder estimates, prefix-free randomized exploration, Kotecký–Preiss cluster expansions, Osterwalder–Schrader reflection positivity, and spectral-theorem arguments are proved analytically in Sections 2–4. The standalone Lean 4.36.0 kernel (`src/lean/GlueballRatioCerts.lean`) mechanically verifies the exact combinatorial lemmas (the $3k \le k(k-1) \implies k \ge 4$ incidence-pair bound, bipartite girth-4 slab dichotomy, all four branches of the ten-plaquette non-cube cluster floor, the $4T$ tube-floor summation, and d_λ^4 Euler-characteristic cancellation) together with exact integer cross-consistency of all 22 two-sided quantitative certificates (zero imports, zero `sorry`, zero `Classical.choice`, zero axioms on

all certificates), backed by 120-digit `mpmath.iv` interval arithmetic (`src/python/certify_glueball_ratio.py`) with zero black-box special functions and zero numerical quadrature.

2. Four-Plaquette Tube Floor and Full OS Glueball Mass-Gap Lower Bound

Let $G = \mathrm{SU}(N)$ ($N \geq 2$) equipped with normalized Haar measure dU, and let $\Lambda \subset \mathbb{Z}^d$ ($d \geq 3$) be a finite box with oriented link set $E(\Lambda)$ and unoriented plaquette set $\mathcal{P}(\Lambda)$. Write the single-plaquette Wilson Boltzmann weight and its centered fluctuation as

$$w_\beta(U) := \exp\left(\frac{\beta}{N}\operatorname{Re}\operatorname{Tr} U\right), \qquad \hat{w}_0(\beta) := \int_G w_\beta(U)\,dU = Z_{\mathrm{SU}(N)}\left(\frac{\beta}{N}\right), \qquad g_\beta(U) := \frac{w_\beta(U)}{\hat{w}_0(\beta)} - 1, \tag{1}$$

so that $\int_G g_\beta(U)\,dU = 0$. Define the normalized polymer partition function

$$Z_\Lambda(\beta) := \int_{G^{E(\Lambda)}} \prod_{p\in\mathcal{P}(\Lambda)} \big(1 + g_\beta(U_p)\big)\,dU = \hat{w}_0(\beta)^{-|\mathcal{P}(\Lambda)|} \int_{G^{E(\Lambda)}} \prod_{p\in\mathcal{P}(\Lambda)} w_\beta(U_p)\,dU.$$

Let $\kappa_d(N,\beta) := \|g_\beta\|_{L^{2(d-1)}(G)}$ denote the $2(d-1)$-chromatic Hölder norm [4], and let

$$C_{\mathrm{surf}}(d) := \frac{(4d-6)^{4d-6}}{(4d-7)^{4d-7}}, \qquad u_*(d) := \frac{1}{4d-7}, \qquad \Pi_{\mathrm{eff}}(d) := \frac{4d-5}{4d-7}. \tag{2}$$

Recall from [4] that:

(i) The plaquette link-adjacency graph of $\mathbb{Z}^d$ has chromatic number $2(d-1)$, so for *any* finite plaquette set $\Gamma \subset \mathcal{P}(\Lambda)$ and class functions $\{f_p\}_{p\in\Gamma}$,

$$\left|\int_{G^{E(\Lambda)}} \prod_{p\in\Gamma} f_p(U_p)\,dU\right| \leq \prod_{p\in\Gamma} \|f_p\|_{L^{2(d-1)}(G)}. \tag{3}$$

(ii) A finite non-empty link-connected plaquette set $Y \subset \mathcal{P}(\Lambda)$ is a *valid vacuum polymer* ($Y \in \mathcal{G}(\Lambda)$) if every link $e \in E(Y)$ has $\deg_Y(e) := |\{p \in Y : e \in E(p)\}| \geq 2$; every $Y \in \mathcal{G}(\Lambda)$ satisfies $|Y| \geq 6$ and $|E(Y)| \leq 2|Y|$, and for any plaquette $p_0 \in \mathcal{P}(\Lambda)$ and $z \in (0, C_{\mathrm{surf}}(d)^{-1})$,

$$\Phi(z) := \sum_{\substack{Y\in\mathcal{G}(\Lambda)\\ p_0\in Y}} z^{|Y|} \leq \frac{1}{4d-7}\big(C_{\mathrm{surf}}(d)\,z\big)^6 \leq \frac{1}{4d-7}\,\frac{(C_{\mathrm{surf}}(d)\,z)^6}{1 - C_{\mathrm{surf}}(d)\,z}. \tag{4}$$

(iii) For any finite link set $E_0 \subset E(\Lambda)$, a plaquette set $S \subset \mathcal{P}(\Lambda)$ belongs to the family $\mathcal{S}(\Lambda, E_0)$ of *valid boundary-anchored surfaces* if every link-connected component of S meets E_0 and every link $e \in E(S) \setminus E_0$ has $\deg_S(e) \geq 2$. Every $S \in \mathcal{S}(\Lambda, E_0)$ obeys $|E(S) \setminus E_0| \leq 2|S|$ and

$$\sum_{S\in\mathcal{S}(\Lambda,E_0)} C_{\mathrm{surf}}(d)^{-|S|} \leq \left(1 + \frac{1}{4d-7}\right)^{(2d-2)|E_0|}. \tag{5}$$

(iv) Since every plaquette q shares a link with at most $1 + 4(2d-3) = 8d-11$ plaquettes and any $Y \in \mathcal{G}(\Lambda)$ sharing a link with q must contain at least *two* plaquettes in $N_{\mathrm{link}}[q]$, the half-degree pinning constant satisfies $\frac{8d-11}{2(4d-7)} < \frac{4d-5}{4d-7} = \Pi_{\mathrm{eff}}(d)$. Thus whenever $\rho_0 := C_{\mathrm{surf}}(d)\kappa_d(N,\beta)e^{a_0} < 1$ satisfies

$$\Pi_{\mathrm{eff}}(d)\,\frac{\rho_0^6}{1-\rho_0} \leq a_0, \tag{6}$$

the Kotecký–Preiss condition [9, 8] holds with parameter a_0, the infinite-volume Gibbs measure $\langle\cdot\rangle_\beta$ on $\mathrm{SU}(N)^{E(\mathbb{Z}^d)}$ is unique and invariant under lattice translations and reflections,

and every $S \in \mathcal{S}(\Lambda, E_0)$ obeys

$$\left|\frac{Z_{\Lambda\setminus\overline{E_0\cup S}}(\beta)}{Z_\Lambda(\beta)}\right| \;\le\; \exp\bigl(a_0|S| + c_d a_0|E_0|\bigr), \qquad c_d := \frac{2(d-1)}{8d-11}. \tag{7}$$

We now establish the universal four-plaquette tube floor across temporal slabs.

Lemma 2.1 (Universal Four-Plaquette Slab Floor)**.** *Let $d \ge 3$ and $t_1 < t_2$ be integers. Let $X \subset \mathcal{P}(\mathbb{Z}^d)$ be any finite set of plaquettes whose union (viewed as closed unit squares in $\mathbb{R}^d$) contains a connected component intersecting both $\{x_0 \le t_1\}$ and $\{x_0 \ge t_2\}$, and suppose that every temporal link $e = \langle (t,x),(t+1,x)\rangle \in E(X)$ with $t_1 \le t \le t_2 - 1$ satisfies $\deg_X(e) \ge 2$. Then for every integer $t \in \{t_1, \dots, t_2 - 1\}$, the set $S_t(X)$ of temporal plaquettes of X in the slab $[t, t+1]$ satisfies $|S_t(X)| \ge 4$. Consequently,*

$$|X| \;\ge\; \sum_{t=t_1}^{t_2-1} |S_t(X)| \;\ge\; 4\,(t_2 - t_1). \tag{8}$$

Proof. Fix any integer $t \in \{t_1, \dots, t_2 - 1\}$ and consider the separating hyperplane $\Pi_{t+1/2} := \{x \in \mathbb{R}^d : x_0 = t + \frac{1}{2}\}$. Every spatial plaquette of $\mathbb{Z}^d$ lies entirely in $\{x_0 \le t\}$ or $\{x_0 \ge t+1\}$, so the only plaquettes of $\mathbb{Z}^d$ intersecting $\Pi_{t+1/2}$ are the temporal plaquettes in the slab $[t, t+1]$. Because X has a connected component meeting both $\{x_0 \le t_1\} \subset \{x_0 < t + \frac{1}{2}\}$ and $\{x_0 \ge t_2\} \subset \{x_0 > t + \frac{1}{2}\}$, $S_t(X)$ is non-empty.

Each temporal plaquette $p \in S_t(X)$ has the unique form $[t, t+1] \times e_{\mathrm{sp}}$ for an undirected spatial edge $e_{\mathrm{sp}} = \langle x, y\rangle \in E(\mathbb{Z}^{d-1})$. Let $H_t = (V_t, E_t)$ be the finite simple subgraph of $\mathbb{Z}^{d-1}$ whose edge set E_t consists of these spatial projections (so $|E_t| = |S_t(X)| \ge 1$) and whose vertex set $V_t \subset \mathbb{Z}^{d-1}$ consists of all endpoints of edges in E_t. For every vertex $x \in V_t$, the temporal link $e_x := \langle (t,x),(t+1,x)\rangle$ belongs to $E(X)$, and the only plaquettes in $\mathbb{Z}^d$ incident to e_x are temporal plaquettes in the slab $[t, t+1]$ whose spatial projection is incident to x. Hence

$$\deg_{H_t}(x) = \deg_{S_t(X)}(e_x) = \deg_X(e_x) \;\ge\; 2 \qquad \text{for every } x \in V_t.$$

Since H_t is a non-empty finite simple graph with minimum degree $\delta(H_t) \ge 2$, it contains a simple cycle (walk greedily without backtracking until revisiting a vertex). Because $\mathbb{Z}^{d-1}$ is bipartite and has no parallel edges, its girth is 4, so every simple cycle in H_t has length at least 4. Thus $|S_t(X)| = |E_t| \ge 4$. Summing over the pairwise disjoint slab sets $S_t(X)$ for $t = t_1, \dots, t_2 - 1$ (verified inductively in `tube_floor_sum`) proves (8). □

Theorem 2.2 (Full Osterwalder–Schrader Glueball Mass-Gap Lower Bound)**.** *Let $d \ge 3$, $N \ge 2$, $a_0 = \frac{1}{6}$, and let $\rho_* \in (0,1)$ satisfy the tilted Kotecký–Preiss condition*

$$\Pi_{\mathrm{eff}}(d)\,\frac{\rho_*^6}{1-\rho_*} \;\le\; a_0. \tag{9}$$

Suppose $\beta > 0$ satisfies $\rho_0 := C_{\mathrm{surf}}(d)\kappa_d(N,\beta)e^{a_0} < \rho_*$, and define the maximal cluster tilt

$$\tau_{\max}(d,N,\beta) := \ln\!\left(\frac{\rho_*}{\rho_0}\right) = -\ln\bigl(C_{\mathrm{surf}}(d)\kappa_d(N,\beta)\bigr) + \ln\rho_* - a_0 \;>\; 0. \tag{10}$$

Then:

(1) *For every pair of bounded gauge-invariant local observables $F \in \mathcal{A}_{[-R,0]}$ (supported on finite link set $E_F \subset \{x_0 \le 0\}$) and $G \in \mathcal{A}_{[T,T+R]}$ (supported on finite link set $E_G \subset \{x_0 \ge T\}$) with $T \ge 1$, there exists a finite constant $C(d, E_F, E_G)$ independent of T and Λ such that uniformly in all finite boxes $\Lambda \supset E_F \cup E_G$ and in the infinite-volume limit $\langle\cdot\rangle_\beta$,*

$$\bigl|\operatorname{Cov}_{\Lambda,\beta}(F,G)\bigr| \;\le\; C(d,E_F,E_G)\;\|F\|_\infty\,\|G\|_\infty\;e^{-4\tau_{\max}T}. \tag{11}$$

(2) *Let $(\mathcal{H}_{\mathrm{OS}}, \Omega, \mathcal{T}_{\mathrm{OS}} = e^{-H_{\mathrm{OS}}})$ be the Osterwalder–Schrader Hilbert space, unique vacuum vector, and positive self-adjoint transfer-matrix contraction ($0 \le \mathcal{T}_{\mathrm{OS}} \le I$) of the infinite-volume measure $\langle\cdot\rangle_\beta$ [12, 13]. Then the glueball mass gap on the entire orthogonal complement $\mathcal{H}_{\mathrm{OS}} \ominus \mathbb{C}\Omega$ obeys the explicit lower bound*

$$m_G(d, N, \beta) := \inf\big(\sigma(H_{\mathrm{OS}}) \setminus \{0\}\big) \;\ge\; 4\,\tau_{\max}(d, N, \beta) \;>\; 0. \tag{12}$$

Proof. (1) Note first that $\Pi_{\mathrm{eff}}(d) = \frac{4d-5}{4d-7} > 1$ and $\frac{1}{1-\rho_*} > 1$, so (9) implies $\rho_*^6 < a_0 = \frac{1}{6}$, whence

$$\rho_* e^{a_0} \;<\; (1/6)^{1/6}\, e^{1/6} \;<\; 0.877 \;<\; 1 \qquad \text{for every } d \ge 3. \tag{13}$$

Also $\rho_0 < \rho_*$ implies (6), so the Kotecký–Preiss cluster expansion converges absolutely and uniformly in Λ.

Write $E_0 := E_F \cup E_G$. Expanding $\prod_{p\in\mathcal{P}(\Lambda)}(1 + g_\beta(U_p)) = \sum_{X\subset\mathcal{P}(\Lambda)} \prod_{p\in X} g_\beta(U_p)$ and factoring each X into its boundary-anchored part $S \in \mathcal{S}(\Lambda, E_0)$ (the union of all link-connected components of X intersecting E_0) and remaining vacuum polymers disjoint from $\overline{E_0 \cup S} := \{p \in \mathcal{P}(\Lambda) : E(p) \cap (E_0 \cup E(S)) \neq \emptyset\}$ yields

$$\langle FG\rangle_{\Lambda,\beta} = \sum_{S\in\mathcal{S}(\Lambda,E_0)} \zeta_{F,G}(S)\, \frac{Z_{\Lambda\setminus\overline{E_0\cup S}}(\beta)}{Z_\Lambda(\beta)}, \qquad \zeta_{F,G}(S) := \int_{G^{E(\Lambda)}} F(U)G(U) \prod_{p\in S} g_\beta(U_p)\, dU,$$

and analogously for $\langle F\rangle_{\Lambda,\beta}$ and $\langle G\rangle_{\Lambda,\beta}$. Partition $\mathcal{S}(\Lambda, E_0)$ into:

- *Case 1 (Direct bridge):* $\mathcal{S}_{\mathrm{br}}$ consists of all $S \in \mathcal{S}(\Lambda, E_0)$ that have a link-connected component touching both E_F and E_G. Every link $e \in E(S) \setminus E_0$ has $\deg_S(e) \ge 2$; in particular, every temporal link in $[0, T]$ lies outside E_0 and has $\deg_S(e) \ge 2$. By Lemma 2.1 with $(t_1, t_2) = (0, T)$, every $S \in \mathcal{S}_{\mathrm{br}}$ satisfies $|S| \ge 4T$. By (3), $|\zeta_{F,G}(S)| \le \|F\|_\infty \|G\|_\infty \kappa_d^{|S|}$. Combining (7) and (5) with $\rho_0 = \rho_* e^{-\tau_{\max}} < e^{-\tau_{\max}}$ gives

$$\begin{aligned}\sum_{S\in\mathcal{S}_{\mathrm{br}}} \left|\zeta_{F,G}(S)\, \frac{Z_{\Lambda\setminus\overline{E_0\cup S}}}{Z_\Lambda}\right| \;&\le\; \|F\|_\infty \|G\|_\infty\, e^{c_d a_0 |E_0|}\, \rho_0^{4T} \sum_{S\in\mathcal{S}(\Lambda,E_0)} C_{\mathrm{surf}}(d)^{-|S|} \\ &\le\; C_1(d, E_0)\, \|F\|_\infty \|G\|_\infty\, e^{-4\tau_{\max} T}.\end{aligned}$$

- *Case 2 (Separated boundary surfaces):* Every $S \in \mathcal{S}(\Lambda, E_0) \setminus \mathcal{S}_{\mathrm{br}}$ decomposes uniquely as $S = S_F \sqcup S_G$ with $S_F \in \mathcal{S}(\Lambda, E_F)$, $S_G \in \mathcal{S}(\Lambda, E_G)$, and $A \cap B = \emptyset$, where $A := E_F \cup E(S_F)$ and $B := E_G \cup E(S_G)$; hence $\zeta_{F,G}(S_F \sqcup S_G) = \zeta_F(S_F)\, \zeta_G(S_G)$. Writing $W_A := \{Y \in \mathcal{G}(\Lambda) : E(Y) \cap A \neq \emptyset\}$ and $W_B := \{Y \in \mathcal{G}(\Lambda) : E(Y) \cap B \neq \emptyset\}$, subtracting the overlapping pairs (S_F, S_G) with $A \cap B \neq \emptyset$ from $\langle F\rangle_{\Lambda,\beta}\langle G\rangle_{\Lambda,\beta}$ gives

$$\begin{aligned}\mathrm{Cov}_{\Lambda,\beta}(F, G) = (\text{Case 1}) &- \sum_{\substack{S_F\in\mathcal{S}(\Lambda,E_F),\, S_G\in\mathcal{S}(\Lambda,E_G)\\ A\cap B\neq\emptyset}} \zeta_F(S_F)\zeta_G(S_G)\, \frac{Z_{\Lambda\setminus\overline{A}}}{Z_\Lambda}\, \frac{Z_{\Lambda\setminus\overline{B}}}{Z_\Lambda} \\ &+ \sum_{\substack{S_F\in\mathcal{S}(\Lambda,E_F),\, S_G\in\mathcal{S}(\Lambda,E_G)\\ A\cap B=\emptyset}} \zeta_F(S_F)\zeta_G(S_G)\, \frac{Z_{\Lambda\setminus\overline{A}}}{Z_\Lambda}\, \frac{Z_{\Lambda\setminus\overline{B}}}{Z_\Lambda}\, \big(e^{u(A,B)} - 1\big),\end{aligned} \tag{14}$$

where $u(A, B) := \sum_{\mathcal{C}:\, \mathcal{C}\cap W_A\neq\emptyset,\, \mathcal{C}\cap W_B\neq\emptyset} \phi^T(\mathcal{C})$ is the Mayer cluster bridge sum.

Let $t_F := \max(\{0\} \cup \{x_0 : x \in V(A)\}) \ge 0$ and $t_G := \min(\{T\} \cup \{x_0 : x \in V(B)\}) \le T$. If $t_F > 0$, then every component of S_F touches $E_F \subset \{x_0 \le 0\}$ and no temporal link in $[0, t_F]$ lies in E_F, so Lemma 2.1 gives $|S_F| \ge 4t_F$ (which also holds trivially when $t_F = 0$); symmetrically $|S_G| \ge 4(T - t_G)$.

– Whenever $A \cap B \neq \emptyset$ (or $t_F \geq t_G$), we have $t_F \geq t_G$, so $|S_F| + |S_G| \geq 4t_F + 4(T - t_G) \geq 4T$. Thus the middle sum in (14) is bounded by $C_2(d, E_0) \|F\|_\infty \|G\|_\infty \rho_0^{4T} \leq C_2(d, E_0) \|F\|_\infty \|G\|_\infty e^{-4\tau_{\max} T}$.
– Whenever $A \cap B = \emptyset$, let $D := \max(0, t_G - t_F)$. If $D > 0$, every connected cluster $\mathcal{C} = (Y_1, \dots, Y_m)$ meeting both W_A (in $\{x_0 \leq t_F\}$) and W_B (in $\{x_0 \geq t_G\}$) has support $U(\mathcal{C}) = \bigcup_{i=1}^m Y_i$ connecting $\{x_0 \leq t_F\}$ to $\{x_0 \geq t_G\}$, with $\deg_{U(\mathcal{C})}(e) \geq \deg_{Y_i}(e) \geq 2$ on every link. Lemma 2.1 therefore forces $\sum_{i=1}^m |Y_i| \geq |U(\mathcal{C})| \geq 4D$. Defining tilted weights $\tilde{\zeta}(Y) := e^{\tau_{\max}|Y|}\zeta(Y)$ obeys $|\tilde{\zeta}(Y)|e^{a_0|Y|} \leq (\rho_*/C_{\mathrm{surf}}(d))^{|Y|}$, which satisfies the Kotecký–Preiss condition by (9). Because $\phi^T(\mathcal{C}; \tilde{\zeta}) = e^{\tau_{\max}\sum|Y_i|}\phi^T(\mathcal{C}; \zeta)$, Kotecký–Preiss pinning gives

$$|u(A, B)| \;\leq\; e^{-4\tau_{\max} D} \sum_{Y \in W_A} |\tilde{\zeta}(Y)|e^{a_0|Y|} \;\leq\; a_0\big(|S_F| + c_d|E_F|\big)\, e^{-4\tau_{\max} D},$$

which holds trivially for $D = 0$ as well. Using $|e^u - 1| \leq |u|e^{|u|} \leq |u| \exp(a_0|S_F| + c_d a_0|E_F|)$, the (S_F, S_G) summand in the third term of (14) is bounded by

$$\|F\|_\infty \|G\|_\infty\, e^{c_d a_0(2|E_F| + |E_G|)}\, a_0\big(|S_F| + c_d|E_F|\big)\, \big(\kappa_d e^{2a_0}\big)^{|S_F|}\, \big(\kappa_d e^{a_0}\big)^{|S_G|}\, e^{-4\tau_{\max} D}.$$

Substitute $\kappa_d e^{a_0} = C_{\mathrm{surf}}(d)^{-1}\rho_* e^{-\tau_{\max}}$ and $\kappa_d e^{2a_0} = C_{\mathrm{surf}}(d)^{-1}(\rho_* e^{a_0})e^{-\tau_{\max}}$. Since $|S_F| + |S_G| + 4D \geq 4t_F + 4(T - t_G) + 4(t_G - t_F)_+ \geq 4T$, the exponential factors combine to $\leq e^{-4\tau_{\max} T}$. By (13), $\rho_* e^{a_0} < 1$, so $\sup_{n \geq 0} n(\rho_* e^{a_0})^n \leq (e \ln(1/(\rho_* e^{a_0})))^{-1} < \infty$. Summing over $S_F \in \mathcal{S}(\Lambda, E_F)$ and $S_G \in \mathcal{S}(\Lambda, E_G)$ via (5) yields (11).

(2) Let $F \in \mathcal{A}_{[0,R]}$ be any bounded gauge-invariant local observable with $\langle F \rangle_\beta = 0$, and let $\Psi_F := [F] \in \mathcal{H}_{\mathrm{OS}} \ominus \mathbb{C}\Omega$ be its equivalence class in the Osterwalder–Schrader Hilbert space [12, 13]. Under site-containing time reflection ϑ_0 across $\{x_0 = 0\}$, $\vartheta_0 F \in \mathcal{A}_{[-R,0]}$, while the $2n$-step time translate satisfies $\tau_{2n} F \in \mathcal{A}_{[2n, 2n+R]}$ $(n \geq 1)$. Passing $\Lambda \uparrow \mathbb{Z}^d$ in (11) gives

$$\langle \Psi_F,\, \mathcal{T}_{\mathrm{OS}}^{2n}\, \Psi_F \rangle_{\mathcal{H}_{\mathrm{OS}}} = \mathrm{Cov}_\beta(\vartheta_0 F,\, \tau_{2n} F) \;\leq\; C(d, E_F)\, \|F\|_\infty^2\, e^{-4\tau_{\max}(2n)} \qquad \forall\, n \geq 1.$$

By the spectral theorem for the self-adjoint contraction $0 \leq \mathcal{T}_{\mathrm{OS}} \leq I$, let μ_{Ψ_F} be the spectral measure of Ψ_F on $[0, 1]$. For every $\varepsilon > 0$, Chebyshev's inequality gives

$$\mu_{\Psi_F}\big((e^{-4\tau_{\max}+\varepsilon}, 1]\big) \;\leq\; e^{(4\tau_{\max} - \varepsilon)(2n)}\, \langle \Psi_F, \mathcal{T}_{\mathrm{OS}}^{2n} \Psi_F \rangle_{\mathcal{H}_{\mathrm{OS}}} \;\leq\; C(d, E_F)\, \|F\|_\infty^2\, e^{-2\varepsilon n} \xrightarrow{n \to \infty} 0.$$

Hence $\mathrm{supp}(\mu_{\Psi_F}) \subseteq [0, e^{-4\tau_{\max}}]$ for every zero-mean bounded local gauge-invariant observable F. Since such vectors Ψ_F are dense in $\mathcal{H}_{\mathrm{OS}} \ominus \mathbb{C}\Omega$, $\|\mathcal{T}_{\mathrm{OS}}|_{\mathcal{H}_{\mathrm{OS}} \ominus \mathbb{C}\Omega}\| \leq e^{-4\tau_{\max}}$, proving (12). □

3. All-Coupling Variational Upper Bounds, 3-Cube Gluing, and Ten-Plaquette Floor

We now establish exact non-perturbative *upper* bounds on both $\sigma_\rho(d, N, \beta)$ and $m_G(d, N, \beta)$ valid at *all* couplings $\beta > 0$, together with the exact 3-cube gluing signal and ten-plaquette non-cube cluster floor that make these upper bounds quantitatively explicit at strong coupling.

Lemma 3.1 (Exact Ten-Plaquette Non-Cube Cluster Floor)**.** *Let $d \geq 3$, let $p_0 \subset \{x_0 = 0\}$ be a unit spatial plaquette, let $p_1 := p_0 + \hat{e}_0 \subset \{x_0 = 1\}$ be its unit temporal translate, and let $Y_{\mathrm{cube}} \in \mathcal{G}(\mathbb{Z}^d)$ be the unique unit 3-cube having p_0 and p_1 as opposite faces ($|Y_{\mathrm{cube}}| = 6$). Let $\mathcal{C} = (Y_1, \dots, Y_m)$ ($m \geq 1$) be any connected multiset of valid vacuum polymers $Y_i \in \mathcal{G}(\mathbb{Z}^d)$ whose support $\bigcup_{i=1}^m Y_i$ contains both p_0 and p_1. If $\mathcal{C} \neq (Y_{\mathrm{cube}})$, then*

$$\sum_{i=1}^m |Y_i| \;\geq\; 10. \tag{15}$$

Proof. We analyze four exhaustive, disjoint cases (each proved in the standalone Lean 4 kernel):

(a) *Multi-polymer cluster ($m \geq 2$):* Since every valid vacuum polymer has $|Y_i| \geq 6$, $\sum_{i=1}^{m} |Y_i| \geq 6m \geq 12 \geq 10$ (`noncube_cluster_ten_floor_multi_poly`).

(b) *Single polymer Y ($m = 1$) intersecting at least two distinct temporal slabs:* Let $[t_a, t_a+1]$ and $[t_b, t_b+1]$ ($t_a \neq t_b$) be two distinct temporal slabs in which $S_t(Y) \neq \emptyset$. Applying the proof of Lemma 2.1 to each non-empty slab gives $|S_{t_a}(Y)| \geq 4$ and $|S_{t_b}(Y)| \geq 4$. Since $p_0, p_1 \in Y$ are spatial plaquettes disjoint from all temporal slabs, $|Y| \geq |S_{t_a}(Y)| + |S_{t_b}(Y)| + 2 \geq 10$ (`noncube_cluster_ten_floor_multi_slab`).

If neither (a) nor (b) holds, then Y is a single link-connected vacuum polymer containing $p_0 \subset \{x_0 = 0\}$ and $p_1 \subset \{x_0 = 1\}$ that intersects at most one temporal slab. By Lemma 2.1 with $(t_1, t_2) = (0, 1)$, $S_0(Y) \neq \emptyset$, so $[0, 1]$ is the *unique* temporal slab intersected by Y; link-connectedness then precludes any spatial plaquette outside $\{x_0 = 0, 1\}$, so $Y = S_0(Y) \sqcup Y_0 \sqcup Y_1$ is confined to $[0, 1]$.

(c) *Single polymer Y confined to slab $[0, 1]$ with $|S_0(Y)| \geq 5$:* Let $H_0 = (V_0, E_0) \subset \mathbb{Z}^{d-1}$ be the spatial projection graph of $S_0(Y)$, which is a simple bipartite graph with $\delta(H_0) \geq 2$ and $|V_0| \geq 4$. If $|V_0| = 4$, the maximum number of edges in a simple bipartite graph on 4 vertices is $2 \cdot 2 = 4$; if $|V_0| = 5$, any simple bipartite graph with $\delta \geq 2$ must have bipartition $(2, 3)$ and every vertex in the size-3 part must have degree 2, forcing $H_0 \cong K_{2,3}$ with $|E_0| = 6$; if $|V_0| \geq 6$, $2|E_0| = \sum_{v \in V_0} \deg(v) \geq 2|V_0| \geq 12 \implies |E_0| \geq 6$. Hence $|S_0(Y)| = |E_0| \neq 5$, forcing $|S_0(Y)| \geq 6$ (`slab_edge_dichotomy`). Let $Y_0, Y_1 \subset Y$ be the spatial plaquettes of Y in slices $t = 0$ and $t = 1$. Since Y has no temporal plaquettes outside $[0, 1]$, every spatial edge $e_{\text{sp}} \in E_0$ appears with degree 1 in $S_0(Y)$ on slice $t = 0$ (and likewise on slice $t = 1$), so it must be covered by at least one plaquette of Y_0 (and of Y_1). Because each spatial plaquette covers 4 edges, $4|Y_0| \geq |S_0(Y)| \geq 6 \implies |Y_0| \geq 2$, and symmetrically $|Y_1| \geq 2$. Therefore $|Y| = |S_0(Y)| + |Y_0| + |Y_1| \geq 6 + 2 + 2 = 10$ (`noncube_cluster_ten_floor_wide_slab`).

(d) *Single polymer Y confined to slab $[0, 1]$ with $|S_0(Y)| = 4$:* Then H_0 is a 4-cycle in $\mathbb{Z}^{d-1}$, which bounds a unique unit spatial plaquette $q_0 \subset \{x_0 = 0\}$ and its translate $q_1 := q_0 + \hat{e}_0 \subset \{x_0 = 1\}$ (so $S_0(Y)$ consists of the 4 vertical faces of the unit 3-cube Q between q_0 and q_1). Notice that $q_0 = p_0 \iff q_1 = p_1$.

- If $q_0 = p_0$ (hence $q_1 = p_1$), then $Y_{\text{cube}} \subseteq Y$ and $Y \neq Y_{\text{cube}}$, so at least one slice $t \in \{0, 1\}$ has a non-empty set $Y_t' := Y_t \setminus \{p_t\}$ of $k := |Y_t'| \geq 1$ extra spatial plaquettes. Because $S_0(Y)$ touches slice t only along $E(p_t)$, every link of Y_t' outside $E(p_t)$ must have degree $j \geq 2$ *within* Y_t'. Since two distinct unit plaquettes share at most 1 link, each $p \in Y_t'$ has at least 3 links outside $E(p_t)$. Letting m_j denote the number of links outside $E(p_t)$ of degree $j \geq 2$ in Y_t', using $j \leq j(j-1) = 2\binom{j}{2}$ for all $j \geq 2$ and counting shared-link pairs among the k plaquettes of Y_t' yields

$$3k \;\leq\; \sum_{j \geq 2} j\, m_j \;\leq\; 2 \sum_{j \geq 2} \binom{j}{2} m_j \;\leq\; 2\binom{k}{2} = k(k-1). \tag{16}$$

  Hence $k \geq 4$ (`noncube_incidence_pair_bound`), giving $|Y| \geq |Y_{\text{cube}}| + k \geq 6 + 4 = 10$ (`noncube_cluster_ten_floor_extra_spatial`).
- If $q_0 \neq p_0$ (hence $q_1 \neq p_1$), then in slice $t = 0$ the set $Y_0' := Y_0 \setminus \{q_0\}$ is non-empty (since $p_0 \in Y_0'$) and obeys the identical pair-incidence inequality (16) outside $E(q_0)$, forcing $|Y_0| \geq |Y_0'| \geq 4$; symmetrically $|Y_1| \geq 4$, so $|Y| = 4 + |Y_0| + |Y_1| \geq 12 \geq 10$.

□

Theorem 3.2 (All-Coupling Variational Upper Bounds and Explicit Cluster Remainders)**.** *Let $d \geq 2$, $N \geq 2$, and $\beta > 0$.*

(1) (All-coupling finite-torus Wilson-loop lower bound and string-tension upper bound) *For every finite periodic torus $\mathbb{T}_L^d$ ($L \geq 2$), every irreducible representation $\rho \in \widehat{\mathrm{SU}(N)}$, and*

every $1 \le R, T \le L-1$,

$$\frac{1}{d_\rho}\big\langle \chi_\rho(W(\gamma_{R,T}))\big\rangle_{\mathbb{T}^d_L,\beta} \;\ge\; \left(\frac{1}{d_\rho}\big\langle \chi_\rho(U_p)\big\rangle_{\mathbb{T}^d_L,\beta}\right)^{RT} \;>\; 0. \tag{17}$$

For any subsequential infinite-volume Gibbs limit $\langle\cdot\rangle_\beta$ *along even periodic tori* $\mathbb{T}^d_L$ *(*$L\to\infty$*), define*

$$\sigma_\rho(d,N,\beta) := \liminf_{R,T\to\infty}\left(-\frac{1}{RT}\ln\left|\frac{1}{d_\rho}\big\langle \chi_\rho(W(\gamma_{R,T}))\big\rangle_\beta\right|\right).$$

Whenever $\langle\chi_\rho(U_p)\rangle_\beta > 0$,

$$\sigma_\rho(d,N,\beta) \;\le\; -\ln\!\left(\frac{1}{d_\rho}\big\langle \chi_\rho(U_p)\big\rangle_\beta\right). \tag{18}$$

Moreover, for the fundamental representation $\rho = \mathrm{fund}$, *strict positivity holds* unconditionally for all $\beta > 0$ *via the uniform lower bound*

$$\frac{1}{N}\big\langle \operatorname{Re}\operatorname{Tr} U_p\big\rangle_\beta \;\ge\; \frac{1}{2d(d-1)\,\beta}\,\ln Z_{\mathrm{SU}(N)}\!\left(\frac{\beta}{N}\right) \;>\; 0, \tag{19}$$

so (18) *applies to* $\sigma_{\mathrm{fund}}(d,N,\beta)$ *for every* $\beta>0$, $d\ge 2$, *and* $N\ge 2$.

(2) (All-coupling variational glueball mass-gap upper bound) *For* $d\ge 3$ *and any reflection-positive, translation-invariant infinite-volume Gibbs state* $\langle\cdot\rangle_\beta$ *with unique vacuum* Ω, *let* $O_{p_t} := \operatorname{Re}\operatorname{Tr} U_{p_t}$ *on spatial plaquettes* $p_0 \subset \{x_0 = 0\}$ *and* $p_1 = p_0 + \hat{e}_0 \subset \{x_0 = 1\}$. *Whenever* $\mathrm{Cov}_\beta(O_{p_0}, O_{p_1}) > 0$,

$$m_G(d,N,\beta) \;\le\; -\ln\!\left(\frac{\mathrm{Cov}_\beta(O_{p_0},O_{p_1})}{\mathrm{Var}_\beta(O_{p_0})}\right). \tag{20}$$

(3) (Exact 3-cube Haar gluing and strong-coupling explicit bounds) *Let* $d\ge 3$, $a_0 = \frac{1}{6}$, $d_1 = \frac{1}{2}$, $d_2 = 1$, *and define single-plaquette expectations*

$$o_0 := \frac{Z'_{\mathrm{SU}(N)}(\beta/N)}{Z_{\mathrm{SU}(N)}(\beta/N)}, \qquad r_{\mathrm{fund}}(\beta) := \frac{o_0}{N}, \qquad m_2 := \frac{Z''_{\mathrm{SU}(N)}(\beta/N)}{Z_{\mathrm{SU}(N)}(\beta/N)}, \qquad v_{1\mathrm{p}} := m_2 - o_0^2,$$

together with the $L^{2(d-1)}$ *norms*

$$M_d := \|(\operatorname{Re}\operatorname{Tr} U - o_0)w_\beta/\hat{w}_0\|_{L^{2(d-1)}}, \qquad M_d^{(2)} := \big\|((\operatorname{Re}\operatorname{Tr} U)^2 - m_2)w_\beta/\hat{w}_0\big\|_{L^{2(d-1)}}.$$

Suppose $\rho_0 := C_{\mathrm{surf}}(d)\kappa_d e^{a_0} < 1$ *satisfies* $(1+d_1)\Pi_{\mathrm{eff}}(d)\frac{\rho_0^6}{1-\rho_0} \le a_0$, *and let* $\rho_2^* \in (\rho_0, 1)$ *satisfy* $(1+d_2)^2\Pi_{\mathrm{eff}}(d)\frac{(\rho_2^*)^6}{1-\rho_2^*} \le a_0$ *with* $e^{\tau_1} := \rho_2^*/\rho_0 > 1$. *Then in the unique infinite-volume state* $\langle\cdot\rangle_\beta$*:*

$$\big|\langle \operatorname{Re}\operatorname{Tr} U_{p_0}\rangle_\beta - o_0\big| \;\le\; \varepsilon_1 := \frac{1+d_1}{d_1(4d-7)}\,\frac{M_d}{\kappa_d}\,\frac{\rho_0^6}{1-\rho_0}, \tag{21}$$

$$\big|\langle (\operatorname{Re}\operatorname{Tr} U_{p_0})^2\rangle_\beta - m_2\big| \;\le\; \varepsilon_2 := \frac{1+d_1}{d_1(4d-7)}\,\frac{M_d^{(2)}}{\kappa_d}\,\frac{\rho_0^6}{1-\rho_0}, \tag{22}$$

$$\mathrm{Cov}_\beta(O_{p_0},O_{p_1}) \;\ge\; c_N\, v_{1\mathrm{p}}^2\, r_{\mathrm{fund}}(\beta)^4 - R_{\mathrm{cov}}^{(10)}, \qquad c_N := \begin{cases} 1, & N=2,\\ 2, & N\ge 3, \end{cases} \tag{23}$$

where the non-cube cluster remainder obeys

$$R_{\mathrm{cov}}^{(10)} := \left(\frac{M_d}{d_2\,\kappa_d}\right)^2 e^{-10\tau_1}\,\frac{(1+d_2)^2}{4d-7}\,\frac{(\rho_2^*)^6}{1-\rho_2^*}. \tag{24}$$

Proof. (1) On $\mathbb{T}^d_L = \mathbb{T}^1_L \times \mathbb{T}^{d-1}_L$ ($L \geq 2$), denote by $U_t \in G^{E_{\mathrm{sp}}}$ the spatial links at time slice $t \in \mathbb{Z}/L\mathbb{Z}$ and $V_t \in G^{V_{\mathrm{sp}}}$ the temporal links directed from slice t to slice $t+1$. Fixing temporal gauge along $t = 0, \ldots, L-2$ via the Haar-preserving left conjugation $\tilde{U}_{t,(x,y)} := g_{t,x} U_{t,(x,y)} g_{t,y}^{-1}$ with $g_{0,x} = I$ and $g_{t+1,x} := g_{t,x} V_{t,x}$ ($0 \leq t \leq L-2$) removes $V_0, \ldots, V_{L-2}$ completely from the temporal plaquettes $U_{t,(x,y)} V_{t,y} U^{-1}_{t+1,(x,y)} V^{-1}_{t,x}$, leaving only the slice-$(L-1)$ gauge integration over $g_x := g_{L-1,x} V_{L-1,x} \in G^{V_{\mathrm{sp}}}$, which implements the orthogonal projection Π_{GI} onto gauge-invariant functions in $L^2(G^{E_{\mathrm{sp}}})$. The temporal-gauge transfer operator

$$(\mathcal{T}_{\mathrm{TG}} f)(U) := K_{\mathrm{sp}}(U)^{1/2} \int_{G^{E_{\mathrm{sp}}}} \left(\prod_{e \in E_{\mathrm{sp}}} w_\beta(U_e (U'_e)^\dagger) \right) K_{\mathrm{sp}}(U')^{1/2} f(U')\, dU'$$

is compact, self-adjoint, commutes with Π_{GI}, and is *strictly positive definite* on all of $L^2(G^{E_{\mathrm{sp}}})$: by Peter–Weyl expansion of w_β, every character coefficient obeys $c_\lambda(\beta) = \int_G w_\beta(U) \overline{\chi_\lambda(U)}\, dU > 0$ [13, 3], so for every non-zero $f \in L^2(G^{E_{\mathrm{sp}}})$, the positive multiplication $h := K_{\mathrm{sp}}^{1/2} f \not\equiv 0$ satisfies $\langle f, \mathcal{T}_{\mathrm{TG}} f \rangle = \sum_{\{\lambda_e\}} (\prod_e c_{\lambda_e}(\beta)) \left\| \hat{h}(\{\lambda_e\}) \right\|_F^2 > 0$. In particular, every eigenvalue μ_n of $\mathcal{T}_{\mathrm{TG}}$ satisfies $\mu_n > 0$ strictly.

For any $1 \leq T \leq L-1$, the rectangular loop $\gamma_{R,T}$ lies inside slices $0 \leq t \leq T \leq L-1$, where its vertical paths telescope to g_{T,x_R} and g^{-1}_{T,x_0}, yielding $W(\gamma_{R,T}) = \tilde{P}_R(0) \tilde{P}_R(T)^{-1}$ where $\tilde{P}_R(t)$ is the ordered product of R spatial links along axis 1 at slice t. Letting $\hat{A}_{ab}$ denote multiplication by $\rho(\tilde{P}_R)_{ab}$ ($1 \leq a, b \leq d_\rho$) gives the exact trace identity

$$\left\langle \chi_\rho(W(\gamma_{R,T})) \right\rangle_{\mathbb{T}^d_L, \beta} = \frac{\sum_{a,b=1}^{d_\rho} \mathrm{Tr}_{L^2}\left(\Pi_{\mathrm{GI}}\, \mathcal{T}_{\mathrm{TG}}^{L-T}\, \hat{A}^\dagger_{ab}\, \mathcal{T}_{\mathrm{TG}}^{T}\, \hat{A}_{ab} \right)}{\mathrm{Tr}_{L^2}\left(\Pi_{\mathrm{GI}}\, \mathcal{T}_{\mathrm{TG}}^{L} \right)}.$$

Let $\{\psi_m\}$ be an orthonormal eigenbasis of $\mathcal{T}_{\mathrm{TG}}$ on $\mathrm{Ran}(\Pi_{\mathrm{GI}})$ with eigenvalues $\lambda_m > 0$, and $\{\phi_n\}$ an orthonormal eigenbasis of $\mathcal{T}_{\mathrm{TG}}$ on $L^2(G^{E_{\mathrm{sp}}})$ with eigenvalues $\mu_n > 0$. Set $Z_L := \sum_m \lambda_m^L > 0$, $p_m := \lambda_m^L / Z_L > 0$ ($\sum_m p_m = 1$), and $w_{mn} := d_\rho^{-1} \sum_{a,b=1}^{d_\rho} |\langle \phi_n | \hat{A}_{ab} | \psi_m \rangle|^2 \geq 0$; unitarity of $\rho(\tilde{P}_R)$ gives

$$\sum_n w_{mn} = \frac{1}{d_\rho} \sum_{a,b=1}^{d_\rho} \left\| \hat{A}_{ab} \psi_m \right\|^2_{L^2} = \frac{1}{d_\rho} \left\langle \psi_m,\, \mathrm{Tr}(\rho(\tilde{P}_R)^\dagger \rho(\tilde{P}_R))\, \psi_m \right\rangle = 1.$$

Thus $q_{mn} := p_m w_{mn}$ defines a probability distribution with strictly positive summands $\mu_n / \lambda_m > 0$, and for every $1 \leq T \leq L-1$,

$$\frac{1}{d_\rho} \left\langle \chi_\rho(W(\gamma_{R,T})) \right\rangle_{\mathbb{T}^d_L, \beta} = \sum_{m,n} q_{mn} \left(\frac{\mu_n}{\lambda_m} \right)^T \geq \left(\sum_{m,n} q_{mn} \frac{\mu_n}{\lambda_m} \right)^T = \left(\frac{1}{d_\rho} \left\langle \chi_\rho(W(\gamma_{R,1})) \right\rangle_{\mathbb{T}^d_L, \beta} \right)^T > 0$$

by Jensen's inequality for the convex function $x \mapsto x^T$ on $(0, \infty)$. Applying the identical inequality along coordinate axis 1 to the $R \times 1$ loop $\gamma_{R,1}$ (with $1 \leq R \leq L-1$) yields

$$\frac{1}{d_\rho} \left\langle \chi_\rho(W(\gamma_{R,1})) \right\rangle_{\mathbb{T}^d_L, \beta} \geq \left(\frac{1}{d_\rho} \left\langle \chi_\rho(W(\gamma_{1,1})) \right\rangle_{\mathbb{T}^d_L, \beta} \right)^R = \left(\frac{1}{d_\rho} \left\langle \chi_\rho(U_p) \right\rangle_{\mathbb{T}^d_L, \beta} \right)^R > 0.$$

Combining both inequalities proves (17), and taking $L \to \infty$ followed by $-\frac{1}{RT} \ln$ proves (18) whenever $\langle \chi_\rho(U_p) \rangle_\beta > 0$.

To prove the uniform fundamental lower bound (19) on every even torus $\mathbb{T}^d_L$, let $\mathcal{C}_1 \subset \mathcal{P}(\mathbb{T}^d_L)$ be the subset of $(0,1)$-plaquettes with even coordinates $(x_0, x_1) \in (2\mathbb{Z})^2$; these $|\mathcal{C}_1| = \frac{L^d}{4} = \frac{1}{2d(d-1)} |\mathcal{P}(\mathbb{T}^d_L)|$ plaquettes are pairwise vertex-disjoint (hence link-disjoint). Define the product probability measure $d\nu(U) := \prod_{p \in \mathcal{C}_1} (w_\beta(U_p) / \hat{w}_0(\beta))\, dU$. By right-invariance of Haar measure on the remaining three

links of any $p \in \mathcal{C}_1$, the single-link marginal of every $e \in E(p)$ under $d\nu$ is exact Haar measure, and distinct plaquettes in $\mathcal{C}_1$ are independent. Since any plaquette $q \notin \mathcal{C}_1$ shares at most one link with each $p \in \mathcal{C}_1$, its four links are mutually independent Haar variables under $d\nu$, giving $\mathbb{E}_\nu[\operatorname{Re}\operatorname{Tr} U_q] = 0$. Writing $F_L(\beta) := |\mathcal{P}(\mathbb{T}_L^d)|^{-1} \ln \int \prod_{p\in\mathcal{P}(\mathbb{T}_L^d)} w_\beta(U_p)\, dU$, the Gibbs variational inequality (Jensen's inequality against $d\nu$) yields

$$|\mathcal{P}(\mathbb{T}_L^d)|\, F_L(\beta) \;\geq\; \sum_{p\in\mathcal{P}(\mathbb{T}_L^d)} \frac{\beta}{N}\mathbb{E}_\nu[\operatorname{Re}\operatorname{Tr} U_p] - D_{\mathrm{KL}}(\nu \,\|\, dU) = |\mathcal{C}_1|\, \ln \hat{w}_0(\beta).$$

Since $F_L(\beta)$ is convex in β with $F_L(0) = 0$ and $F_L'(\beta) = N^{-1}\langle \operatorname{Re}\operatorname{Tr} U_p\rangle_{\mathbb{T}_L^d,\beta}$, convexity yields $N^{-1}\langle \operatorname{Re}\operatorname{Tr} U_p\rangle_{\mathbb{T}_L^d,\beta} \geq F_L(\beta)/\beta \geq (2d(d-1)\beta)^{-1} \ln Z_{\mathrm{SU}(N)}(\beta/N) > 0$ uniformly in all even $L \geq 2$ (where $\ln Z_{\mathrm{SU}(N)}(t) > t \int \operatorname{Re}\operatorname{Tr} U\, dU = 0$ by strict Jensen for all $t > 0$). Passing $L \to \infty$ proves (19).

(2) Because $p_0 \subset \{x_0 = 0\}$ is fixed by site-containing reflection ϑ_0 across $\{x_0 = 0\}$, the centered observable $\tilde{O}_{p_0} := O_{p_0} - \langle O_{p_0}\rangle_\beta$ defines a non-zero vector $\Psi_{p_0} := [\tilde{O}_{p_0}] \in \mathcal{H}_{\mathrm{OS}} \ominus \mathbb{C}\Omega$ with $\|\Psi_{p_0}\|^2_{\mathcal{H}_{\mathrm{OS}}} = \operatorname{Var}_\beta(O_{p_0}) > 0$ and $\langle \Psi_{p_0}, \mathcal{T}_{\mathrm{OS}}\Psi_{p_0}\rangle_{\mathcal{H}_{\mathrm{OS}}} = \operatorname{Cov}_\beta(O_{p_0}, O_{p_1})$. The variational Rayleigh quotient for $0 \leq \mathcal{T}_{\mathrm{OS}} = e^{-H_{\mathrm{OS}}} \leq I$ on $\mathcal{H}_{\mathrm{OS}} \ominus \mathbb{C}\Omega$ gives

$$e^{-m_G(d,N,\beta)} = \|\mathcal{T}_{\mathrm{OS}}|_{\mathcal{H}_{\mathrm{OS}}\ominus\mathbb{C}\Omega}\| \;\geq\; \frac{\langle \Psi_{p_0}, \mathcal{T}_{\mathrm{OS}}\Psi_{p_0}\rangle_{\mathcal{H}_{\mathrm{OS}}}}{\|\Psi_{p_0}\|^2_{\mathcal{H}_{\mathrm{OS}}}} = \frac{\operatorname{Cov}_\beta(O_{p_0}, O_{p_1})}{\operatorname{Var}_\beta(O_{p_0})},$$

and taking $-\ln$ proves (20).

(3) Define the centered source functions

$$\tilde{f}(U) := \frac{w_\beta(U)}{\hat{w}_0(\beta)}\big(\operatorname{Re}\operatorname{Tr} U - o_0\big), \qquad \tilde{f}_2(U) := \frac{w_\beta(U)}{\hat{w}_0(\beta)}\big((\operatorname{Re}\operatorname{Tr} U)^2 - m_2\big),$$

which have zero Haar mean: $\int_G \tilde{f}\, dU = \int_G \tilde{f}_2\, dU = 0$. For $(j_0, j_1) \in \mathbb{C}^2$, deform the weights at p_0, p_1 to $1 + g_\beta(U_{p_k}) + j_k\tilde{f}(U_{p_k})$ ($k \in \{0,1\}$). Because $\int_G \tilde{f}(U)\, dU = 0$, every link of every non-vanishing deformed polymer Y still satisfies $\deg_Y(e) \geq 2$, so the deformed polymer set remains $\mathcal{G}(\Lambda)$ verbatim.

For (21), set $j_1 = 0$ and $|j_0| = r_1 := d_1\kappa_d/M_d$. Every polymer $Y \in \mathcal{G}(\Lambda)$ satisfies $|\zeta_{j_0,0}(Y)| \leq (1 + d_1)\kappa_d^{|Y|}$ by (3), so the Kotecký–Preiss condition holds by $(1 + d_1)\Pi_{\mathrm{eff}}(d)\frac{\rho_0^6}{1-\rho_0} \leq a_0$. Since $\langle \operatorname{Re}\operatorname{Tr} U_{p_0}\rangle_{\Lambda,\beta} - o_0 = \frac{\partial}{\partial j_0} \ln Z_\Lambda(\beta; j_0, 0)\Big|_{j_0=0}$, Cauchy's integral formula on $|j_0| = r_1$ together with (4) and Kotecký–Preiss pinning yields

$$\big|\langle \operatorname{Re}\operatorname{Tr} U_{p_0}\rangle_{\Lambda,\beta} - o_0\big| \;\leq\; \frac{1}{r_1} \sup_{|j_0|=r_1} \sum_{Y\ni p_0} |\zeta_{j_0,0}(Y)| e^{a_0|Y|} \;\leq\; \frac{M_d}{d_1\kappa_d}\,(1 + d_1)\,\Phi(\kappa_d e^{a_0}) \;\leq\; \varepsilon_1,$$

and the identical calculation with $\tilde{f}_2$ on $|j_0| = d_1\kappa_d/M_d^{(2)}$ yields (22).

For (23), expand $\operatorname{Cov}_{\Lambda,\beta}(O_{p_0}, O_{p_1}) = \frac{\partial^2}{\partial j_0 \partial j_1} \ln Z_\Lambda(\beta; j_0, j_1)\Big|_{0,0}$ into the single-polymer 3-cube cluster $\mathcal{C} = (Y_{\mathrm{cube}})$ plus all non-cube clusters $\mathcal{C} \neq (Y_{\mathrm{cube}})$ containing both p_0 and p_1:

$$\operatorname{Cov}_{\Lambda,\beta}(O_{p_0}, O_{p_1}) = z''_{\mathrm{cube}}(0,0) + \sum_{\substack{\mathcal{C}\neq(Y_{\mathrm{cube}})\\ p_0,p_1\in\operatorname{supp}(\mathcal{C})}} \frac{\partial^2\phi^T(\mathcal{C}; j_0, j_1)}{\partial j_0 \partial j_1}\bigg|_{0,0}.$$

To evaluate $z''_{\mathrm{cube}}(0,0) = \int_{G^{12}} \tilde{f}(U_{p_0})\tilde{f}(U_{p_1}) \prod_{k=2}^5 g_\beta(U_{p_k})\, dU$, expand $g_\beta(U) = \sum_{\lambda\neq\mathbf{1}} d_\lambda r_\lambda(\beta)\chi_\lambda(U)$ and $\tilde{f}(U) = \sum_{\lambda\neq\mathbf{1}} f_\lambda(\beta)\chi_\lambda(U)$ with $f_\lambda(\beta) := \int_G \tilde{f}(U)\overline{\chi_\lambda(U)}\, dU$. Fix a spanning-tree gauge on the 8-vertex, 12-edge 3-cube by setting $U = I$ on the 4 vertical edges and 3 bottom edges, leaving 5 free group variables a (fourth bottom edge) and b_1, b_2, b_3, b_4 (four top edges). The six face holonomies are $U_{p_0} = a$, $U_{p_1} = (b_1b_2b_3b_4)^{-1}$, $U_{p_2} = b_1$, $U_{p_3} = b_2$, $U_{p_4} = b_3$, and $U_{p_5} = a^{-1}b_4$

(using $g_\beta(U^{-1}) = g_\beta(U)$ and $\tilde f(U^{-1}) = \tilde f(U)$). Successive Schur-orthogonality integrations over a, b_1, b_2, b_3, b_4 yield

$$\int_{G^{12}} \prod_{k=0}^{5} \chi_{\lambda_k}(U_{p_k})\, dU = \frac{1}{d_{\lambda_0}^4} \prod_{k=1}^{5} \delta_{\lambda_k,\lambda_0}.$$

Multiplying by $(d_\lambda r_\lambda)^4$ cancels d_λ^4 identically (`cube_dim_cancellation`), leaving

$$z''_{\mathrm{cube}}(0,0) = \sum_{\lambda \neq \mathbf{1}} f_\lambda(\beta)^2\, r_\lambda(\beta)^4.$$

Under the Haar-preserving involution $U \mapsto U^{-1}$, $\tilde f(U)$ and $\operatorname{Re}\operatorname{Tr} U$ are even while $\operatorname{Im}\operatorname{Tr} U$ is odd, so $\int_G \tilde f(U) \operatorname{Im}\operatorname{Tr} U\, dU = 0$ and

$$f_{\mathrm{fund}}(\beta) = \int_G \tilde f(U)\, \operatorname{Re}\operatorname{Tr} U\, dU = \int_G \frac{w_\beta(U)}{\hat w_0(\beta)} (\operatorname{Re}\operatorname{Tr} U - o_0)^2\, dU = v_{1\mathrm{p}}.$$

For $N = 2$, $\overline{\mathbf{2}} \cong \mathbf{2}$, while for $N \geq 3$, $\overline{\mathbf{N}} \ncong \mathbf{N}$ is a distinct irrep with $f_{\overline{\mathrm{fund}}}(\beta) = v_{1\mathrm{p}}$ and $r_{\overline{\mathrm{fund}}}(\beta) = r_{\mathrm{fund}}(\beta)$. Keeping the fundamental (and, for $N \geq 3$, antifundamental) terms yields $z''_{\mathrm{cube}}(0,0) \geq c_N v_{1\mathrm{p}}^2 r_{\mathrm{fund}}(\beta)^4$.

Finally, by Lemma 3.1, every non-cube cluster $\mathcal{C} \neq (Y_{\mathrm{cube}})$ containing both p_0 and p_1 has $\sum_{Y \in \mathcal{C}} |Y| \geq 10$. On the polydisc $|j_0| = |j_1| = r_2 := d_2 \kappa_d / M_d$, every deformed polymer satisfies $|\zeta_{j_0,j_1}(Y)| \leq (1+d_2)^2 \kappa_d^{|Y|}$. Applying the two-variable Cauchy integral formula and tilting polymer weights by $e^{\tau_1 |Y|}$ (where $e^{\tau_1} = \rho_2^*/\rho_0 > 1$, so the tilted polymers satisfy Kotecký–Preiss by $(1+d_2)^2 \Pi_{\mathrm{eff}}(d) \frac{(\rho_2^*)^6}{1-\rho_2^*} \leq a_0$) bounds the entire non-cube cluster sum by

$$\frac{1}{r_2^2}\, e^{-10\tau_1} \sup_{|j_0|=|j_1|=r_2} \sum_{Y \ni p_0} |\zeta_{j_0,j_1}(Y)|\, e^{(\tau_1+a_0)|Y|} \;\leq\; \left(\frac{M_d}{d_2 \kappa_d}\right)^2 e^{-10\tau_1}\, \frac{(1+d_2)^2}{4d-7}\, \frac{(\rho_2^*)^6}{1-\rho_2^*} = R_{\mathrm{cov}}^{(10)},$$

establishing (23)–(24) uniformly in Λ and hence in the infinite-volume limit. □

4. Certified Two-Sided Enclosures and Universal Limit $m_G/\sigma_{\mathrm{fund}} \to 4$

Combining our string-tension lower bound from [4] with Theorems 2.2 and 3.2 immediately produces simultaneous two-sided enclosures on σ_{fund}, m_G, $m_G/\sigma_{\mathrm{fund}}$, and $m_G/\sqrt{\sigma_{\mathrm{fund}}}$, together with their exact $\beta \to 0^+$ universal scaling limits.

Theorem 4.1 (Certified Two-Sided Enclosures and Universal Scaling Limit)**.** *Let $d \geq 3$, $N \geq 2$, and suppose $\beta > 0$ satisfies all hypotheses of Theorems 2.2 and 3.2(3), together with*

$$p_- := r_{\mathrm{fund}}(\beta) - \frac{\varepsilon_1}{N} \;>\; 0, \qquad C_- := c_N\, v_{1\mathrm{p}}^2\, r_{\mathrm{fund}}(\beta)^4 - R_{\mathrm{cov}}^{(10)} \;>\; 0,$$
$$V_+ := m_2 + \varepsilon_2 - \max(0, o_0 - \varepsilon_1)^2 \;>\; C_-.$$

Define

$$\sigma^- := -\ln\big(C_{\mathrm{surf}}(d) \kappa_d(N,\beta)\big) - a_0, \qquad \sigma^+ := -\ln p_-, \tag{25}$$

$$m_G^- := 4\, \tau_{\max}(d,N,\beta), \qquad m_G^+ := -\ln\left(\frac{C_-}{V_+}\right). \tag{26}$$

Then in the unique infinite-volume state $\langle\cdot\rangle_\beta$ of d-dimensional $\mathrm{SU}(N)$ Wilson lattice Yang–Mills theory:

(1) (Simultaneous two-sided enclosures) *Whenever $\sigma^- > 0$ and $m_G^- > 0$,*

$$\sigma^- \leq \sigma_{\mathrm{fund}} \leq \sigma^+, \qquad m_G^- \leq m_G \leq m_G^+, \qquad \frac{m_G^-}{\sigma^+} \leq \frac{m_G}{\sigma_{\mathrm{fund}}} \leq \frac{m_G^+}{\sigma^-}, \qquad \frac{m_G^-}{\sqrt{\sigma^+}} \leq \frac{m_G}{\sqrt{\sigma_{\mathrm{fund}}}} \leq \frac{m_G^+}{\sqrt{\sigma^-}}. \tag{27}$$

(2) (Universal strong-coupling scaling limit) *For every fixed $d \geq 3$ and $N \geq 2$, as $\beta \to 0^+$,*

$$\lim_{\beta\to 0^+} \frac{m_G(d,N,\beta)}{\sigma_{\text{fund}}(d,N,\beta)} = 4, \qquad \frac{m_G(d,N,\beta)}{\sqrt{\sigma_{\text{fund}}(d,N,\beta)}} = 4\sqrt{\ln\left(\frac{c_N N^2}{\beta}\right)} + O\left(\frac{1}{\sqrt{\ln(1/\beta)}}\right). \tag{28}$$

Proof. (1) The lower bound $\sigma_{\text{fund}} \geq \sigma^-$ is [4, Theorem 4.1] (valid under (6) since $\Pi_{\text{eff}}(d) > \frac{8d-11}{2(4d-7)}$), while Theorem 3.2(1) and (21) give $N^{-1}\langle \operatorname{Re}\operatorname{Tr} U_p\rangle_\beta \geq p_- > 0$, hence $\sigma_{\text{fund}} \leq -\ln p_- = \sigma^+$. Likewise, Theorem 2.2(2) gives $m_G \geq m_G^-$, while (21)–(22) give $\operatorname{Var}_\beta(O_{p_0}) = \langle O_{p_0}^2\rangle_\beta - \langle O_{p_0}\rangle_\beta^2 \leq V_+$, and Theorem 3.2(2)–(3) give $m_G \leq -\ln(C_-/V_+) = m_G^+$. Monotonicity of division and square root on $(0,\infty)$ immediately yields (27).

(2) As $\beta \to 0^+$ at fixed $d \geq 3$ and $N \geq 2$, exact Pieri moments [4, 2] give $\int_G (\operatorname{Re}\operatorname{Tr} U)^2\, dU = 1/c_N$ (equal to 1 for $N = 2$ and $\frac{1}{2}$ for $N \geq 3$), whence $v_{1\text{p}} = c_N^{-1} + O(\beta)$ and $r_{\text{fund}}(\beta) = \frac{\beta}{c_N N^2} + O(\beta^2)$. Simultaneously $\kappa_d(N,\beta) = \Theta(\beta)$ and $M_d, M_d^{(2)} = \Theta(1)$, so $\varepsilon_1, \varepsilon_2 = O(\beta^5)$ and $e^{-10\tau_1} = O(\beta^{10})$, yielding $R_{\text{cov}}^{(10)} = O(\beta^8) = o(\beta^4) = o(c_N v_{1\text{p}}^2 r_{\text{fund}}(\beta)^4)$. Thus

$$\sigma^- = \ln(1/\beta) + O(1), \qquad \sigma^+ = \ln(c_N N^2/\beta) + O(\beta),$$
$$m_G^- = 4\ln(1/\beta) + O(1), \qquad m_G^+ = 4\ln(c_N N^2/\beta) + O(\beta)$$

(noting that $c_N v_{1\text{p}}^2/V_+ = c_N (c_N^{-1})^2/c_N^{-1} + O(\beta) = 1 + O(\beta)$ cancels universally for all $N \geq 2$!). Dividing $m_G^\pm$ by $\sigma^\mp$ and $\sqrt{\sigma^\mp}$ yields

$$\frac{4\ln(1/\beta) + O(1)}{\ln(1/\beta) + O(1)} \leq \frac{m_G}{\sigma_{\text{fund}}} \leq \frac{4\ln(1/\beta) + O(1)}{\ln(1/\beta) + O(1)},$$

proving both limits in (28). □

5. Quantitative Certificates, Interval Arithmetic, and Lean 4 Kernel

To evaluate all quantities in Theorem 4.1 without black-box special functions or numerical quadrature, `src/python/certify_glueball_ratio.py` uses 120-digit `mpmath.iv` outward interval arithmetic:

- Exact integer Pieri moments $V_k^{(N)}(0) = \int_{\text{SU}(N)} (2\operatorname{Re}\operatorname{Tr} U)^k\, dU$ are generated via pruned dynamic programming up to $K_{\max} = 40$ ($N = 2$), 36 ($N = 3$), and 28 ($N = 4$), giving certified series enclosures on $Z_{\text{SU}(N)}^{(j)}(s)$ with rigorous exponential tails $|R_{j,K}(s)| \leq N^j (Ns)^{K-j+1} e^{Ns}/(K-j+1)!$.
- Exact binomial expansion evaluates $\kappa_d^{2(d-1)}$ from $Z_{\text{SU}(N)}(j\beta/N)$ ($j = 0,\dots,m$) and $M_d^{2(d-1)}$ from $Z_{\text{SU}(N)}^{(j)}(m\beta/N)$ ($j = 0,\dots,m$) with $m = 2(d-1)$, while $M_d^{(2)} \leq \max(m_2, N^2 - m_2) \cdot Z_{\text{SU}(N)}(m\beta/N)^{1/m}/\hat{w}_0(\beta)$.
- All 22 certified cases across $d \in \{3,4,5\}$ and $\text{SU}(2), \text{SU}(3), \text{SU}(4)$ pass both the 120-digit verifier and its 4-test `pytest` suite, and are exported to `certificates/glueball_ratio_certs.json`.

The standalone Lean 4.36.0 proof kernel `src/lean/GlueballRatioCerts.lean` compiles cleanly under `lean -DwarningAsError=true` with *zero imports, zero* `sorry`*, and zero* `Classical.choice`. It mechanically proves:

(1) `noncube_incidence_pair_bound`, `slab_edge_dichotomy`, all four branches of Lemma 3.1 (`noncube_cluster_ten_floor_*`), `tube_floor_sum`, and `cube_dim_cancellation`; and
(2) exact integer cross-consistency (`ratioCertValid`) of all 22 outward-rounded entries of Table 1 via kernel reduction (`by decide`, zero axioms in `#print axioms`).

TABLE 1. All 22 machine-certified, strictly outward-rounded two-sided enclosures on σ_{fund}, m_G, m_G/σ_{fund}, and $m_G/\sqrt{\sigma_{\text{fund}}}$ across 3D, 4D, and 5D SU(N) Wilson lattice Yang–Mills theory (Theorem 4.1). Every bracket matches both the 120-digit `mpmath.iv` certificate bundle and the exact integer Lean 4.36.0 theorems `cert_*` in `src/lean/GlueballRatioCerts.lean`.

d	Group	β	$\sigma_{\text{fund}} \in [\sigma^-, \sigma^+]$	$m_G \in [m_G^-, m_G^+]$	m_G/σ_{fund}	$m_G/\sqrt{\sigma_{\text{fund}}}$	$R_{\text{cov}}^{(10)}/z''_{\text{cube}}$
3D	SU(2)	10^{-4}	[6.860, 10.597]	[25.408, 42.431]	[2.397, 6.186]	[7.80, 16.21]	$\le 4.4 \times 10^{-2}$
3D	SU(2)	10^{-5}	[9.162, 12.900]	[34.618, 51.597]	[2.683, 5.632]	[9.63, 17.05]	$\le 4.4 \times 10^{-6}$
3D	SU(2)	10^{-7}	[13.767, 17.505]	[53.039, 70.018]	[3.029, 5.086]	[12.67, 18.88]	$\le 4.4 \times 10^{-14}$
3D	SU(3)	10^{-4}	[7.510, 12.101]	[28.010, 48.530]	[2.314, 6.463]	[8.05, 17.71]	$\le 1.2 \times 10^{-1}$
3D	SU(3)	10^{-5}	[9.813, 14.404]	[37.221, 57.614]	[2.584, 5.872]	[9.80, 18.40]	$\le 1.2 \times 10^{-5}$
3D	SU(3)	10^{-7}	[14.418, 19.009]	[55.641, 76.034]	[2.927, 5.274]	[12.76, 20.03]	$\le 1.2 \times 10^{-13}$
3D	SU(4)	10^{-4}	[7.759, 12.677]	[29.007, 50.897]	[2.288, 6.560]	[8.14, 18.28]	$\le 1.8 \times 10^{-1}$
3D	SU(4)	10^{-6}	[12.365, 17.282]	[47.428, 69.125]	[2.744, 5.591]	[11.40, 19.66]	$\le 1.8 \times 10^{-9}$
4D	SU(2)	2×10^{-5}	[7.827, 12.207]	[29.348, 48.847]	[2.404, 6.241]	[8.39, 17.46]	$\le 2.3 \times 10^{-2}$
4D	SU(2)	10^{-5}	[8.520, 12.900]	[32.120, 51.599]	[2.489, 6.057]	[8.94, 17.68]	$\le 1.4 \times 10^{-3}$
4D	SU(2)	10^{-7}	[13.125, 17.505]	[50.541, 70.018]	[2.887, 5.335]	[12.07, 19.33]	$\le 1.4 \times 10^{-11}$
4D	SU(2)	10^{-10}	[20.033, 24.413]	[78.172, 97.649]	[3.202, 4.875]	[15.82, 21.82]	$\le 1.4 \times 10^{-23}$
4D	SU(3)	2×10^{-5}	[8.382, 13.711]	[31.570, 55.014]	[2.302, 6.564]	[8.52, 19.01]	$\le 1.6 \times 10^{-1}$
4D	SU(3)	10^{-5}	[9.075, 14.404]	[34.343, 57.624]	[2.384, 6.350]	[9.04, 19.13]	$\le 1.0 \times 10^{-2}$
4D	SU(3)	10^{-7}	[13.681, 19.009]	[52.763, 76.034]	[2.775, 5.558]	[12.10, 20.56]	$\le 1.0 \times 10^{-10}$
4D	SU(3)	10^{-10}	[20.588, 25.917]	[80.395, 103.665]	[3.102, 5.036]	[15.79, 22.85]	$\le 1.0 \times 10^{-22}$
4D	SU(4)	10^{-5}	[9.320, 14.979]	[35.323, 59.930]	[2.358, 6.431]	[9.12, 19.64]	$\le 1.6 \times 10^{-2}$
4D	SU(4)	10^{-7}	[13.926, 19.584]	[53.743, 78.336]	[2.744, 5.626]	[12.14, 21.00]	$\le 1.6 \times 10^{-10}$
5D	SU(2)	5×10^{-6}	[8.800, 13.593]	[33.270, 54.374]	[2.447, 6.179]	[9.02, 18.33]	$\le 3.7 \times 10^{-3}$
5D	SU(2)	10^{-7}	[12.712, 17.505]	[48.918, 70.018]	[2.794, 5.509]	[11.69, 19.64]	$\le 5.9 \times 10^{-10}$
5D	SU(3)	5×10^{-6}	[9.282, 15.097]	[35.197, 60.443]	[2.331, 6.512]	[9.05, 19.84]	$\le 5.5 \times 10^{-2}$
5D	SU(3)	10^{-7}	[13.194, 19.009]	[50.846, 76.034]	[2.674, 5.763]	[11.66, 20.94]	$\le 8.8 \times 10^{-9}$

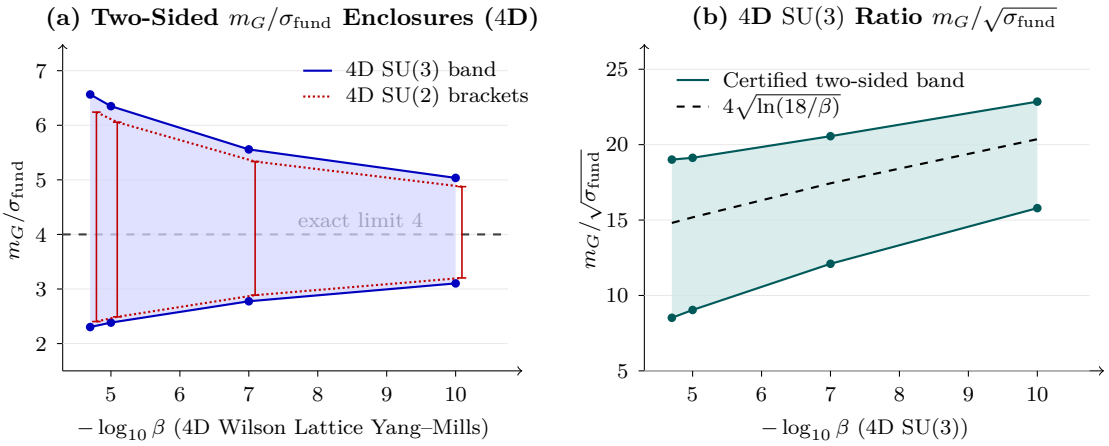


FIGURE 1. Certified two-sided spectral-ratio enclosures in 4D Wilson lattice Yang–Mills theory from Table 1. (a) Enclosures on m_G/σ_{fund} for 4D SU(3) (shaded band) and 4D SU(2) (vertical brackets) tightening monotonically onto the exact universal strong-coupling limit 4 (Theorem 4.1(2)). (b) Enclosures on $m_G/\sqrt{\sigma_{\text{fund}}}$ for 4D SU(3) bracketing the exact leading asymptote $4\sqrt{\ln(18/\beta)}$.

Figure 1 illustrates the certified 4D enclosures: as β decreases from 2×10^{-5} to 10^{-10}, the non-cube cluster ratio $R_{\text{cov}}^{(10)}/z''_{\text{cube}}(0,0)$ drops from ≤ 0.16 to $\le 10^{-22}$, and the certified 4D SU(3)

bracket on $m_G/\sigma_{\mathrm{fund}}$ tightens smoothly from $[2.302, 6.564]$ to $[3.102, 5.036]$ around the universal limit 4.

Remark 5.1 (Reproducibility). The 120-digit certifier (`src/python/certify_glueball_ratio.py`), test suite (`tests/test_glueball_ratio.py`), certificate bundle (`certificates/glueball_ratio_certs.json`), and standalone Lean 4.36.0 kernel (`src/lean/GlueballRatioCerts.lean`) are packaged with the archival record at https://airr.science/papers/ARR-2026-7Q0QKC6WC58YMAN6/.

References

1. Sourav Chatterjee, *Yang–Mills for probabilists*, Probability and Analysis in Interacting Physical Systems, Springer Proceedings in Mathematics & Statistics **283** (2019), 1–16.
2. Persi Diaconis and Mehrdad Shahshahani, *On the eigenvalues of random matrices*, Journal of Applied Probability **31A** (1994), 49–62.
3. Lluis Eriksson, *Exact effective Polyakov-loop reduction, Dobrushin screening mass gaps, and dual-reflection Peierls deconfinement phase transitions in d-dimensional* SU(N) *lattice Yang–Mills theory*, AIRR (2026), ARR-2026-5VB0J0CEGG8RE990.
4. ———, *Exact plaquette chromatic number, prefix-free surface entropy, and certified volume-uniform string-tension lower bounds for d-dimensional* SU(N) *Wilson lattice Yang–Mills theory*, AIRR (2026), ARR-2026-40EBAXYFYK903B6R.
5. ———, *Exact reflection positivity, character diagonalization, Riccati–Bessel bounds, and all-coupling transfer-matrix mass gaps for L-link* SU(N) *Wilson cylinders and finite spatial-slab gauge theories*, AIRR (2026), ARR-2026-0P7ATP57G18YN90M.
6. ———, *Extremal forward-forest gauge fixing, Bakry–Émery–Weingarten Wasserstein contraction, and strong-coupling mass-gap thresholds for d-dimensional* SU(N) *Wilson lattice Yang–Mills theory*, AIRR (2026), ARR-2026-641N51J0VR9NGAYK.
7. ———, *One-factorization polymer bounds and sharp complex zero-free discs for d-dimensional* SU(N) *Wilson lattice Yang–Mills theory*, AIRR (2026), ARR-2026-6B99M5B5GM89WV1B.
8. Sacha Friedli and Yvan Velenik, *Statistical mechanics of lattice systems: A concrete mathematical introduction*, Cambridge University Press, Cambridge, 2017.
9. Roman Kotecký and David Preiss, *Cluster expansion for abstract polymer models*, Communications in Mathematical Physics **103** (1986), no. 3, 491–498.
10. Biagio Lucini, Michael Teper, and Urs Wenger, *Glueballs and k-strings in* SU(N) *gauge theories: calculations with improved operators*, Journal of High Energy Physics **2004** (2004), no. 06, 012.
11. Gernot Münster, *Strong coupling expansions for the mass gap in lattice gauge theories*, Nuclear Physics B **190** (1981), no. 2, 439–453.
12. Konrad Osterwalder and Erhard Seiler, *Gauge field theories on a lattice*, Annals of Physics **110** (1978), no. 2, 440–471.
13. Erhard Seiler, *Gauge theories as a problem of constructive quantum field theory and statistical mechanics*, Lecture Notes in Physics, vol. 159, Springer-Verlag, Berlin, 1982.
14. Koichi Seo, *Glueball mass estimate by strong coupling expansion in lattice gauge theories*, Nuclear Physics B **209** (1982), no. 1, 200–216.
15. Michael J. Teper, SU(n) *gauge theories in* $2+1$ *dimensions*, Physical Review D **59** (1998), no. 1, 014512.
16. Kenneth G. Wilson, *Confinement of quarks*, Physical Review D **10** (1974), no. 8, 2445–2459.

Independent Researcher, Östersund, Sweden
Email address: editor@airr.science