

The arithmetic of sampling static quantum disorder: exact aliases, finite witnesses, and clock robustness

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Abstract

Whether a dephasing experiment can exclude static Hamiltonian disorder depends on the arithmetic of its measurement times. We prove that at any finite set of rationally independent positive times, every real dephasing semigroup with strictly positive off-diagonal rates is reproduced exactly by one finite ensemble of static diagonal Hamiltonians. At most $1 + kd(d - 1)$ Hamiltonians suffice for k times and d levels; no uniform energy bound is asserted. We also construct an explicit bounded-energy static ensemble reproducing the same semigroup on an entire sufficiently coarse uniform grid. In the opposite direction, every rate matrix outside the cut cone admits a separating test on finitely many odd harmonics of one clock period. The proof combines a circle-distance cut representation with an explicit Fourier-tail bound, and applies to arbitrary static Hermitian ensembles through their coherence-transfer entries. A qutrit example uses 20 times and 40,500 shots to bound each error probability by 5%. Consequently, exact finite-time detectability can disappear under arbitrarily small independent perturbations of the sampling times. A spectral-width assumption restores robustness with an explicit timing error budget. Alternatively, fresh bounded time randomization tolerates independent additive clock error without an energy prior; a certified qutrit design needs 53,000 shots. All results concern fresh single-use channel experiments, not interventions within one noise trajectory. The contribution is an identifiability and experimental-design theory; no quantum environment or solution of an Erdős problem is claimed.

1 Question and relation to existing work

An ensemble of time-independent Hamiltonians can produce irreversible- looking coherence decay. A Cauchy-distributed fixed frequency even gives exact exponential decay. Hamiltonian-ensemble representations and their Fourier compatibility constraints are established tools [4, 5, 6, 7]. Nevertheless, an incompatibility of a *whole time-dependent family* need not be visible in a specified finite collection of its channels. This paper asks which sampling designs preserve or destroy that distinction.

The answer has three parts. Generic finite menus admit exact static aliases, carefully related finite menus can exclude every static model, and coarse infinite grids can admit aliases again. These statements are compatible: they use different arithmetic relations between times. Our constructive separation also exposes a discontinuity relevant to clock calibration. Without an energy bound, independent timing errors cannot be treated as a uniformly small perturbation of the static null.

The mathematical ingredients have classical antecedents. The torus-density argument is the continuous Kronecker theorem; the convex-hull and finite-support arguments are standard finite-dimensional convex analysis [8, 9]. The phase-correlation body, finite convex representations, and interiority of the identity are established in Dykema–Juschenko [1, Proposition 2.5, Remark 2.7, Section 4] and Kribs et al. [2, Theorem 27, Remark 28]. Wrapped Cauchy kernels and circular

copulas are well studied [10]. Cut-cone and L^1 metric representations are classical [11, Chapter 4, pp. 37–52]. Our earlier manuscript [13] gives the continuous-time cut-cone classification and a diagonal spectral surrogate. Those results are restated with proofs here, and are not counted as new independent discoveries. The new results developed here are the exact finite-menu alias theorem, the explicit simultaneous grid alias, the complete finite harmonic certificates, and their timing-robustness consequences, including a bounded-time randomized repair and a lower bound on the energy cost of nearby aliases. We have not located these combined statements in the cited literature; this is a scoped priority assessment, not an exhaustive novelty certificate.

2 Access model and observables

Fix an integer $d \geq 2$ and a distinguished basis of $\mathbb{C}^d$, write $E_{ij} = |i\rangle\langle j|$, and set $\hbar = 1$. A static ensemble has a probability law μ on finite Hermitian $d \times d$ matrices and channels

$$\Psi_t(A) = \mathbb{E}_\mu[e^{-itH} A e^{itH}], \quad t \geq 0. \quad (1)$$

The law is the same at every queried time; it cannot depend on the fresh time, input, or readout choice. A shot prepares a fresh system, applies one such channel, and measures it. Channel queries are memoryless between uses: the hidden Hamiltonian is drawn afresh with a common law, independently of the newly chosen time, pair, preparation phase and readout. The channel model makes no statement about multiple interventions within the same realization of H .

For a real symmetric matrix Γ with zero diagonal, the reference is

$$\Phi_t(E_{ij}) = e^{-t\Gamma_{ij}} E_{ij}. \quad (2)$$

We assume complete positivity: Γ is conditionally negative semidefinite, or equivalently $\Gamma_{ij} = \|v_i - v_j\|^2$ for suitable real vectors v_i [12]. In particular, $\Gamma_{ij} \geq 0$. This is a random diagonal unitary channel at each individual time: if G is a standard real Gaussian vector, the phases $\sqrt{2t}\langle v_i, G \rangle$ realize (2). This representation changes its phase law with time and does not itself give a static Hamiltonian law.

For an arbitrary channel define

$$T_{ij}(t) = \langle i | \Psi_t(E_{ij}) | j \rangle, \quad C_{ij}(t) = \operatorname{Re} T_{ij}(t) \quad (i \neq j). \quad (3)$$

Population preservation is not assumed for the static null.

Lemma 2.1 (Diagonal spectral surrogate). *For every law in (1) there is one joint law of real random variables $(\varepsilon_1, \dots, \varepsilon_d)$ such that*

$$T_{ij}(t) = \mathbb{E} e^{-it(\varepsilon_i - \varepsilon_j)} \quad (i \neq j, t \in \mathbb{R}). \quad (4)$$

If $\operatorname{diam} \operatorname{spec}(H) \leq B$ almost surely, this law can be chosen with $|\varepsilon_i - \varepsilon_j| \leq B$ almost surely.

Proof. Conditionally on H , independently sample each ε_i from the spectral probability measure $\langle i | P_H(\cdot) | i \rangle$. Then $\mathbb{E}[e^{-it\varepsilon_i} | H] = (e^{-itH})_{ii}$, and conditional independence gives $\mathbb{E}[e^{-it(\varepsilon_i - \varepsilon_j)} | H] = (e^{-itH})_{ii} \overline{(e^{-itH})_{jj}}$. The latter is the transfer entry in (3) before averaging over H . All sampled energies lie in the same conditional spectrum. $\square$

The surrogate reproduces these entries, not necessarily the whole channel. It is nevertheless enough for every separating statistic below.

Lemma 2.2 (Bounded measurement of a transfer entry). *Set $P = E_{ii} + E_{jj}$, $X = E_{ij} + E_{ji}$, and $Y = -iE_{ij} + iE_{ji}$. Uniformly choose (A, s) from $\{X, Y\} \times \{-1, 1\}$, prepare $(P + sA)/2$, measure A after the channel, and multiply the outcome by s . The resulting variable O belongs to $[-1, 1]$ and has mean $C_{ij}(t)$.*

Proof. The mean is $\frac{1}{4} \operatorname{Tr}[X \Psi_t(X) + Y \Psi_t(Y)] = \operatorname{Re} T_{ij}(t)$. The identity uses only linearity and Hermiticity preservation. The bounded-outcome assertion additionally uses a physical channel, positive input states, and observables with spectrum in $[-1, 1]$. $\square$

3 Exact aliases on generic finite time menus

Let $\mathcal{C}_d = \text{conv}\{zz^* : z \in \mathbb{C}^d, |z_i| = 1\}$, viewed in the real affine space of Hermitian matrices with diagonal one. These are the correlation matrices of random diagonal unitary channels. Its dimension is $d(d-1)$.

Lemma 3.1 (Convex density and the phase body). *If S is dense in a compact set $K \subset \mathbb{R}^m$, then $\overline{\text{conv } S} = \text{conv } K$ and $\text{ri}(\text{conv } K) \subseteq \text{conv } S$. The identity matrix is an interior point of $\mathcal{C}_d$.*

Proof. The closure assertion follows by approximating each vertex of a finite convex combination. For the relative-interior assertion, take a small simplex surrounding the point within the affine hull, approximate its vertices by points of $\text{conv } S$, and retain containment under a small perturbation. Expanding the vertex combinations gives membership in $\text{conv } S$. For the second assertion, Haar-averaging zz^* gives I . A supporting affine functional at I would give a nonnegative trigonometric polynomial with Haar integral zero, hence the zero polynomial. The distinct characters $z_i \bar{z}_j$, $i \neq j$, are linearly independent, so this functional is constant on the full affine space. Thus there is no nonconstant supporting functional at I . $\square$

Lemma 3.2 (Interior of real dephasing data). *If $\Gamma_{ij} > 0$ for $i \neq j$, every correlation matrix $M(t) = (e^{-t\Gamma_{ij}})_{ij}$, $t > 0$, is interior to $\mathcal{C}_d$. No strict conditional negative definiteness is required.*

Proof. Use the Gaussian phase representation from Section 2. Suppose a nonnegative affine functional on $\mathcal{C}_d$ vanishes at $M(t)$. Its continuous, nonnegative value on the phase vectors $z_i(x) = e^{-i\langle v_i, x \rangle}$ has zero integral against a Gaussian with everywhere-positive density on $\mathbb{R}^r$. It therefore vanishes for every x . Averaging against a Gaussian of any other variance shows that it vanishes at $M(s)$ for every $s > 0$. Since all off-diagonal rates are positive, $M(s) \rightarrow I$ as $s \rightarrow \infty$. This contradicts Lemma 3.1 unless the functional is constant. $\square$

Theorem 3.3 (Finite-menu alias theorem). *Let $t_1, \dots, t_k > 0$ satisfy*

$$\sum_{\ell=1}^k n_\ell t_\ell = 0, \quad n \in \mathbb{Z}^k \implies n = 0. \quad (5)$$

The closure of the static diagonal-ensemble data at these times is $\mathcal{C}_d^k$. Every tuple in the interior of this product is realized exactly by a static diagonal ensemble supported on at most $1 + kd(d-1)$ Hamiltonians. In particular, this holds for the channels (2) whenever all off-diagonal rates are positive.

Proof. The orbit $s \mapsto (e^{-it_1 s}, \dots, e^{-it_k s})$ is dense in $\mathbb{T}^k$. For completeness, the time average of each nonconstant torus character along this orbit tends to zero by (5). Approximation by trigonometric polynomials implies convergence of the time averages of continuous functions to their Haar integrals. A nonempty open set cannot be missed by the orbit. This is the continuous Kronecker argument.

Choose each energy ε_i independently as a coordinate in $\mathbb{R}$, with $\varepsilon_d = 0$ as a gauge. The resulting phase arrays are dense in $(\mathbb{T}^{d-1})^k$. Taking the phase correlation matrix at each time is a continuous map. Its image closure is the product of the rank-one phase-correlation sets, whose convex hull is $\mathcal{C}_d^k$: independent convex combinations in the factors give a joint combination by multiplying their weights. Lemma 3.1 proves exact interior representation by finite convex combinations of actual static energy vectors. Carathéodory's theorem bounds their number by the affine dimension plus one. Lemma 3.2 gives the final assertion. $\square$

Corollary 3.4 (Statistical indistinguishability). *Under the hypotheses of Theorem 3.3, no test restricted to fresh single-use queries at those k times can distinguish the reference from every static ensemble with type-I and type-II errors whose sum is less than one. This remains true with arbitrary input states, ancillas, measurements, and adaptation between queries.*

Proof. The alias and reference supply exactly the same channel at every allowed query. Sequential composition with the experimenter's operations gives identical outcome laws. Rejection probabilities are therefore equal. $\square$

For a boundary tuple only approximation is asserted. It still excludes a uniform positive finite-query testing gap: if each entry differs by at most ϵ , a crude diamond-norm bound for the Schur-channel difference is $d^2\epsilon$, obtained by summing its elementary $A \mapsto E_{ii}AE_{jj}$ maps. An N -query hybrid argument bounds outcome total variation by $Nd^2\epsilon/2$.

Remark 3.5. The support bound counts Hamiltonians, not their spectral width. The construction need not keep energies small as the time menu approaches a rational relation. This distinction is essential in Section 7. Zero off-diagonal rates can be handled by first merging equal vectors v_i ; the stated theorem avoids that bookkeeping.

4 An explicit alias on an infinite coarse grid

The next construction is independent of the density argument and gives a bounded-energy law with infinitely many matching observations. For $0 \leq r < 1$, the Poisson kernel on the circle is

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2} = 1 + 2 \sum_{n \geq 1} r^n \cos(n\theta). \quad (6)$$

Theorem 4.1 (Coarse-grid alias). *Let $\Gamma_{ij} > 0$ for $i \neq j$, and put $r_{ij} = e^{-\Delta \Gamma_{ij}}$ for a spacing $\Delta > 0$. If*

$$\sum_{i < j} \frac{2r_{ij}}{1 + r_{ij}} \leq 1, \quad (7)$$

then the density, relative to Haar probability measure on $\mathbb{T}^d$,

$$q(\theta) = 1 + \sum_{i < j} (P_{r_{ij}}(\theta_i - \theta_j) - 1) \quad (8)$$

defines a static diagonal ensemble reproducing (2) at every $t = n\Delta$, $n = 0, 1, 2, \dots$. Its energies may be chosen in $[-\pi/\Delta, \pi/\Delta)$, hence with spectral width at most $2\pi/\Delta$. Writing $M = \binom{d}{2}$ and $\gamma_{\min} = \min_{i < j} \Gamma_{ij}$, the sufficient condition

$$\Delta \geq \frac{\log(2M - 1)}{\gamma_{\min}} \quad (9)$$

implies (7); for $d = 2$ any positive spacing works.

Proof. Every summand has Haar mean zero. The minimum of P_r is $(1 - r)/(1 + r)$, so (7) makes q nonnegative. Distinct positive multiples of distinct roots $e_i - e_j$, with $i < j$, are distinct Fourier frequencies, even up to sign. Equation (6) and orthogonality therefore give

$$\int e^{-in(\theta_i - \theta_j)} q(\theta) dm(\theta) = r_{ij}^{|n|} \quad (n \neq 0).$$

Choose centered representatives of the phases and set $H = \text{diag}(\theta_1, \dots, \theta_d)/\Delta$. Integer multiples of Δ remove the representative ambiguity. All populations are fixed, so equality of the coherence factors is equality of the whole channels. Finally, $r_{ij} \leq (2M - 1)^{-1}$ implies each summand in (7) is at most $1/M$. $\square$

This sufficient condition is not asserted to be necessary or optimal. The Poisson kernel itself is standard, as are circular models with wrapped Cauchy marginals [10]. The particular sum in (8) is used because its disjoint Fourier supports prescribe *all* integer moments of *all* pair differences simultaneously. It need not be a wrapped multivariate Cauchy distribution.

5 Finite harmonic certificates are complete

For a subset $S \subseteq \{1, \dots, d\}$ let $\delta_S(i, j) = |1_S(i) - 1_S(j)|$. The cut cone CUT_d consists of all nonnegative combinations of these cut semimetrics. It is a closed, finitely generated cone. All pair sums below are over $i < j$.

Let $\rho(\theta) = \min_{m \in \mathbb{Z}} |\theta - 2\pi m| \in [0, \pi]$ be geodesic distance to zero on the circle. Its absolutely convergent Fourier series is

$$\rho(\theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\cos(n\theta)}{n^2}. \quad (10)$$

Lemma 5.1 (Circular cut representation). *For arbitrary phases $\theta_1, \dots, \theta_d$, the matrix $(\rho(\theta_i - \theta_j))_{ij}$ belongs to CUT_d . Hence for every static ensemble and every $\Delta > 0$ the matrix with off-diagonal entries*

$$D_{ij}(\Delta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \geq 1, n \text{ odd}} \frac{C_{ij}(n\Delta)}{n^2} \quad (11)$$

and diagonal zero belongs to CUT_d .

Proof. For the rotating half-circle $I_\alpha = [\alpha, \alpha + \pi)$,

$$\rho(\theta_i - \theta_j) = \frac{1}{2} \int_0^{2\pi} |1_{I_\alpha}(\theta_i) - 1_{I_\alpha}(\theta_j)| d\alpha.$$

Partitioning the integral by its finitely many induced subsets gives a nonnegative cut combination. Apply this identity to $\theta_i = \Delta \varepsilon_i \pmod{2\pi}$ from Lemma 2.1 and take expectations. Absolute convergence justifies (11). $\square$

For the reference semigroup these entries are $F(\Delta \Gamma_{ij})$, where

$$F(s) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \geq 1, n \text{ odd}} \frac{e^{-ns}}{n^2}, \quad (12)$$

$$F'(s) = \frac{2}{\pi} \log \coth(s/2), \quad F(s) = \frac{2}{\pi} s \log(1/s) + O(s) \quad (s \downarrow 0). \quad (13)$$

Here $F(0) = 0$. To verify the derivative, sum $\sum_{n \geq 1, n \text{ odd}} e^{-ns}/n = \frac{1}{2} \log \coth(s/2)$ and integrate from zero. Consequently, for every fixed $\gamma \geq 0$,

$$\lim_{\Delta \downarrow 0} \frac{\pi F(\Delta \gamma)}{2\Delta \log(1/\Delta)} = \gamma. \quad (14)$$

Proposition 5.2 (Continuous-time classification, with a circle proof). *The real semigroup (2) is static exactly when $\Gamma \in \text{CUT}_d$. Necessity still holds if only all its real off-diagonal transfer entries agree with an arbitrary static Hermitian ensemble for every $t \geq 0$.*

Proof. Necessity follows from Lemma 5.1, positive rescaling, (14), and closedness of the cut cone. For sufficiency write $\Gamma = \sum_S a_S \delta_S$ with $a_S \geq 0$. Let independent centered Cauchy variables Z_S have characteristic functions $\mathbb{E} e^{-itZ_S} = e^{-a_S|t|}$ and set $\varepsilon_i = \sum_S 1_S(i) Z_S$. Then $\mathbb{E} e^{-it(\varepsilon_i - \varepsilon_j)} = e^{-|t|\Gamma_{ij}}$, which gives the required diagonal ensemble. $\square$

For $K \geq 1$ define $\mathcal{O}_K = \{1, 3, \dots, 2K-1\}$ and

$$b_K = \frac{4}{\pi} \sum_{n \geq 2K+1, n \text{ odd}} \frac{1}{n^2} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \in \mathcal{O}_K} \frac{1}{n^2}. \quad (15)$$

Replace the infinite sum in (11) by $n \in \mathcal{O}_K$ to obtain $D_{ij}^{(K)}(\Delta)$. Uniformly over every static energy law, $|D_{ij}^{(K)} - D_{ij}| \leq b_K$.

Theorem 5.3 (Complete finite harmonic separation). *Let $\Gamma \notin \text{CUT}_d$. Choose a real pair functional w such that*

$$\sum_{i < j} w_{ij} \delta_S(i, j) \leq 0 \quad \text{for every } S, \quad \sum_{i < j} w_{ij} \Gamma_{ij} > 0, \quad (16)$$

and put $W = \sum_{i < j} |w_{ij}|$. There exist $\Delta > 0$ and a finite K such that the statistic

$$S_{K,\Delta} = \sum_{i < j} w_{ij} D_{ij}^{(K)}(\Delta) \quad (17)$$

has null upper bound $U = b_K W$ for every static Hermitian ensemble, while its reference value V satisfies $g := V - U > 0$. Only the K times $\Delta, 3\Delta, \dots, (2K - 1)\Delta$ are needed. Conversely, a rate matrix in CUT_d cannot be excluded by any valid static-null test on the whole reference family.

Proof. A separator exists because CUT_d is closed and convex. The null bound follows from Lemma 5.1 and $|w \cdot (D^{(K)} - D)| \leq b_K W$. By (14), $w \cdot (F(\Delta \Gamma_{ij}))$ is positive for sufficiently small Δ . Fix such a spacing and increase K until twice the truncation bound $b_K W$ is smaller than this positive number. Then $V > b_K W$. The converse follows from Proposition 5.2. $\square$

This proves existence and supplies computable certificates; it does not claim polynomial complexity in dimension or an optimal number of times. For any proposed w , its validity can be checked on the finitely many cuts. Thus the general certificate does not require trusting the output of a cone-membership solver.

6 Finite-shot experiment and a certified example

Write the statistic in the form

$$S_{K,\Delta} = c + \sum_{i < j, n \in \mathcal{O}_K} a_{ijn} C_{ij}(n\Delta), \quad c = \frac{\pi}{2} \sum_{i < j} w_{ij}, \quad (18)$$

$$a_{ijn} = -\frac{4w_{ij}}{\pi n^2}, \quad A = \sum |a_{ijn}| = \frac{4W}{\pi} \sum_{n \in \mathcal{O}_K} \frac{1}{n^2}.$$

Choose (i, j, n) with probability $|a_{ijn}|/A$ and perform the four-phase experiment of Lemma 2.2 at $n\Delta$. Set $Z = \text{sgn}(a_{ijn})O$. Then $|Z| \leq 1$ and $\mathbb{E}Z = (S_{K,\Delta} - c)/A$.

Proposition 6.1 (Shot guarantee). *Given $g = V - U > 0$, reject the static null when $c + A\bar{Z} > (U + V)/2$. For independent fresh shots, each error probability is at most*

$$\exp\left(-\frac{Ng^2}{8A^2}\right). \quad (19)$$

The null guarantee also holds with predictable drift if, conditionally on the past, the current Hamiltonian law is chosen before the independent random measurement settings and obeys the same static null bound.

Proof. The normalized separation is g/A and each decision threshold is at least $g/(2A)$ from the corresponding mean. Hoeffding's inequality [3] for $[-1, 1]$ -valued variables gives (19). Its conditional exponential-moment proof gives the stated drift version. $\square$

Consider the qutrit rates

$$\Gamma = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{pmatrix}. \quad (20)$$

They arise from the generator $-[D, [D, \cdot]]$ with $D = \text{diag}(0, 1, 2)$ and so define physical channels. The triangle functional $w_{13} = 1$, $w_{12} = w_{23} = -1$ is nonpositive on every cut, but $w \cdot \Gamma = 2$. Thus the whole family has no static representation.

Corollary 6.2 (Twenty-time certificate). *For (20), take $K = 20$ and $\Delta = 13/100$ in the time units fixed by (20). Then*

$$\begin{aligned} b_{20} &\in (0.01591218, 0.01591219), & A &\in (4.66465243, 4.66465244), \\ g &\in (0.11365, 0.11366). \end{aligned}$$

Here $U = 3b_{20} \simeq 0.04773654$, $V = U + g \simeq 0.16138983$, and the threshold for $c + A\bar{Z}$ is approximately 0.10456319. The maximum queried time is $39\Delta = 5.07$. A total of 40,500 shots is sufficient for each error probability to be less than 0.05.

Proof. Here $W = 3$, $c = -\pi/2$, and the exact margin is

$$g = \frac{4}{\pi} \sum_{n \in \mathcal{O}_{20}} \frac{2e^{-13n/100} - e^{-52n/100} + 3}{n^2} - 2\pi. \quad (21)$$

The companion verifier brackets this finite expression using rational arithmetic: alternating arctangent sums in Machin's formula bracket π , and positive Taylor sums with a geometric tail bracket each exponential. The displayed conservative bounds imply $40500g^2/(8A^2) > 3$; the positive series for e^3 gives $e^3 > 20$. Equation (19) completes the proof. $\square$

The same example has an exact alias on *every* grid time $n \log 4$. Indeed $r_{12} = r_{23} = 1/4$ and $r_{13} = 1/256$, and (7) becomes $4/5 + 2/257 = 1038/1285 < 1$. The density (8) is therefore at least $247/1285$. This gives an explicit pair of physically different models whose entire coarse-grid channel data are identical, although the 20-time certificate excludes every static model on its finer, deliberately related menu.

7 Clock fragility and a quantitative repair

The set of time vectors satisfying (5) has full Lebesgue measure and is dense: its complement is a countable union of integer hyperplanes.

Corollary 7.1 (No unrestricted independent-jitter guarantee). *Fix a real dephasing semigroup with positive off-diagonal rates and any finite nominal menu of positive times. Every open neighborhood of that menu contains actual timings at which one static finite ensemble reproduces the reference channels exactly. Consequently, no test has a uniform positive separation over an unrestricted static null and an independent nonzero uncertainty interval for each time setting.*

Proof. Choose a rationally independent vector in the uncertainty box and apply Theorem 3.3 and Corollary 3.4. $\square$

The corollary concerns fixed but independently uncertain menu entries. It does not assert that all clock errors destroy detection. A common multiplicative clock error preserves harmonic relations. Nor does it cover a continuum of freshly randomized times whose law is independent of the hidden Hamiltonian; that is a different experiment, as in [13, 14].

Theorem 7.2 (Timing stability). *Suppose the static null satisfies $\text{diam spec}(H) \leq B$ almost surely. If every implemented time is nonnegative and differs from its intended time by at most η , with the timing error independent of the input/readout phase and the hidden Hamiltonian, the statistic (18) has null upper bound $U + AB\eta$. Set $G = \max_{i < j} \Gamma_{ij}$. The value under the reference (2) is at least $V - AG\eta$. Thus a positive certified gap remains whenever*

$$g_\eta = g - A(B + G)\eta > 0. \quad (22)$$

Using the midpoint of the adjusted bounds in the decision rule gives each error probability at most $\exp[-Ng_\eta^2/(8A^2)]$.

Proof. Lemma 2.1 and $|e^{ix} - e^{iy}| \leq |x - y|$ imply $|C_{ij}(t) - C_{ij}(s)| \leq B|t - s|$. The reference factors satisfy $|e^{-t\Gamma_{ij}} - e^{-s\Gamma_{ij}}| \leq G|t - s|$ for nonnegative times. Sum the biases against the absolute coefficients in (18) and apply Proposition 6.1. The same argument permits random timing errors bounded almost surely, provided the setting-independent Hamiltonian assumption is retained. $\square$

For the qutrit example, $G = 4$. The sufficient calibration requirement is $\eta < g/[A(B + 4)]$; setting $B = 10$ gives $\eta < 0.00174$ in the chosen time units. This is a bound conditional on a separately justified spectral-width prior, not a bound inferred from the same finite data. The high-energy aliases in Theorem 3.3 explain why such an assumption matters.

7.1 Fresh clock randomization without a spectral-width prior

The bandwidth hypothesis can be replaced by a different, explicit experimental resource: a fresh continuously randomized base time. This changes the access model from a finite menu to a finite family of time distributions. It does not contradict Corollary 7.1.

Choose $0 < h < \Delta$ and draw S uniformly from $[\Delta - h, \Delta + h]$ on each shot, independently of the hidden Hamiltonian and other settings. For a selected harmonic n , use the intended time nS . Retain the coefficients in (18). The ideal static bound is still $U = b_K W$: condition on S and apply Theorem 5.3. Define

$$c_h(n, \gamma) = \begin{cases} \frac{e^{-n\gamma(\Delta-h)} - e^{-n\gamma(\Delta+h)}}{2n\gamma h}, & \gamma > 0, \\ 1, & \gamma = 0, \end{cases} \quad (23)$$

$$V_h = c + \sum_{i < j, n \in \mathcal{O}_K} a_{ijn} c_h(n, \Gamma_{ij}), \quad L_h = \frac{4W}{\pi h} \sum_{n \in \mathcal{O}_K} \frac{1}{n^3}. \quad (24)$$

Theorem 7.3 (Bounded-time dither with independent additive error). *For each selected pair and harmonic, suppose the actual time is $nS + E$, where $|E| \leq \eta < \Delta - h$ almost surely. Its conditional error law may depend on the pair and harmonic, but is independent of S , the hidden Hamiltonian, and the input/readout phase. No spectral-width assumption is made. The static null upper bound is $U + \eta L_h$, and the reference lower bound is $V_h - AG\eta$. Hence the gap*

$$g_{h,\eta} = V_h - U - \eta(L_h + AG) \quad (25)$$

is certified whenever positive. With fresh independent shots, midpoint thresholding gives each error probability at most $\exp[-Ng_{h,\eta}^2/(8A^2)]$. For every rate matrix outside CUT_d there exist K, Δ, h and a strictly positive η for which this gap is positive.

Proof. The total variation distance between a uniform law on an interval of length $2nh$ and its translate by e is $\min(1, |e|/(2nh))$. A function bounded in $[-1, 1]$ therefore changes its expectation by at most $|e|/(nh)$. Conditional on the chosen settings, apply this to the bounded mean outcome of the physical four-phase experiment, and then average over the independent error. Summing the absolute coefficients bounds the null shift by $\eta \sum |a_{ijn}|/(nh) = \eta L_h$ without controlling frequencies. For the reference, the Lipschitz bound from Theorem 7.2 gives a shift at most $AG\eta$. Integrating the ideal exponential over the base interval gives (23). The shot bound follows as before. Finally, first select a positive finite-menu gap, then use $V_h \rightarrow V$ as $h \downarrow 0$, and last choose $\eta > 0$ small enough. $\square$

For the qutrit certificate, take $h = 1/20$ and $\eta = 1/10000$. Exact rational exponential bounds give $g_{h,\eta} \in (0.09992, 0.09993)$ and $53000g_{h,\eta}^2/(8A^2) > 3$. Thus 53,000 shots suffice for both

errors below 5%, with all actual times in $[0, 7.0201]$ and no spectral-width prior. These are 20 randomized time intervals, not 20 exact times. The error model includes fixed setting-dependent offsets and independent bounded additive jitter. It does not cover adversarial errors depending on the realized base time, rounding to a finite menu, or an unknown time distribution merely assumed to be uniform. More generally, a verified total-variation bound on each implemented time law can replace the additive-error assumption in the proof.

Corollary 7.4 (Energy cost near a separating finite menu). *Any bounded-width static ensemble that exactly aliases the reference at perturbed versions of a finite separating menu, with maximum perturbation $\eta > 0$, has essential spectral width*

$$B \geq \frac{g}{A\eta} - G.$$

In particular, finite-support aliases from Theorem 3.3 must have widths diverging at least as $1/\eta$ when approaching that menu.

Proof. Exact agreement forces the adjusted null upper bound in Theorem 7.2 to be at least its reference lower bound. $\square$

8 Verification, interpretation, and open questions

The companion code supplies an exact rational certificate for Corollary 6.2 and the randomized-clock example, checks the Fourier moments and positivity of the explicit qutrit grid density, and tests the phase estimator and spectral surrogate against direct unitary evolution in several dimensions. A finite linear-programming example illustrates interpolation to floating-point precision at binary approximations of two intended rationally independent times. That numerical interpolation is an illustration; Theorem 3.3 supplies its existence proof. No proof-assistant formalization or laboratory validation is claimed.

The results are stronger than simply observing ordinary frequency aliasing. They concern one positive joint energy law matching many transition channels simultaneously, and identify an arithmetic contrast between entire classes of experimental menus. At the same time, the proof ingredients are elementary and the scope is specific. The null assumes a fresh single-use channel on each shot. Multi-time interventions, echo controls within a single trajectory, or measurements of the hidden Hamiltonian may provide information unavailable in this model. Likewise, failure of a static explanation can be caused by classical time-dependent noise; it does not certify nonclassical environmental correlations.

Several quantitative problems remain. The coarse-grid positivity criterion is only sufficient. Beyond the inverse-error lower bound, the smallest bandwidth of a finite-menu alias near a harmonic menu is not determined. Neither the 20-time design nor its shot bound is claimed optimal. Quantitative Diophantine approximation may bound alias energies, while sharper trigonometric majorants and statistical estimators may reduce certificate cost. These are open extensions of the stated theorems, not results silently assumed in them.

Reproducibility and assistance. The release package contains the source, verification code and output, public source inventory, and a scoped literature audit. The author used OpenAI Codex for research assistance, mathematical drafting, code, and verification. AI assistance and any subsequent model assessments do not constitute human peer review. Publication as a working paper is distinct from editorial acceptance or validation of a claimed priority.

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