One-Spike Inverse Self-Commutators and Exact Three-versus-Four-Kick Curvature Synthesis

Singular rigidity, sharp stability, and a universal switching gain

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Abstract

For a traceless Hermitian matrix $F \in M_d(\mathbb{C})$, let

$$\kappa_d(F) = \min\{\|H\|_{\mathrm{HS}}\,\|K\|_{\mathrm{HS}} : H = H^*,\ K = K^*,\ -i[H, K] = F\}.$$

We solve this inverse problem on the complete spectral cone having one positive eigenvalue. If the nonzero spectrum is $(P, -b_1, \dots, -b_n)$ with $b_1 \geq \cdots \geq b_n > 0$ and $\sum_j b_j = P$, then

$$\kappa_d(F) = \sum_{j=1}^{n} j b_j.$$

Every balanced optimum has the same nonzero squared singular values $2\sum_{j=\ell}^{n} b_j$, $1 \leq \ell \leq n$, and hence rank n, independently of ambient zero padding. The deficit from the fixed-mass maximum is one half of the pairwise dispersion of the b_j and is sharply equivalent to their ℓ^1 distance from uniformity.

We then solve two fixed-target matrix-polygon problems in every finite dimension. For a balanced m-kick loop $x_1 + \cdots + x_m = 0$, define

$$H_{\mathrm{geo}} = \frac{i}{2}\sum_{k>j}[x_k, x_j], \qquad S_2 = \sum_j \|x_j\|_{\mathrm{HS}},$$

and let $\mathcal{A}_m(F)$ be the infimum of S_2^2 subject to $H_{\mathrm{geo}} = F$. For every nonzero traceless Hermitian target,

$$\mathcal{A}_3(F) = 12\sqrt{3}\,\kappa_d(F), \qquad \mathcal{A}_4(F) = 16\,\kappa_d(F).$$

The three-kick identity consolidates the author's earlier low-dimensional cases, while the four-kick identity extends the previously known sign-paired equality face to every target. Thus a fourth kick lowers the exact optimal squared perimeter by the target- and dimension-independent factor $4/(3\sqrt{3})$, a reduction of $1 - 4/(3\sqrt{3}) \approx 23.02\%$. The four-kick lower bound uses an exact Gram-determinant argument; a tempting stronger parallelogram inequality is false and is not used. Exact-arithmetic replays and an independent implementation accompany the manuscript.

1 Question, answer, and scope

Finite-dimensional self-commutator existence is classical [1, 21, 15]. It does not determine the least Hilbert–Schmidt resource of Hermitian factors generating a prescribed target. We study that inverse metric problem and its consequence for balanced matrix loops. Throughout,

$$\|X\|_{\mathrm{HS}}^2 = \mathrm{Tr}(X^* X)$$

is unnormalized, $F = F^*$ is traceless, and scalar parts of Hermitian factors are removed because they do not affect commutators.

The surrounding literature contains forward Frobenius commutator inequalities [3, 4, 8], unrestricted-factor inverse problems [22], and the weighted-shift constructions used in self-commutator existence theory [15, 2]. The present optimization keeps both factors Hermitian. Its one-spike answer is extracted from a small universal family of Horn inequalities rather than from a dimension-by-dimension enumeration.

The loop problem is motivated by second-order product formulas and geometric terms in controlled or Zeno dynamics [5, 7]. Polygonal isoperimetry itself is classical [18]; our fixed-target quantity retains the complete matrix F, not merely one norm of it. Building on the author's earlier dimension-three, dimension-four, and dimension-five inverse formulas and loop corollaries [13, 10, 9], on earlier balanced-kick mechanisms [12], and on an earlier sign-paired matrix-polygon equality face [11], the scoped package is:

(i) an exact all-dimensional formula on the one-positive or one-negative spectral cone;

(ii) rigidity of the singular spectrum and rank of every balanced optimum;

(iii) a sharp ℓ^1 stability theorem at the uniform one-spike ray;

(iv) a dimension-free consolidation of the triangular law and an all-target four-kick law extending the sign-paired case, including a strict universal switching gain.

We do not claim novelty for the earlier low-dimensional slices, weighted-shift self-commutator constructions, Horn feasibility, group-commutator loops, or general Gini/mean-deviation inequalities. We do not claim a closed formula for κ_d when both sign multiplicities exceed one, a classification of all optimizing matrices, an infinite-dimensional result, or a general exact formula for five or more kicks. Priority is stated conservatively: the Horn and weighted-shift machinery is standard, and the accompanying search is not a proof of novelty.

2 Balanced self-commutators

Definition 2.1. For traceless Hermitian $F \in M_d(\mathbb{C})$ set

$$\kappa_d(F) = \inf\{\|H\|_{\mathrm{HS}} \|K\|_{\mathrm{HS}} : H = H^*,\ K = K^*,\ -i[H,K] = F\}. \tag{1}$$

Lemma 2.2 (one-matrix reduction)**.** *For every traceless Hermitian F,*

$$\kappa_d(F) = \frac{1}{2}\min\{\|C\|_{\mathrm{HS}}^2 : CC^* - C^*C = 2F\}. \tag{2}$$

The minimum is attained. Moreover, an optimal Hermitian pair may be chosen so that

$$\langle H, K\rangle_{\mathrm{HS}} = 0, \qquad \|H\|_{\mathrm{HS}} = \|K\|_{\mathrm{HS}} = \sqrt{\kappa_d(F)}. \tag{3}$$

Proof. If $F = 0$, both minima are zero, attained by $H = K = C = 0$, and the final assertion is immediate. Hence assume $F \neq 0$. If $-i[H,K] = F$, reciprocal scaling of H, K preserves the commutator and their norm product, and can make their norms equal. For $C = H + iK$,

$$CC^* - C^*C = -2i[H,K] = 2F, \qquad \|C\|_{\mathrm{HS}}^2 = \|H\|_{\mathrm{HS}}^2 + \|K\|_{\mathrm{HS}}^2.$$

Conversely, the Hermitian real and imaginary parts X, Y of any feasible C obey $-i[X,Y] = F$ and $\|X\|_{\mathrm{HS}} \|Y\|_{\mathrm{HS}} \le \|C\|_{\mathrm{HS}}^2/2$. This proves (2). For general feasibility, order an eigenbasis with all positive eigenvalues first, then zeros, then negative eigenvalues. Every partial sum s_j of this ordering is nonnegative and the final sum is zero. The finite weighted shift $Ce_{j+1} = \sqrt{2s_j}\, e_j$ then satisfies $(CC^* - C^*C)/2 = F$. Finite-dimensional compactness gives attainment.

Take a minimizing Hermitian pair and balance its nonzero norms. Replacing K by

$$K - \frac{\langle H, K\rangle_{\mathrm{HS}}}{\|H\|_{\mathrm{HS}}^2} H$$

preserves the commutator. Unless the inner product vanishes, it strictly reduces the product after rebalancing, contradicting optimality. This gives (3). □

The Horn description of spectra of sums of Hermitian matrices [17, 19, 20, 16] applies to $CC^* + (-C^*C) = 2F$. We only need a nested family for which the Littlewood–Richardson coefficient is visibly one.

3 The complete one-spike cone

Assume that F has one positive eigenvalue. Suppressing arbitrary zero padding, write its nonzero spectrum as

$$(P, -b_1, \dots, -b_n), \qquad b_1 \geq \cdots \geq b_n > 0, \qquad \sum_{j=1}^{n} b_j = P. \tag{4}$$

The order in (4) is adapted to the constructor; the decreasing spectral order is $(P, 0, \dots, 0, -b_n, \dots, -b_1)$.

Theorem 3.1 (exact cost and optimizer rigidity)**.** *For* (4),

$$\boxed{\kappa_d(F) = \sum_{j=1}^{n} j b_j.} \tag{5}$$

If $C = H + iK$ comes from any balanced optimal pair, then the nonzero squared singular values of C, in decreasing order, are

$$p_\ell = 2\sum_{j=\ell}^{n} b_j, \qquad 1 \leq \ell \leq n. \tag{6}$$

Consequently every such C has rank exactly n. The same statements hold, with the same singular data, when the negative spectral subspace has rank one.

Proof. Let $d = n + 1 + z$, where z is the zero multiplicity, and let $p_1 \geq \cdots \geq p_d \geq 0$ be the eigenvalues of CC^* for a minimizer in (2). If $p_d = t > 0$, write the polar decomposition $C = US^{1/2}$ with $S = C^*C$; invertibility makes U unitary. Then $C' = U(S - t\mathbf{1})^{1/2}$ satisfies

$$C'C'^* = CC^* - t\mathbf{1}, \qquad C'^*C' = C^*C - t\mathbf{1}.$$

It preserves the self-commutator and strictly reduces the norm. Hence $p_d = 0$ at an optimum. The decreasing spectra in Horn's sum problem are

$$\alpha = (p_1, \dots, p_{d-1}, 0), \quad \beta = (0, -p_{d-1}, \dots, -p_1), \quad \gamma = 2\,\mathrm{spec}(F).$$

For $1 \leq \ell \leq n$, take

$$I_\ell = \{1, \dots, \ell\}, \qquad J_\ell = K_\ell = \{1, d - \ell + 2, \dots, d\}.$$

These are valid Horn triples: the partition associated with I_ℓ is zero and therefore $c^{\lambda(K_\ell)}_{0,\lambda(J_\ell)} = 1$. The corresponding inequality telescopes to

$$p_\ell \geq 2\left(P - \sum_{j=1}^{\ell-1} b_j\right) = 2\sum_{j=\ell}^{n} b_j. \tag{7}$$

It follows that

$$\frac{1}{2}\sum_{r=1}^{d} p_r \geq \sum_{\ell=1}^{n}\sum_{j=\ell}^{n} b_j = \sum_{j=1}^{n} j b_j.$$

For attainment, use an eigenbasis ordered as in (4), put $t_\ell = \sum_{j=\ell}^{n} b_j$, and define the weighted shift

$$Ce_{\ell+1} = \sqrt{2t_\ell}\, e_\ell \quad (1 \leq \ell \leq n), \qquad C = 0 \quad \text{on the zero-padded complement.} \tag{8}$$

Direct multiplication gives $(CC^* - C^*C)/2 = F$ and $\|C\|_{\mathrm{HS}}^2/2 = \sum_\ell t_\ell = \sum_j j b_j$. Thus the lower bound is exact. At any optimum, equality of the sum forces equality in every nonnegative inequality (7), while all remaining p_r vanish. This proves (6) and the rank claim. Replacing F by $-F$ replaces C by C^*. □

Remark 3.2 (what rigidity does not say). The theorem fixes the singular spectrum and rank of every balanced optimum; it does not assert that every optimum is entrywise a weighted shift. Unitary and phase freedoms, and possibly further equality geometry, remain.

4 Sharp stability at the uniform ray

For fixed P and n, the general rank-adaptive upper endpoint is $((n+1)/2)P$. On the one-spike cone its deficit has an elementary dispersion form. Comparisons between empirical Gini mean difference and mean absolute deviation are known [6]; the theorem below records the exact constants and equality families in this ordered, nonnegative, fixed-mass spectral normalization.

Theorem 4.1 (Gini identity and sharp trace-distance stability). *Let $u = P/n$ and*

$$D = \frac{n+1}{2}P - \kappa_d(F), \qquad L = \sum_{j=1}^{n} |b_j - u|.$$

Then

$$\boxed{D = \frac{1}{2}\sum_{1\leq i<j\leq n} (b_i - b_j).} \tag{9}$$

For $n \geq 2$,

$$\boxed{\frac{n}{4}L \leq D \leq \frac{n-1}{2}L.} \tag{10}$$

Both constants are optimal. In particular, the uniform opposite-sign block $b_1 = \cdots = b_n = P/n$ is the unique fixed-mass maximizer of κ_d.

Proof. Substitution of (5) gives

$$D = \frac{1}{2}\sum_{j=1}^{n}(n+1-2j)b_j,$$

which is (9). Put $y_j = b_j - u$ and let k be the last index with $y_k \geq 0$. The total positive and negative deviation masses both equal $L/2$. Keeping only the cross pairs in the nonnegative ordered sum gives

$$2D \geq \sum_{i\leq k<j} (y_i - y_j) = \frac{n}{2}L,$$

which is the lower bound. Equality holds for every two-level deviation profile: the y_i are constant on $i \le k$ and on $i > k$.

For the upper bound, the coefficients $w_j = (n+1-2j)/2$ satisfy $|w_j| \le (n-1)/2$, and hence

$$D = \sum_j w_j y_j \le \frac{n-1}{2} \sum_j |y_j|.$$

Equality is obtained when only y_1 and y_n are nonzero and opposite. Finally $D = 0$ exactly when every pairwise difference vanishes. $\square$

The exact formula also orders the entire cone without solving any further optimization problem. Recall that a decreasing vector b majorizes a decreasing vector c of the same mass if $\sum_{j=1}^{k} b_j \ge \sum_{j=1}^{k} c_j$ for every $k < n$.

Corollary 4.2 (strict Schur concavity on the one-spike cone)**.** *Let $b, c \in (0, \infty)^n$ be decreasing and have the same sum P, and let F_b, F_c be one-spike targets with opposite-sign blocks b, c. If b majorizes c, then*

$$\kappa_d(F_b) \le \kappa_d(F_c).$$

Equality holds if and only if $b = c$. Thus making the opposite-sign spectral mass more unequal strictly lowers the inverse cost, while the uniform block is its unique fixed-mass maximum.

Proof. Writing $B_k = \sum_{j=1}^{k} b_j$ and using Theorem 3.1,

$$\kappa_d(F_b) = \sum_{j=1}^{n} j b_j = nP - \sum_{k=1}^{n-1} B_k.$$

The corresponding identity for c and the defining prefix inequalities give the result. Equality of the costs forces equality of every prefix, hence $b = c$. $\square$

5 Exact finite-switch curvature synthesis

Let $x_1, \ldots, x_m$ be Hermitian matrices with $\sum_j x_j = 0$ and define

$$H_{\text{geo}}(x) = \frac{i}{2} \sum_{k>j} [x_k, x_j], \qquad S_2(x) = \sum_{j=1}^{m} \|x_j\|_{\text{HS}}. \tag{11}$$

For a traceless Hermitian target F, set

$$\mathcal{A}_m(F) = \inf\{S_2(x)^2 : \sum_j x_j = 0,\ H_{\text{geo}}(x) = F\}. \tag{12}$$

Scalar components can be deleted from every edge without changing the target and while weakly decreasing S_2, so the definition is already gauge-minimal.

5.1 Three kicks

Theorem 5.1 (exact triangular action)**.** *For every finite-dimensional traceless Hermitian F,*

$$\boxed{\mathcal{A}_3(F) = 12\sqrt{3}\,\kappa_d(F).} \tag{13}$$

Proof. The zero target is attained by the zero loop, so assume $F \neq 0$. Balance gives $x_3 = -x_1 - x_2$ and $F = -i[x_1, x_2]/2$. In the real Hilbert space of Hermitian matrices, remove from x_2 its projection onto x_1. The commutator is unchanged, so

$$\kappa_d(F) \leq \frac{1}{2}\sqrt{\|x_1\|_{\mathrm{HS}}^2 \, \|x_2\|_{\mathrm{HS}}^2 - \langle x_1, x_2\rangle_{\mathrm{HS}}^2} =: \Delta.$$

Here Δ is the Euclidean area of the triangle with edge vectors x_1, x_2, x_3. The sharp triangular isoperimetric inequality gives $S_2^2 \geq 12\sqrt{3}\,\Delta$, proving the lower bound.

Choose H, K as in (3), put $\alpha = 2/\sqrt[4]{3}$, and set

$$x_1 = \alpha H, \qquad x_2 = \alpha\left(-\frac{1}{2}H + \frac{\sqrt{3}}{2}K\right), \qquad x_3 = -x_1 - x_2.$$

The three norms equal $\alpha\sqrt{\kappa_d(F)}$ and direct calculation gives $H_{\mathrm{geo}} = F$. Thus $S_2^2 = 9\alpha^2\kappa_d(F) = 12\sqrt{3}\,\kappa_d(F)$. □

5.2 Four kicks and the Gram correction

Theorem 5.2 (exact quadrilateral action)**.** *For every finite-dimensional traceless Hermitian F,*

$$\boxed{\mathcal{A}_4(F) = 16\,\kappa_d(F).} \tag{14}$$

Proof. The zero target is attained by the zero loop, so assume $F \neq 0$. For any balanced four-tuple, put

$$D_1 = x_1 + x_2, \qquad D_2 = x_2 + x_3.$$

A direct expansion gives the diagonal identity

$$H_{\mathrm{geo}} = -\frac{i}{2}[D_1, D_2]. \tag{15}$$

Write

$$a = \|D_1\|_{\mathrm{HS}}^2, \qquad b = \|D_2\|_{\mathrm{HS}}^2, \qquad c = \langle D_1, D_2\rangle_{\mathrm{HS}}, \qquad q = \sqrt{ab - c^2}.$$

Shearing D_2 by its component parallel to D_1 leaves the commutator unchanged, so (15) implies

$$\kappa_d(F) \leq \frac{q}{2}. \tag{16}$$

Balance also gives $D_1 + D_2 = x_2 - x_4$ and $D_1 - D_2 = x_1 - x_3$. Therefore

$$\begin{aligned} S_2 &\geq \|D_1 + D_2\|_{\mathrm{HS}} + \|D_1 - D_2\|_{\mathrm{HS}}, \\ S_2^2 &\geq 2(a+b) + 2\sqrt{(a-b)^2 + 4(ab - c^2)} \\ &\geq 8q \geq 16\kappa_d(F). \end{aligned} \tag{17}$$

The penultimate step uses $a + b \geq 2q$ and $\sqrt{(a-b)^2 + 4q^2} \geq 2q$.

For attainment, choose the orthogonal equal-norm optimum H, K from (3), set $D_1 = \sqrt{2}H$, $D_2 = \sqrt{2}K$, and take

$$x_1 = \frac{D_1 - D_2}{2}, \quad x_2 = \frac{D_1 + D_2}{2}, \quad x_3 = -x_1, \quad x_4 = -x_2. \tag{18}$$

Then every edge has norm $\sqrt{\kappa_d(F)}$, $H_{\mathrm{geo}} = F$, and $S_2^2 = 16\kappa_d(F)$. □

Remark 5.3 (why the determinant is necessary). It is false in general that $(\|D_1 + D_2\| + \|D_1 - D_2\|)^2$ is at least $4(\|D_1\|^2 + \|D_2\|^2)$; nearly parallel diagonals give counterexamples. The invariant controlled by the commutator is the Gram area q, and (17) is the valid sharp inequality.

Corollary 5.4 (universal three-versus-four switching gain)**.** *For every nonzero traceless Hermitian target in every finite dimension,*

$$\boxed{\frac{\mathcal{A}_4(F)}{\mathcal{A}_3(F)} = \frac{4}{3\sqrt{3}} < 1.} \tag{19}$$

Equivalently, the fourth kick reduces the optimum squared perimeter by $1 - 4/(3\sqrt{3}) \approx 23.02\%$*. On the one-spike cone this becomes*

$$\mathcal{A}_3(F) = 12\sqrt{3}\sum_{j=1}^{n} jb_j, \qquad \mathcal{A}_4(F) = 16\sum_{j=1}^{n} jb_j.$$

The displayed optimal square uses four equal-norm edges and an internal factor of rank n*.*

6 Relation to earlier records and development provenance

The dimension-three, dimension-four, and dimension-five inverse-cost papers already contain the corresponding fixed-dimensional formulas and triangular loop identities [13, 10, 9]. The operational-curvature paper already uses balanced three- and four-kick commutator loops [12], and the matrix-isoperimetry paper establishes the exact four-edge value on the sign-paired spectral face [11]. Accordingly, Theorem 5.1 is an all-dimensional consolidation, while the new scope of Theorem 5.2 is its extension from that equality face to arbitrary traceless Hermitian targets.

An earlier local, unpublished development package titled *One-Spike Inverse Self-Commutators in Every Dimension* contains the one-spike formula, rigidity, and stability lemmas from which this consolidated paper was developed. Its hashes and chronology are preserved in the accompanying priority audit. A prepared submission sheet is not treated as evidence of public disclosure. Public searches of the author's ARR archive, repositories, and release assets through 30 August 2026 did not locate that exact manuscript; this negative evidence is not an exhaustive priority guarantee.

The complementary paper *Sharp Rank-Adaptive Bounds for Inverse Self-Commutators* is now public as ARR-2026-1D2QV1RP1292JREW [14]. It proves the general rank-adaptive bounds, the uniform one-spike endpoint, both equality classifications, and the exact sign-cut leakage identity. It does not contain the arbitrary one-spike formula, every-optimizer singular rigidity, the sharp stability theorem, or the all-target kick laws proved here. The present paper should therefore be read as an exact interior-cone and synthesis companion, not as a replacement for that general extremal theorem.

7 Examples and boundary of the formula

Qubit. For spectrum $(P, -P)$, $\kappa_2(F) = P$ and the formulas give $\mathcal{A}_3 = 12\sqrt{3}P$ and $\mathcal{A}_4 = 16P$.

A zero-padded repeated spectrum. For nonzero spectrum $(6, -3, -2, -1)$ in any $d \geq 4$,

$$\kappa_d = 3 + 2 \cdot 2 + 3 \cdot 1 = 10, \qquad \operatorname{spec}_{>0}(CC^*) = (12, 6, 2).$$

Thus every balanced optimum has rank three, while the exact loop costs are $120\sqrt{3}$ and 160.

Why one-spike is a real restriction. For $F = \mathrm{diag}(5, 1, -3, -3)$, a direct four-level Horn certificate gives $\kappa_4(F) \geq \lambda_1 - \lambda_3 = 8$. Equality is attained by

$$C = E_{12} + 3E_{14} + \sqrt{3}E_{23} + \sqrt{3}E_{43}, \qquad CC^* - C^*C = 2F, \qquad \frac{1}{2}\|C\|_{\mathrm{HS}}^2 = 8.$$

Collapsing the two positive levels into a fictitious spike would instead produce $3 + 2 \cdot 3 = 9$. Hence (5) is not an unrestricted spectral formula. The universal loop laws remain valid because they retain $\kappa_d(F)$ rather than replacing it by the one-spike expression.

8 Reproducibility, limitations, and conclusion

The reproduction package contains two independent exact-arithmetic routes. One checks tail identities, stability bounds and equality families over rational compositions. The other checks the weighted-shift and matrix-loop identities directly, includes nearly parallel diagonals that falsify the discarded inequality, and verifies the corrected Gram chain. These programs test algebraic consequences and fixtures; they do not certify Horn's theorem or replace the displayed arbitrary-dimensional proofs.

The result is finite-dimensional and uses distinct roles for three notions: factor product κ_d, loop perimeter S_2, and squared perimeter $\mathcal{A}_m$. No claim is made about quadratic edge energy, operator-norm action, open paths, coincident-time limits, or higher Magnus orders. The priority audit found classical mechanisms close to both halves of the paper and exact self-overlap in low-dimensional slices. Novelty therefore remains a scoped literature-search conclusion rather than a mathematical theorem.

The main structural point is that a difficult inverse matrix invariant can be read operationally. On an entire high-rank spectral cone it is an explicit ordered moment of the negative eigenvalues and rigidifies all optimal singular data. Without any spectral restriction, it is exactly the resource converted by the optimal triangle and square. The resulting fourth-switch gain is universal, and the inverse cost is strictly Schur-concave on the one-spike simplex, even though it is not known in closed form on general sign patterns.

Author and AI disclosure

Lluis Eriksson directed the research, selected the claims, and accepts responsibility for the manuscript. OpenAI Codex assisted with algebraic exploration, literature triage, adversarial proof checking, exact replay, typesetting, and drafting. No external peer review or formal proof certification is claimed.

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