From the Gibbs Weight to the Spectral Gap: A Machine-Checked Transfer-Matrix Identification and Spectral-Gap Chain for the Open $\mathbb{Z}_2$ Ising Chain

Verbatim Lean 4 signatures, non-circular measure/operator separation, and exact scope boundaries

Lluis Eriksson

23 November 2026 (v2)

Abstract

We report a complete formal verification in Lean 4 with `mathlib` of the transfer-matrix identification and spectral-gap chain for the one-dimensional open-boundary $\mathbb{Z}_2$ Ising spin chain (one $\mathbb{Z}_2$ variable per time slice, 2×2 transfer operator): from the Boltzmann path weight $W_\beta(\sigma) = \prod_{t=0}^{n-1} e^{\beta s(\sigma_t,\sigma_{t+1})}$ and an independently defined normalized transfer operator T_β on $\mathbb{C}^2$, through the exact partition function $Z_n = 2(e^\beta + e^{-\beta})^n$ and the Hermitian identification theorem $\mathbb{E}_n[\overline{A(\sigma_0)}\, B(\sigma_n)] = \langle A\Omega,\, T_\beta^n(B\Omega)\rangle_{\mathbb{C}}$ (using Mathlib's inner product conjugate-linear in the first slot), to the exact two-point correlation $\mathbb{E}_n[f(\sigma_0)f(\sigma_n)] = (\tanh\beta)^n$ of the sign observable f. For $\beta \geq 0$, the non-vacuum spectral radius is $\tanh\beta$, the transfer-operator spectral gap is $1-\tanh\beta > 0$, and for $\beta > 0$ the Hamiltonian mass is $-\log(\tanh\beta) > 0$. For $\beta < 0$, the signed eigenvalue remains $\tanh\beta$ while the contraction rate is $|\tanh\beta|$ and the true spectral gap is $1 - |\tanh\beta|$; at $\beta = 0$, $T_0 = |\Omega\rangle\langle\Omega|$ is rank-one and no finite Hamiltonian satisfies $T_0 = e^{-H}$.

This v2 revision answers every material objection raised against v1 (`ARR-2026-7PRDH0DHH280HRKH` v1). We print the verbatim Lean 4 definitions and theorem signatures so the complex conjugation in Theorem 4.1 and every hypothesis can be audited directly from the PDF. We narrow the title and claims explicitly: the formalized model is the 1D open Ising spin chain (not a pure-link 1D gauge theory, which has no plaquettes and is gauge-trivial), and the companion reflection-positivity result (`ARR-2026-10JJK0VCCS87VTQH`) is *not* a logical dependency of `z2_identification` or `z2_two_point_tanh`. Moreover, `z2_pairing_nondegenerate` proves definiteness only on time-zero single-slice observables; on the full positive-time algebra, nonzero observables such as $F(\sigma_0,\sigma_1) = f(\sigma_1) - (\tanh\beta)f(\sigma_0)$ have zero reflected norm, so the full Osterwalder–Seiler GNS quotient is genuinely nontrivial and is treated separately (`ARR-2026-2TBRDN2DE598KSXD`). Nothing here is a claim about SU(N), spatial-volume uniformity, the continuum limit, or the Yang–Mills mass gap.

1 Introduction and scope correction (v2)

Constructive lattice field theory specifies a Euclidean probability measure, whereas a mass gap is a spectral property of a self-adjoint transfer operator. Connecting the two requires an *identification theorem* expressing Euclidean expectations in the measure as matrix elements of powers of the transfer operator, proved without circularity (i.e., without defining the measure from the operator).

1.1 Exact model and what is proved

For the open-boundary 1D $\mathbb{Z}_2$ Ising spin chain on $n+1$ time slices $\sigma : \{0, \dots, n\} \to \{0, 1\}$, we define solely from exp and the bond sign $s(i, j) = +1$ $(i = j)$, -1 $(i \neq j)$:

$$W_\beta(\sigma) = \prod_{t=0}^{n-1} e^{\beta s(\sigma_t, \sigma_{t+1})}, \qquad Z_n(\beta) = \sum_\sigma W_\beta(\sigma),$$
$$\mathbb{E}_n[\overline{A(\sigma_0)}\, B(\sigma_n)] = \frac{1}{Z_n(\beta)} \sum_\sigma \overline{A(\sigma_0)}\, B(\sigma_n)\, W_\beta(\sigma).$$

Independently, on $\mathbb{C}^2$ with Mathlib's Euclidean inner product $\langle x, y\rangle_{\mathbb{C}} = \sum_i \overline{x_i} y_i$ (conjugate-linear in the first slot, linear in the second), we define

$$T_\beta = a(\beta)\, \mathbf{1} + b(\beta)\, S, \qquad a(\beta) = \frac{e^\beta}{e^\beta + e^{-\beta}}, \quad b(\beta) = \frac{e^{-\beta}}{e^\beta + e^{-\beta}},$$

where S swaps the two coordinates. The verified chain establishes:

1. $Z_n(\beta) = 2(e^\beta + e^{-\beta})^n$ (computed by induction, not cancelled symbolically);
2. $T_\beta \Omega = \Omega$ for $\Omega = 2^{-1/2}(1, 1)$, as a consequence of $a + b = 1$;
3. $\|T_\beta - |\Omega\rangle\langle\Omega|\| \leq a(\beta) - b(\beta) = \tanh\beta < 1$ for $\beta \geq 0$;
4. the exact Hermitian identification $\mathbb{E}_n[\overline{A(\sigma_0)}\, B(\sigma_n)] = \langle A\Omega, T_\beta^n(B\Omega)\rangle_{\mathbb{C}}$ for all $\beta \in \mathbb{R}$, $n \in \mathbb{N}$, and $A, B : \{0, 1\} \to \mathbb{C}$;
5. the exact sign-observable correlation $\mathbb{E}_n[f(\sigma_0) f(\sigma_n)] = (\tanh\beta)^n$ and vanishing one-point function $\mathbb{E}_n[f(\sigma_0)] = 0$;
6. time-zero pairing non-degeneracy: $\mathbb{E}_0[\overline{A(\sigma_0)} A(\sigma_0)] = 0 \implies A = 0$;
7. an additional corollary (`z2_clustering_of_transfer`) feeding T_β into the abstract operator-gap-to-clustering criterion `connCorrC_le_of_gap`.

1.2 Explicit boundaries and clarifications over v1

1. **Spin chain, not 1D pure-link gauge theory.** An open one-dimensional pure-link gauge theory has no plaquettes and every link configuration is gauge-equivalent. The formalized law is the 1D open $\mathbb{Z}_2$ Ising spin chain (equivalently, a 1+1D ladder gauge strip in temporal gauge, though no gauge-fixing reduction is formalized in this module). The v1 title "$\mathbb{Z}_2$ Lattice Gauge Chain" has been corrected accordingly.

2. **Transfer-matrix identification, not full positive-time OS reconstruction.** The companion reflection-positivity theorem `z2Wilson_reflectionPositive` (`ARR-2026-10JJK0VCCS87VTQH`) is *not* imported or logically consumed by `Z2Identification.lean`. What is machine-checked here is a non-circular time-zero transfer-matrix identification and spectral-gap chain.

3. **Temporal vs. spatial uniformity.** The exact identity $(\tanh\beta)^n$ holds uniformly across all temporal separations $n \in \mathbb{N}$ for the family of open chains, whereas spatial-volume uniformity does not arise because each spatial slice is a single site ($\dim \mathbb{C}^2 = 2$).

4. **Mathematics is textbook.** The 1D Ising transfer matrix is classical [2]; the contribution is a zero-`sorry`, zero-project-axiom Lean 4 verification demonstrating clean non-circular interface separation. Nothing here bears on SU(N), continuum limits, or the Clay Yang–Mills problem [3].

2 Verbatim Lean 4 definitions and signatures

To make fidelity between prose and formalization directly auditable from this PDF, we quote verbatim the core declarations in `YangMills/OS/Z2Identification.lean` and `YangMills/OS/Z2Transfer.lean` at checkpoint `bd2c18394ce8a55ec73413c11a9f311bbf002248`:

```
-- YangMills/OS/Z2Identification.lean (measure side; no transfer operator imported)
def z2Sign (i j : Fin 2) : Real := if i = j then 1 else -1
noncomputable def z2Bond (beta : Real) (i j : Fin 2) : Real :=
  Real.exp (beta * z2Sign i j)
noncomputable def z2PathWeight (beta : Real) {n : Nat}
    (sigma : Fin (n + 1) -> Fin 2) : Real :=
  Finset.prod Finset.univ fun t : Fin n =>
    z2Bond beta (sigma t.castSucc) (sigma t.succ)
noncomputable def z2Partition (beta : Real) (n : Nat) : Real :=
  Finset.sum Finset.univ fun sigma : Fin (n + 1) -> Fin 2 =>
    z2PathWeight beta sigma
noncomputable def z2PathSum (beta : Real) (n : Nat) (A B : Fin 2 -> Complex) : Complex :=
  Finset.sum Finset.univ fun sigma : Fin (n + 1) -> Fin 2 =>
    (starRingEnd Complex) (A (sigma 0)) * B (sigma (Fin.last n))
      * ((z2PathWeight beta sigma : Real) : Complex)
noncomputable def z2Expect (beta : Real) (n : Nat) (A B : Fin 2 -> Complex) : Complex :=
  z2PathSum beta n A B / ((z2Partition beta n : Real) : Complex)

-- Dictionary & main theorems
noncomputable def z2Obs (A : Fin 2 -> Complex) : EuclideanSpace Complex (Fin 2) :=
  ((invSqrtTwo : Real) : Complex) • WithLp.toLp 2 A

theorem z2_identification (beta : Real) (n : Nat) (A B : Fin 2 -> Complex) :
    z2Expect beta n A B = inner Complex (z2Obs A) (((z2TransferOp beta) ^ n) (z2Obs B))

theorem z2_two_point_tanh (beta : Real) (n : Nat) :
    z2Expect beta n z2SignObs z2SignObs = ((Real.tanh beta : Real) : Complex) ^ n

theorem z2_one_point_sign_eq_zero (beta : Real) (n : Nat) :
    z2Expect beta n z2SignObs (fun _ => (1 : Complex)) = 0

theorem z2_pairing_nondegenerate (beta : Real) {A : Fin 2 -> Complex}
    (h : z2Expect beta 0 A A = 0) : A = fun _ => 0

-- YangMills/OS/Z2Transfer.lean (spectral bounds & abstract clustering corollary)
theorem z2TransferOp_gap {beta : Real} (hbeta : 0 <= beta) :
    ||z2TransferOp beta - vacuumProj z2Vac|| <= z2A beta - z2B beta
theorem z2_spectral_gap_pos (beta : Real) : 0 < 1 - Real.tanh beta
theorem z2_mass_pos {beta : Real} (hbeta : 0 < beta) : 0 < -Real.log (Real.tanh beta)
theorem z2_clustering_of_transfer {beta : Real} (hbeta : 0 <= beta)
    (v : EuclideanSpace Complex (Fin 2)) :
```

```
Exists fun C : Real => forall n : Nat,
  ||connCorrC (z2TransferOp beta) z2Vac v n|| <= C * (z2A beta - z2B beta) ^ n
```

Remark 2.1 (Non-circularity check). The measure-side dependency closure is `z2Expect` $\to$ {`z2PathSum`, `z2Partition`} $\to$ `z2PathWeight` $\to$ `z2Bond` $\to$ {exp, `z2Sign`}, containing no reference to `z2TransferOp`, $a(\beta)$, or $b(\beta)$. The two sides meet at a single bridge lemma `z2Bond_eq` expressing $k_\beta(i,j) = (e^\beta + e^{-\beta})(a\delta_{ij} + b(1-\delta_{ij}))$. (Sharing a primitive bond kernel would not automatically be circular, but keeping the definition closures strictly disjoint makes non-circularity checkable by printing six definitions.)

3 Spectral quantities across $\beta \in \mathbb{R}$

Theorem 3.1 (Vacuum invariance and non-vacuum norm bound)**.** *For every $\beta \in \mathbb{R}$, $T_\beta \Omega = \Omega$ where $\Omega = 2^{-1/2}(1,1)$ (`z2TransferOp_fix`), and $(T_\beta - |\Omega\rangle\langle\Omega|) = (\tanh\beta)(\mathbf{1} - |\Omega\rangle\langle\Omega|)$. In particular:*

1. *For $\beta \geq 0$, $\|T_\beta - |\Omega\rangle\langle\Omega|\| \leq \tanh\beta < 1$ (`z2TransferOp_gap`), and the transfer-operator spectral gap is $1 - \tanh\beta > 0$ (`z2_spectral_gap_pos`).*
2. *For $\beta > 0$, $0 < \tanh\beta < 1$, so $T_\beta = e^{-H}$ with strictly positive Hamiltonian mass $m(\beta) = -\log(\tanh\beta) > 0$ (`z2_mass_pos`).*
3. *For $\beta < 0$, the non-vacuum eigenvalue is $\tanh\beta < 0$: the non-vacuum spectral radius (contraction rate) is $|\tanh\beta| = -\tanh\beta < 1$, and the true spectral gap of T_β is $1 - |\tanh\beta| \in (0,1)$ (whereas $1 - \tanh\beta > 1$, proved positive for all β in `z2_spectral_gap_pos`, is* not *the spectral gap when $\beta < 0$).*
4. *At $\beta = 0$, $\tanh 0 = 0$, $T_0 = |\Omega\rangle\langle\Omega|$ is rank-one, correlations vanish identically for all $n \geq 1$, and no finite Hermitian H on $\mathbb{C}^2$ satisfies $T_0 = e^{-H}$ (the mass diverges to $+\infty$). Restricting `z2_mass_pos` to $\beta > 0$ avoids Lean's junk value `Real.log 0 = 0`.*

4 The identification theorem and exact correlation

Theorem 4.1 (Hermitian transfer-matrix identification)**.** *For every $\beta \in \mathbb{R}$, every $n \in \mathbb{N}$, and all $A, B : \{0,1\} \to \mathbb{C}$,*

$$\mathbb{E}_n\big[\overline{A(\sigma_0)}\, B(\sigma_n)\big] \;=\; \langle A\Omega,\; T_\beta^n (B\Omega)\rangle_{\mathbb{C}},$$

where $\langle x, y\rangle_{\mathbb{C}} = \sum_{i\in\{0,1\}} \overline{x_i}\, y_i$ is Mathlib's Euclidean inner product, conjugate-linear in the first argument (`z2_identification`).

Proof, as formalized. Split the sum over $\sigma : \mathrm{Fin}(n+1) \to \mathrm{Fin}\,2$ at the first spin $\sigma = (s,\tau)$. Factoring out the first bond $k_\beta(s,\tau_0) = (e^\beta + e^{-\beta})(T_\beta)_{s,\tau_0}$ and using self-adjointness of T_β yields by induction on n (`z2PathSum_succ`):

$$\sum_\sigma \overline{A(\sigma_0)}\, B(\sigma_n)\, W_\beta(\sigma) = (e^\beta + e^{-\beta})^n \,\langle A,\, T_\beta^n B\rangle_{\mathbb{C}}.$$

Setting $A = B \equiv 1$ and using $T_\beta \mathbf{1} = \mathbf{1}$ gives $Z_n(\beta) = 2(e^\beta + e^{-\beta})^n$ (`z2Partition_eq`). Dividing cancels $(e^\beta + e^{-\beta})^n$ and leaves $\frac{1}{2}\langle A, T_\beta^n B\rangle_{\mathbb{C}} = \langle 2^{-1/2}A,\, T_\beta^n (2^{-1/2}B)\rangle_{\mathbb{C}} = \langle A\Omega,\, T_\beta^n (B\Omega)\rangle_{\mathbb{C}}$. (Without the complex conjugate on $A(\sigma_0)$, the identity would already fail at $n = 0$ for $A \equiv i$, $B \equiv 1$, where $\mathbb{E}_0[i \cdot 1] = i \neq -i = \langle i\Omega,\, \Omega\rangle_{\mathbb{C}}$.) $\square$

Corollary 4.2 (Exact sign two-point function)**.** *For* $f(0) = +1$, $f(1) = -1$ *(real-valued, so* $\overline{f} = f$*), every* $\beta \in \mathbb{R}$*, and every* $n \in \mathbb{N}$*,*

$$\mathbb{E}_n[f(\sigma_0)f(\sigma_n)] = (\tanh\beta)^n, \qquad \mathbb{E}_n[f(\sigma_0)] = 0$$

(`z2_two_point_tanh`, `z2_one_point_sign_eq_zero`).

Remark 4.3 (Portability to general finite kernels). For a general finite symmetric kernel K, path induction yields $Z_n = \mathbf{1}^\top K^n \mathbf{1}$. The power formula $Z_n = |S|\,\lambda^n$ and the uniform vacuum $\Omega = |S|^{-1/2}\mathbf{1}$ require *constant row sums* ($K\mathbf{1} = \lambda\mathbf{1}$), not symmetry alone.

5 Time-zero definiteness, the positive-time OS quotient, and the abstract criterion

Remark 5.1 (Time-zero definiteness vs. the full positive-time OS quotient). The $n = 0$ instance of Theorem 4.1 gives $\mathbb{E}_0[\overline{A(\sigma_0)}A(\sigma_0)] = \|A\Omega\|^2$, so `z2_pairing_nondegenerate` proves that the pairing is strictly positive definite on *time-zero single-slice observables* $A(\sigma_0)$. This does *not* mean that the standard Osterwalder–Seiler reflection pairing on the full positive-time algebra $\{\sigma_0, \sigma_1, \dots\}$ is definite: under site reflection across $t = 0$, the nonzero positive-time observable $F(\sigma_0, \sigma_1) := f(\sigma_1) - (\tanh\beta) f(\sigma_0)$ satisfies $\mathbb{E}[F \mid \sigma_0] = 0$ for every finite β, hence has zero reflected norm $\mathbb{E}[\overline{(\theta F)}F] = 0$ by Markov conditional independence across σ_0. Thus the full positive-time GNS quotient is nontrivial even for the 1D Ising chain (which is why a nontrivial degenerate GNS quotient is formalized separately in `ARR-2026-2TBRDN2DE598KSXD`).

Remark 5.2 (Algebraic structure of T_β and role of the abstract clustering criterion). With $P = |\Omega\rangle\langle\Omega|$ and $r = \tanh\beta = a - b$, algebraically $T_\beta = P + r(\mathbf{1} - P) = r\mathbf{1} + (1 - r)P$ (correcting an erroneous algebraic claim in v1's Remark 7.1). The distinction from earlier toy witnesses is *constructive*, not algebraic: T_β is built from the Boltzmann bond weights $e^{\pm\beta}$ rather than taking a target rate r as input. Finally, `z2_two_point_tanh` is proved directly from the eigenfunction $T_\beta f = (\tanh\beta) f$; the separate corollary `z2_clustering_of_transfer` feeds T_β into the abstract dense-family clustering criterion `connCorrC_le_of_gap` (`ARR-2026-31ED4MTKC18JPSDE`) for arbitrary $v \in \mathbb{C}^2$ and $\beta \geq 0$, rather than serving as a dependency of Corollary 4.2.

6 Reproducibility and verification

All Lean files live in `https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME` (archival release `yang-mills-archive-v1.0`, Lean `v4.29.0-rc6`, pinned Mathlib `07642720480157414db592fa85b626dafb71355b`; original module freeze `bd2c18394ce8a55ec73413c11a9f311bbf002248`). Reproduce and inspect the axiom readout of every declaration in Section 2 via:

```
git clone -c core.longpaths=true \
  https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME.git
cd THE-ERIKSSON-PROGRAMME && git checkout yang-mills-archive-v1.0
lake exe cache get && lake build YangMillsCore && lake env lean oracle_check.lean
```

GitHub Actions (`lean-build.yml` and `ci.yml`) verifies continuously that `YangMillsCore` compiles with zero `sorry`, zero project axioms, and `#print axioms` restricted to `[propext, Classical.choice, Quot.sound]`.

Disclosure

Large language models assisted Lean 4 formalization and manuscript editing under author supervision; every theorem in Section 2 is checked by the Lean 4 kernel. Author Lluis Eriksson administers AIRR (https://airr.science/); no independent editorial admission or human peer review is claimed.

References

[1] K. Osterwalder and E. Seiler, *Gauge field theories on a lattice*, Annals of Physics **110** (1978), 440–471.

[2] B. Simon, *The Statistical Mechanics of Lattice Gases, Volume I*, Princeton University Press, 1993.

[3] A. Jaffe and E. Witten, *Quantum Yang–Mills theory*, Clay Mathematics Institute Millennium Prize Problem description, 2000.

[4] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, CADE-28, LNCS 12699 (2021), 625–635.