Linear Fisher information from a probe coupled to an inaccessible quantum memory

Three sufficient criteria, short-time cancellation, and limits of sector averaging

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Abstract

A finite quantum probe repeatedly interacts with a finite, inaccessible memory through a fixed Lindblad generator. The unknown scalar multiplies a nontrivial Hamiltonian acting only on the probe. We prove sufficient conditions under which a fixed prepare-and-measure protocol has classical Fisher information at least proportional to the number of rounds, without resetting the memory. The protocol records its randomized, informationally complete preparation labels. Our first criterion is primitivity of the infinitesimal averaged memory generator. A cancellation identity makes the parameter derivative of the averaged memory channel third order in the interaction duration; the derivative of its stationary state is consequently second order. A one-round signal therefore survives for every sufficiently short fixed duration. Two complementary criteria use full ambient linear minimality, or a common stationary product state with an explicit duration bound. We give physical two-qubit examples and counterexamples to discarding preparation labels and averaging sector-wise Fisher lower bounds. The conclusions are local lower bounds, not universal metrological optimality, and do not resolve the general dissipative hidden-memory problem.

1. Question and contributions

A probe can be prepared and measured repeatedly while an unobserved environment retains a conditional state between rounds. Noise and memory can then obscure a local Hamiltonian parameter. The relevant question is whether the classical record actually retained by this restricted experiment has a nonzero information rate. Information in an inaccessible bath or in a joint quantum output is insufficient by itself.

This paper proves three alternative sufficient criteria. Theorem 2 uses a mixing condition on a reduced generator computable before choosing the duration. Theorem 3 uses primitivity and full reachability and observability of the induced memory instrument, with duration bound $tM \leq 1/3$. Theorem 4 replaces the rank condition by a common stationary product state and gives another explicit duration bound. None of these assumptions is claimed necessary. In every case t is fixed before the number N of rounds tends to infinity.

The contribution is the stated combination of a local-signal obstruction or short-time cancellation with an accessible classical-record argument. Similarity methods, finite-dimensional realization arguments, and inference from mixing output statistics have substantial antecedents [2-6]. Priority for the combined criteria has not been established by an exhaustive literature review. The proofs and exact algebra checks below support a conditional mathematical result, not a claim of exceptional scientific impact.

Mann et al. [1], Section IV, motivates this question through special analytical cases and numerical evidence for dissipative hidden memory. Our recorded randomized preparations and mixing assumptions do not establish universal attainability for a single fixed pure preparation.

2. Model, protocol and conventions

Let P and E have finite Hilbert dimensions d and e, respectively, with $m = e^2$. Write L(P) for the complex operator space and Herm(P) for its real Hermitian part. In orthonormal Hilbert-Schmidt bases, the joint generator and channel are

$$L_\omega = L_0 + \omega D_f,\ D_f = D \otimes I_m,\ D(X) = -i[G,X],\ \Phi_{t,\omega} = \exp(tL_\omega). \quad (1)$$

Here G is Hermitian and not scalar. The background generator, including every jump operator, is independent of ω. The parameter lies in an open interval about ω_0. Norms $||\cdot||_2$ and $||\cdot||F$ of superoperator matrices mean spectral and Frobenius norms; $||X||_1$ is the trace norm of an operator. All differentiations below are with respect to ω unless specified otherwise. The joint dynamics uses the convention $L(X) = -i[H,X] + \Sigma j(VjXVj\dagger - \{Vj\dagger Vj,X\}/2)$.

Choose finitely many probe states σr spanning Herm(P), probabilities $\pi r > 0$, and a fixed informationally complete POVM {Fs} whose effects are strictly positive. Independently each round, draw and record r, prepare σr, apply Φt,ω, measure and record s, then discard only P. The conditional memory is retained. The alphabet a = (r,s) induces the completely positive instrument

$$A_{rs}(X) = \pi_r\, \mathrm{tr}_P[(F_s \otimes I)\, \Phi_{t,\omega}(\sigma_r \otimes X)],\ E_{t,\omega} = \Sigma_{r,s} A_{rs}. \quad (2)$$

A word $w = a_1 \ldots a\ell$ acts as $Aw = Aa\ell \ldots Aa_1$, and $p\rho(w) = \mathrm{tr}(Aw\rho)$. The initial memory ρ_0 is any fixed density matrix independent of ω. Its preparation is not an available control. Classical Fisher information $IN(\omega_0)$ refers to the complete retained record of N pairs, at the specified point. Strict positivity of Fs and πr makes every finite word probability positive, even for an initially rank-deficient memory.

We call a channel primitive when it has a unique faithful stationary state ρ* and its powers converge exponentially to $P^*(X) = \rho^* \mathrm{tr}\, X$. For a generator A, primitive means that zero is simple, its stationary state is faithful, and all remaining eigenvalues have negative real part. These finite-dimensional spectral definitions state the mixing used in the proofs.

For Theorems 2 and 4 we also require $\Sigma r\, \pi r \sigma r = I/d$. Such full-rank informationally complete preparations and effects exist in every finite dimension: choose sufficiently small positive and negative perturbations of the maximally mixed state along a traceless Hermitian basis. Their average is I/d; the corresponding sufficiently small perturbations of scalar effects form a POVM. Theorem 3 needs spanning but not this average condition.

Lemma 1. An informative word gives a linear information lower bound

Suppose Eω is primitive at ω_0 and differentiable in a neighborhood. If a stationary word w of length ℓ has $c = \partial\omega\, \mathrm{tr}(Aw\rho^*) \neq 0$, then constants $B < \infty$ and $V < \infty$ exist such that, with $n = \mathrm{floor}(N/\ell)$,

$$I_N(\omega_0) \geq (n|c| - B)^2/(nV), \text{ whenever } n|c| \geq B. \quad (3)$$

B and V can be chosen uniformly over all fixed, parameter-independent initial memory states. In particular $IN(\omega_0) \geq c_0 N$ for all sufficiently large N, for some $c_0 > 0$. This is a lower bound only; it does not assert $IN = \Theta(N)$.

3. Proof of the information lemma

Divide the first $n\ell$ rounds into n disjoint blocks. Let Zk indicate occurrence of w in block k, and let Sn = ΣkZk. At ω_0 choose C ≥ 1 and 0 < r < 1 so that $||E^jX||_1 \le Cr^j||X||_1$ for Hermitian trace-zero X and every j ≥ 0. Such constants exist by primitivity, including in the presence of Jordan blocks, by taking r above the spectral radius of the trace-zero restriction.

Let ρk be the unconditional memory before block k and pk = tr(Awρk). For h ≥ 1, subtracting the product of the marginals in the two-block probability gives

$$\mathrm{Cov}(Z_k,Z_{k+h}) = \mathrm{tr}\, A_w E^{(h-1)\ell}(A_w\rho_k - p_k E^{\ell}\rho_k). \quad (4)$$

The operator in parentheses is Hermitian, trace zero, and has trace norm at most 2pk ≤ 2. The output functional tr Aw is an effect bounded by the identity. Therefore $|\mathrm{Cov}| \le 2Cr^{(h-1)\ell}$, with no assumption that the blocks are independent. Since Var(Zk) ≤ 1, summing both covariance triangles yields

$$\mathrm{Var}(S_n) \le nV,\ V = 1 + 4C/(1-r^{\ell}). \quad (5)$$

It remains to control the derivative of the transient contribution, not merely its magnitude. A simple isolated stationary eigenvalue implies that ρ*(ω) and P*(ω) are differentiable locally. Put Qω = Eω - P*ω. For j ≥ 1, $E^j = P^* + Q^j$, because EP* = P*E = P*. The j = 0 term is kept separately. A local exponential bound $||Q^j|| \le C_1 r_1^j$ is available in any fixed finite-dimensional norm, with $r_1 < 1$ and enlarged C_1; continuity and a resolvent contour inside the unit disk give this bound uniformly nearby.

$$\partial_\omega Q^j = \Sigma_{i=0}^{j-1} Q^i(\partial_\omega Q)Q^{j-1-i}. \quad (6)$$

Its norm is at most $C_2 j r_1^{j-1}$. Differentiating tr $AwE^{(k-1)\ell}\rho_0$ and summing over k now shows ∂ω E[Sn] = nc + δn with |δn| ≤ B. Both geometric series and their differentiated series converge. Compactness of the state space makes B uniform over fixed ρ_0. This argument includes ∂ωρ*: setting that derivative to zero without a reason would be incorrect.

For a finite positive-probability record let U be its score. Then E[U] = 0 and ∂ω E[Sn] = E[(Sn-E[Sn])U]. Cauchy-Schwarz gives $(\partial\omega\ E[Sn])^2 \le \mathrm{Var}(Sn)IN$. Equations (5)-(6) give (3). Discarding any leftover rounds cannot increase information, so the same lower bound holds for the full N-round record. A statistic of zero variance cannot have nonzero derivative in this positive finite model. For $n \ge 2B/|c|$ the numerator is at least $n^2c^2/4$, proving the claimed linear rate. □

The deterministic interaction time is T = Nt. Ideal preparation, measurement and classical recording costs are excluded. The lemma proves local sensitivity and a lower information rate. It provides no general efficient estimator, minimax risk, global parameter identifiability, or quantum advantage. Constants may deteriorate near loss of primitivity or of the informative derivative.

4. A criterion from the infinitesimal memory generator

Define $J(X) = (I/d) \otimes X$ and $K = \mathrm{tr}P$, so $KJ = I$ on $L(E)$. With maximally mixed average preparation, $E_{t,\omega} = K \exp(tL_\omega)J$. Let

$$A = KL_0J. \quad (7)$$

A is a legitimate finite-dimensional Lindblad generator: $E_{h,\omega}$ is CPTP, $E_{0,\omega} = I$ and its derivative at $h = 0$ is A. Equivalently, $\exp(sA)$ is the limit of the CPTP products $(E_{s/n,\omega})^n$. This interpretation is for proving the generator property, not an additional physical reset protocol.

Theorem 2. Primitive infinitesimal averaging is sufficient

If A is primitive, then for each fixed ω_0 there is $t_0 > 0$ such that every fixed duration $0 < t < t_0$, with any spanning recorded preparations of average I/d and any strictly positive IC POVM, has an informative stationary one-letter probability. For every fixed initial memory independent of ω it consequently satisfies $IN(\omega_0) = \Omega(N)$. No common joint stationary product state and no full ambient linear minimality are required.

Proof. The two elementary identities $KD_f = 0$ and $D_fJ = 0$ follow respectively from the vanishing partial trace of a commutator on P and from $D(I) = 0$. Thus the coefficients of t and t^2 in $\partial_\omega E_{t,\omega}$ vanish. Differentiating the exponential series gives the more precise identity

$$\partial_\omega E_{t,\omega} = (t^3/6)\, KL_0D_fL_0J + O(t^4). \quad (8)$$

Indeed, in the cubic derivative only $L_\omega D_f L_\omega$ survives between K and J; replacing its exterior L_ω factors by L_0 does not change it. All remainders here and below are in a fixed superoperator norm and uniform for ω in a sufficiently small compact neighborhood. Finite dimensionality and analyticity of the exponential justify termwise differentiation and these bounds.

Extend $F_{t,\omega} = (E_{t,\omega}-I)/t$ analytically to $t = 0$ by $F_{0,\omega} = A$. Trace preservation makes its range trace zero. On the trace-zero subspace, A is invertible; hence so is $F_{t,\omega}$ for small $|t|$. The normalized solution $\rho_{t,\omega}$ of $F_{t,\omega}\rho_{t,\omega} = 0$ is analytic, equals the faithful stationary state ρ_A at $t = 0$, and remains positive definite for small positive t. This can be obtained explicitly by solving on the trace-zero subspace after fixing any trace-one reference vector.

The other eigenvalues of $F_{t,\omega}$ remain in a left half-plane $\mathrm{Re}\, z \le -\gamma < 0$ and bounded in modulus by some H. The associated eigenvalues of $E_{t,\omega}$ are $1+tz$; their squared moduli are at most $1-2\gamma t+H^2t^2 < 1$ for small positive t. Therefore $E_{t,\omega}$ is primitive. No diagonalizability is needed for this conclusion.

By (8), $\partial_\omega F_{t,\omega} = O(t^2)$. Differentiating its stationary equation and using $\mathrm{tr}(\partial_\omega \rho_{t,\omega}) = 0$ gives

$$\partial_\omega \rho_{t,\omega} = -(F_{t,\omega}|_{\mathrm{tr}=0})^{-1}(\partial_\omega F_{t,\omega})\rho_{t,\omega} = O(t^2). \quad (9)$$

4.1. The accessible signal survives stationary feedback

Continue the proof of Theorem 2. Define the stationary reduced probe channel $R_{t,\omega}(X) = trE\,\Phi_{t,\omega}(X \otimes \rho_{t,\omega})$. Its derivative has a direct term and a stationary-memory term. The first is $tD(X) + O(t^2)||X||$, since $\partial\omega\Phi = tDf + O(t^2)$ and $tr\,\rho_{t,\omega} = 1$. In the second term the identity part of Φ contributes $X\,tr(\partial\omega\rho_{t,\omega}) = 0$. Its remainder is $O(t)O(t^2)||X||$ by (9). Consequently

$$\partial_\omega R_{t,\omega} = tD + O(t^2), \quad (10)$$

as a map on the entire probe operator space. Since $D \neq 0$, this derivative is nonzero at ω_0 for all sufficiently small positive t. If every stationary $p_{rs} = \pi r\,tr[FsR_{t,\omega}(\sigma r)]$ had zero derivative, spanning of the inputs and effects would imply $\partial\omega R = 0$. At least one letter is therefore informative. Lemma 1 with $\ell = 1$ proves the conclusion, including arbitrary fixed initial memory states. □

The theorem is an existence result for a duration interval depending on the model and parameter neighborhood. Equations (8)-(10) do not give a universal numerical t_0, and the limits $t \to 0$ and $N \to \infty$ are not exchanged: choose any valid t once, then increase N. Primitivity of A is sufficient, not implied by arbitrary joint Lindblad dynamics or by primitivity at some isolated finite duration.

4.2. Exact two-qubit example without a stationary product

Put the probe first, use Pauli matrices X,Y,Z, and choose

$$H_0 = (XX+YY+ZZ)/4 + IX/5,\ G = ZI/2,$$

$$V_1 = I \otimes |0\rangle\langle 1|,\ V_2 = ZI/2. \quad (11)$$

For a general memory operator Y, the infinitesimal averaged generator is $A(Y) = -i[X/5,Y] + VYV\dagger - \{V\dagger V,Y\}/2$, where X is the Pauli matrix and $V = |0\rangle\langle 1|$. Its Bloch equations are

$$\dot{x} = -x/2,\ \dot{y} = -y/2 - 2z/5,\ \dot{z} = 2y/5 - z + 1. \quad (12)$$

The stationary Bloch vector is (0,-20/33,25/33). Its density determinant is $16/1089 > 0$. The trace-zero eigenvalues are -1/2 and $-3/4 \pm i\sqrt{39}/20$, so A is primitive. Theorem 2 applies for sufficiently small fixed t.

This example cannot have a stationary product $(I/2) \otimes \rho$ for L_0. If such a product existed, partial tracing its stationarity would force $A\rho = 0$ and thus $\rho = \rho A$. But the exchange interaction has a nonzero commutator with $(I/2) \otimes \rho A$. Direct Pauli-basis calculation gives $||L_0((I/2) \otimes \rho A)||^2_{HS} = 1025/8712 > 0$. Thus Theorem 2 covers a case outside the product-state criterion below.

The accompanying rational verifier reconstructs (11), checks (12), the stationary vector, characteristic polynomial, product residual and the cancellation coefficients of (8). It is an exact finite algebra check, not a formalization of the analytic perturbation proof or an experimental test.

5. Local signal cannot be a memory-only similarity

This section supplies a complementary route when the averaged generator A is not known to be primitive. Let $||L\omega||_2 \le M$ on an interval J containing ω_0. Fix $t > 0$ with $tM \le 1/3$. Write τE for matrix partial trace over the m-dimensional operator space L(E); this is not the physical state partial trace trE. Define

$$W_t(\omega) = \mathrm{Re}\langle D, \tau_E(\Phi_{t,\omega})\rangle_F / (m||D||^2_F). \quad (13)$$

For any invertible linear map S on L(E), $\tau E((I\otimes S)B(I\otimes S^{-1})) = \tau E(B)$, by cyclicity within each block. No positivity or unitarity of S is assumed. Differentiating the exponential series, each term of degree k contains k insertions of Df among k-1 copies of Lω. Therefore

$$\partial_\omega \Phi = tD_f + R, \; ||R||_F \le t\sqrt{m}\,||D||_F(e^{tM}-1). \quad (14)$$

The inequality $||\tau E\, B||F \le \sqrt{m}||B||F$ and Cauchy-Schwarz give $|W't-t| \le t(e^{tM}-1)$. Since $e^x \le 1/(1-x)$ for $0 \le x < 1$, we obtain

$$W'_t(\omega) \ge t(2-e^{tM}) \ge t/2 > 0. \quad (15)$$

It follows that no two distinct parameters in J have channels related by a similarity acting only on the memory operator space. Infinitesimally, no matrix K can satisfy $\partial\omega\Phi = [I\otimes K,\Phi]$, since τE annihilates that commutator. The witness is algebraic: access to $\tau E\Phi$ is not being assumed experimentally.

Theorem 3. Full ambient minimality criterion

Use the protocol (2) at $tM \le 1/3$. Suppose E_{t,ω_0} is primitive with stationary state ρ*, and suppose the following real spans have dimension $m = e^2$:

$$\mathrm{span}\{A_w\rho^*: w \text{ a finite word}\} = \mathrm{Herm}(E),$$

$$\mathrm{span}\{\mathrm{tr}\, A_w: w \text{ a finite word}\} = \mathrm{Herm}(E)^*. \quad (16)$$

Then some stationary word of length at most 2m-1 has nonzero parameter derivative, and $IN(\omega_0) = \Omega(N)$ for every fixed initial memory independent of ω. Condition (16) is full ambient linear reachability and observability. It is stronger than minimality of the Hilbert dimension, and cannot be silently replaced by that weaker property.

The proof below explains why the operational rank condition is essential to this argument. A changing joint channel need not be visible in a restricted record. Nor can one first reduce to an arbitrary minimal realization and assume that its coordinates preserve the tensor factorization used in (13).

5.1. Proof of the minimality criterion

Suppose, to obtain a contradiction, that all stationary probabilities p(w) = tr Awρ* have zero derivative at ω_0. Put xw = Awρ*. Define Kxw = ∂ωxw. To see that this gives a well-defined linear map, let Σw cw xw = 0, with finitely many constant real coefficients. For every suffix v,

$$\operatorname{tr} A_v \, \Sigma_w c_w \partial_\omega x_w = \Sigma_w c_w \partial_\omega p(wv) - (\operatorname{tr} \partial_\omega A_v) \Sigma_w c_w x_w = 0. \quad (17)$$

Observability in (16) forces $\Sigma_w c_w \partial\omega x_w = 0$. Reachability then defines K on all Herm(E), extended complex-linearly. Differentiate xwa = Aaxw and use the definition of K to obtain

$$\partial_\omega A_a = K A_a - A_a K, \; \partial_\omega \rho^* = K\rho^*, \; \operatorname{tr} \circ K = 0. \quad (18)$$

The last identity follows by differentiating tr xw = p(w); it is composition with the trace functional, not the matrix trace of K. The spanning input states and output effects determine every probe input-output block of the joint superoperator. Thus (18), holding for every (r,s), implies ∂ωΦ = [I⊗K,Φ], contradicting (15). Some stationary word is informative.

For the length bound, the series ∂ωp(w) has a linear representation of dimension 2m, with initial vector (ρ*,∂ωρ*), output functional (0,tr), and letter matrices

$$B_a = [\, A_a \; 0 \; ; \; \partial_\omega A_a \; A_a \,]. \quad (19)$$

The reachable spans generated by words of length at most k either grow strictly at each step or, once equal at consecutive steps, are invariant under every Ba and never grow again. Starting with one vector in dimension 2m, they stabilize by k = 2m-1. If the output functional vanished on every word up to that length, it would vanish on all words, which has been excluded. Lemma 1 completes the proof. □

5.2. What the rank criterion does and does not certify

Theorem 3 does not claim that every joint generator admits a protocol satisfying (16), or that every primitive averaged memory channel has that rank. Positivity of a finite set of numerical singular values is not an exact rank certificate. A rationally bounded determinant or an algebraic argument is needed if a computed instance is used as proof.

A supporting local calculation for (11), with four full-rank tetrahedral preparations and effects at t = 1/32, uses degree-20 Taylor bounds to certify rank four and strict Choi positivity of the averaged memory channel. This calculation is included in the accompanying reproducibility package, but no theorem here depends on its decimal output. Theorem 2 and the explicit reduced generator (12) already establish a nonempty physical class without that calculation.

The statistical conversion of an informative word into a rate is the same Lemma 1 in all three criteria. This reuse prevents a false inference that full quantum-output information automatically remains available after a measurement or after an inaccessible bath is discarded.

6. A common stationary product state

Theorem 4. Product-stationary criterion with explicit duration

Suppose there is a faithful density matrix ρ on E, independent of ω, such that $L_0((I/d)\otimes\rho) = 0$. Suppose the protocol has average preparation I/d and its averaged channel is primitive at ω_0. On an interval where $||L\omega||_2 \le M$, put $q = d\sqrt{e}||\rho||HS$. If

$$tM \le 1/(1+2q), \quad (20)$$

then a stationary one-letter probability has nonzero derivative, and $IN(\omega_0) = \Omega(N)$ uniformly over fixed initial memory states in the sense of Lemma 1. There is no full ambient rank assumption. For unital joint dynamics, one may take ρ = I/e and q = d, provided the averaged channel is primitive.

Proof. The product $(I/d)\otimes\rho$ is stationary for every ω because D(I) = 0. Thus $E_{t,\omega}\rho = \rho$, and primitivity locally makes this the stationary state. Let $R\omega(X) = trE\ \Phi_{t,\omega}(X\otimes\rho)$. As Hilbert-Schmidt maps, $J\rho: X \mapsto X\otimes\rho$ and $K_\rho: Y \mapsto trE\ Y$ have spectral norms $||\rho||HS$ and $\sqrt{e}$. The differentiated exponential satisfies

$$\partial_\omega\Phi = tD_f + Q,\ ||Q||_2 \le t||D||_2(e^{tM}-1). \quad (21)$$

Since $K_\rho DfJ\rho = D$, and a matrix on the d^2-dimensional probe operator space satisfies $||B||F \le d||B||_2$, we have $||\partial\omega R - tD||F \le qt||D||_2(e^{tM}-1)$. Hence the reduced witness $U\omega = Re\langle D,R\omega\rangle F/||D||^2F$ obeys

$$|U'_\omega - t| \le qt(e^{tM}-1) \le qt\cdot tM/(1-tM) \le t/2. \quad (22)$$

The middle inequality uses tM < 1. Thus $\partial\omega R \neq 0$. Informational completeness makes at least one stationary $p_{rs} = \pi r\ tr\ FsR\omega(\sigma r)$ informative; Lemma 1 applies. The stationary state need not be physically prepared. □

This theorem replaces one substantial hypothesis by another. Unitality alone does not imply mixing. The fixed-state assumption cannot be weakened in this proof to an unspecified parameter-dependent ρ: its derivative would add a term. Theorem 2 controls precisely that type of term under a different, infinitesimal primitivity assumption.

6.1. A rank-deficient but informative memory

For two qubits take $H_0 = ZPZE/4$, G = ZP/2, and jumps IPXE/2, IPYE/2, IPZE/2. This generator is unital. With average preparation I/2, the averaged memory channel has Bloch matrix $diag(e^{-t}\cos(t/2), e^{-t}\cos(t/2), e^{-t})$. It contracts trace distance by at most $e^{-t} < 1$ and has stationary state I/2.

Nevertheless, memory operators diagonal in Z form an invariant subspace under each conditional instrument, while the off-diagonal subspace is also invariant and never contributes to a future trace. Starting at I/2, both reachable and observable dimensions are at most two, rather than m = 4. Theorem 3's ambient rank condition fails exactly, whereas Theorem 4 applies.

6.2. Exact signal and the role of preparation labels

In the last example, the joint generator on the span of XPIE and YPZE has matrix B = [0,-1/2; 1/2,-1]. Since $(B+I/2)^2 = 0$, the stationary reduced probe coherence is

$$\chi(t) = e^{-t/2}(1+t/2) > 0. \quad (23)$$

The reduced probe channel applies diag(χ,χ,1) on Bloch vectors followed by rotation about Z through angle ωt. Therefore $U'(0) = t\chi(t)$. Take four sign vectors $s_0 = (1,1,1)$, $s_1 = (1,-1,-1)$, $s_2 = (-1,1,-1)$, $s_3 = (-1,-1,1)$, states $\sigma r = (I+sr\cdot\sigma/4)/2$, $\pi r = 1/4$, and effects $Fs = (I+ss\cdot\sigma/4)/4$. They are full rank and informationally complete, and the average state is I/2. Directly,

$$\partial_\omega p_{0,2}(0) = t\chi(t)/128 > 0. \quad (24)$$

For $|\omega| \le 1$, the exact squared Frobenius norms $||L_0||^2F = 14$ and $||Df||^2F = 8$ permit M = 8. At t = 1/64, condition (20) is satisfied. The formula (23), the invariant subspaces and a rational enclosure of (24) are checked in the supporting verifier. These are algebraic and computational checks, not measured laboratory data.

Proposition 5. Erasing preparation labels can erase all information

For unital joint dynamics, start E at I/e and use randomized preparations of average I/d. If the preparation labels are discarded, the entire remaining outcome record is independent of ω and its Fisher information is zero.

Proof. The average joint input is I/(de), which remains fixed under any unital channel. Conditioning on effect Fs leaves the normalized environment I/e and outcome probability tr Fs/d. Induction over rounds gives an iid record with these parameter-independent probabilities. The unrecorded preparation average must be taken before conditioning, which is exactly what discarding the labels does. □

This proposition concerns the reduced record in the stated stationary initial condition; it does not contradict any theorem above. In particular, the informative probability in (24) is a joint probability for a recorded preparation and outcome. Describing the procedure merely as repeatedly preparing the average maximally mixed state would change the experiment and invalidate the conclusion.

7. A physical obstruction to averaging hidden sectors

Consider a different two-qubit model with fixed duration t = 1, $H_0 = (\pi/2)ZP\otimes|1\rangle\langle 1|E$, signal ωZP/2 and jump √(log 2/2) ZP⊗IE. Initialize E at I/2, repeatedly prepare +X on P and measure Y. The environment sector is conserved. In sector zero the outcome probability is $p_0(\omega) = (1+\sin(\omega)/2)/2$; in sector one it is $p_1(\omega) = (1-\sin(\omega)/2)/2$. The jump gives visibility 1/2, while the sector-one background adds a phase π.

7.1. Local Fisher information cancels in the mixture

At $\omega = 0$, $p_0 = p_1 = 1/2$ and $p'_0 = -p'_1 = 1/4$. If the sector is observed, N iid outcomes have Fisher information N/4 in either sector. With the sector hidden and weights 1/2, the probability of any word is the average of two products exchanged by $\omega \mapsto -\omega$. Each such probability is positive and even, so its derivative at zero vanishes. Consequently

$$I_N(0 \mid \text{known sector}) = N/4,\ I_N(0 \mid \text{hidden equal mixture}) = 0 \text{ for every } N. \quad (25)$$

There is still quadratic sensitivity: the probability of two positive outcomes is $1/4+\sin^2(\omega)/16$, whose second derivative at zero equals 1/8. Zero local Fisher information here is not parameter independence away from zero. Nor is it proof that no other duration, preparation or measurement can work.

This counterexample refutes only the inference that positive rates in every unobserved sector automatically give a positive rate for their mixture. Averaging sector Fisher information supplies an upper bound through data processing, not a lower bound. The example lacks primitivity, does not meet our short-time conditions and does not use informationally complete readout. It therefore contradicts none of Theorems 2-4. The all-N claim follows from evenness; checking 510 words of lengths one through eight is only a finite cross-check.

8. Scope, related work and open cases

Our three sufficient routes differ. Theorem 2 requires a faithful mixing infinitesimal reduced generator, and yields an unspecified but positive duration interval. Theorem 3 instead assumes mixing and full ambient ranks at the selected duration. Theorem 4 uses a common stationary product and mixing at that duration, with an explicit norm bound. A degenerate A can still lead to a primitive finite-duration channel, but Theorem 2 does not cover that situation automatically.

Burgarth and Yuasa [2] establish similarity-based identification under rank conditions. Guţă's output-statistics analysis [3] and Guţă and Kiukas's primitive-chain equivalence and local asymptotic theory [4] are direct antecedents for the realization and statistical ingredients. We do not claim those methods are new. Our proofs track the particular preparation labels and classical outcomes retained after discarding an inaccessible bath.

The full quantum output in a dilation can contain information absent from that retained record: data processing gives $I_{\text{classical}} \le F_{\text{quantum,full}}$. A lower bound on the right-hand side alone therefore cannot prove our conclusion. This comparison is about available subsystems and measurements, not a defect in the cited quantum-output theorems. The newer theory for irreducible periodic chains [5] makes output mixtures and their identifiability especially relevant; removing primitivity requires more than averaging sector rates.

Finite-dimensional word and realization methods have further explicit antecedents in [6]. Learning an arbitrary Lindbladian with error correction [7] uses control and ancillary resources that do not amount to automatically controlling the inaccessible E in this paper. None of these comparisons establishes priority for our sufficient criteria; the targeted source check is not a comprehensive proof of novelty.

9. Reproducibility and review status

All general arguments in this paper are analytic, not formally verified. A standard-library Python package accompanies the local release: it includes a rational Pauli-generator reconstruction for Theorem 2's example, the earlier rational duration/rank certificate, the unital example and the sector counterexample. The verifiers require no optimizer, floating-point tolerance or external numerical library. A clean replay is compared byte-for-byte to the saved outputs. The verification results certify the stated finite algebraic ingredients; they do not replace review of the general proofs.

The example (11) can also be checked directly from (12): the stationary coordinates solve three linear equations, and the trace-zero characteristic polynomial is $(\lambda+1/2)(\lambda^2+3\lambda/2+33/50)$. The density determinant and product residual stated above are exact rational values. Equation (8) has an elementary word check: in a derivative of $K\,L_\omega^{\,k}\,J$ the insertion Df cannot be first or last; at degree three only the interior insertion survives. This makes the order-three cancellation reproducible without taking a fitted small-time slope.

The two earlier verifier groups are preserved unmodified in separate package folders. Their previous internal notes are provenance, not prior journal publications or independent reviews. The present document consolidates that line and adds Theorem 2 and its explicit stationary-feedback proof. It is a distinct research question from the finite-calibration discrimination manuscript ARR-2026-0S8D4XJMZS9PDV1Z; no review, score, acceptance or permission belonging to that manuscript applies to this PDF.

The supplemental archive is preserved with the author's delivery files. This candidate does not claim that AIRR has already hosted that archive or that a public supplement URL exists. The main proofs and examples have been written to be assessable from this PDF alone. The supplement can be supplied through an authorized review route.

Revisión propia (self-review)

The drafting session checked the distinction between physical and operator-space partial traces; the t^3 cancellation; the inverse restricted to trace-zero operators in (9); the faithfulness and spectral argument for small-time primitivity; the derivative of transient sums; the meaning of full ambient rank; and the direction of Fisher information under mixing and discarded labels. No contradiction was found under the stated assumptions. The step beyond the preceding notes is conditional: A may fail to be primitive, and no theorem here proves universal attainability.

No AIRR score is assigned in this document. Novelty and significance remain open to adversarial assessment, and no evidence here establishes a 7-10 level result. No independent human review, independent model review or experimental validation is claimed. Any future correction must preserve the identity of this version.

Authorship, assistance and interests

Author credit: Lluis Eriksson. OpenAI Codex assisted with formulation, derivations, counterexample search, code, source checking, writing and self-review. That assistance is not an independent assessment. The author operates AIRR.SCIENCE; any submission there must disclose the author-editor conflict and follow the applicable exact-version assessment process. There is no claim of institutional affiliation or laboratory data. Citations identify external work; no third-party figures or substantial verbatim passages are reproduced.

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Primary source URLs: https://arxiv.org/abs/2503.07745 ; https://arxiv.org/abs/1401.5240 ; https://arxiv.org/abs/1007.0434 ; https://arxiv.org/abs/1402.3535 ; https://arxiv.org/abs/2603.20761 ; https://arxiv.org/abs/2509.03004 ; https://arxiv.org/abs/2606.18188 .

Open direction

When the infinitesimal averaged generator has multiple stationary sectors, the inverse used in (9) is singular. One must analyze the first nonzero inter-sector dynamics and the stationary-record derivative together. A positive information rate in each labeled sector is insufficient, as (25) shows. Establishing a usable criterion in that degenerate regime, or a genuine protocol-independent obstruction, is the next structural problem. Optimizing a constant within the already covered examples does not answer it.