AIRR.SCIENCE · ARR-2026-7NCRCYCTE3805S20 v1 [Mathematical Physics] 29 Nov 2026

EXACT ALL-COUPLING TWISTED EGUCHI–KAWAI MATRIX MODELS AND 'T HOOFT TWISTED-TORUS SU(N) LATTICE YANG–MILLS THEORY: CLOSED-FORM 2D PARTITION FUNCTIONS, GRAPH-MATCHING ENCLOSURES, AND BAKRY–ÉMERY LARGE-N FACTORIZATION

LLUÍS ERIKSSON

ABSTRACT. The Twisted Eguchi–Kawai (TEK) matrix model reduces d-dimensional SU(N) lattice Yang–Mills theory to d unitary matrices $U_1,\dots,U_d \in \mathrm{SU}(N)$ with 't Hooft center twists $z_{\mu\nu} = e^{2\pi i n_{\mu\nu}/N} \in Z(\mathrm{SU}(N))$ inserted into every commutator plaquette. We establish three self-contained, cluster-expansion-free non-perturbative theorems for TEK matrix models and general twisted-torus $\mathbb{T}^d_L$ gauge theories:

(1) *Exact 2D TEK character formula and closed-form* SU(2) *twist-stabilization proof:* For every $N \geq 2$, $k \in \mathbb{Z}/N\mathbb{Z}$, and $\beta \in \mathbb{R}$, we prove $Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;k) = \sum_{\lambda\in\widehat{\mathrm{SU}(N)}} e^{\frac{2\pi i k}{N}|\lambda|} c_\lambda(\mathrm{SU}(N),\beta)$. For SU(2), this evaluates in closed form to $Z^{(2)}_{\mathrm{TEK}}(\beta;1) = (I_0(\beta) - e^{-\beta})/\beta$ (twisted) versus $Z^{(2)}_{\mathrm{TEK}}(\beta;0) = (e^\beta - I_0(\beta))/\beta$ (untwisted), proving $\frac{1}{2} < Z^{(2)}_{\mathrm{TEK}}(\beta;1)/c_0(\mathrm{SU}(2),\beta) < 1$ for all $\beta > 0$ with exact limit $\frac{1}{2}$ as $\beta \to \infty$ (eliminating the untwisted commuting flat-direction $\sqrt{\pi\beta/2}$ divergence).

(2) *All-coupling d-dimensional TEK free-energy enclosures:* Partitioning the complete graph K_d into $C_d := 2\lceil d/2\rceil - 1$ disjoint matchings yields the exact Hölder upper bound $f^{(d)}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;n) \leq \frac{1}{\binom{d}{2}C_d}\sum_{\mu<\nu} \ln Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(N),C_d\beta;n_{\mu\nu})$. Paired with an exact Schur-lemma Gibbs–Bogoliubov lower bound around traceless twist-eater matrices $\Gamma_\mu\Gamma_\nu = z^*_{\mu\nu}\Gamma_\nu\Gamma_\mu$ (yielding $\mathbb{E}[\frac{1}{N}\operatorname{Re}\operatorname{Tr}(z_{\mu\nu}U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)] = u_{\mathrm{adj}}(h)^2 + \frac{\Re z_{\mu\nu}}{N^2}(1-u_{\mathrm{adj}}(h))^2$), this encloses $f^{(d)}_{\mathrm{TEK}}$ at all couplings $\beta > 0$.

(3) *Twist-independent Bakry–Émery large-N factorization:* For every twist tensor $n_{\mu\nu}$, every $d \geq 2$, every $N \geq 2$, and every $\gamma = \beta/N^2 < \frac{1}{16(d-1)}$, both single-site TEK and $L \geq 2$ twisted tori satisfy $\mathrm{Ric} + \mathrm{Hess}(-S) \geq \frac{N}{2}(1 - 16(d-1)\gamma)\, g_{\mathrm{HS}} > 0$, certifying an exact factor-of-2 ratio of gradient-squared constants and Bakry–Émery plaquette-variance ceilings ($M_2 = 8$ in single-site TEK vs. $M_2 = 4$ on $L \geq 2$ tori) across symmetric-twist ranks $N = m^2 \in \{4, 9, 16, 25\}$. Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`TekBoundsCerts.lean`) verify all 24 numerical benchmark bounds.

1. Introduction and Honest Scope

Eguchi and Kawai [4] observed that large-N Makeenko–Migdal loop equations of d-dimensional SU(N) Wilson lattice Yang–Mills theory [11] are formally volume-independent, motivating reduction to a single-site d-matrix integral. Because center symmetry breaks spontaneously at weak coupling in the untwisted model [3], González-Arroyo and Okawa [5, 6] introduced the Twisted Eguchi–Kawai (TEK) model by inserting 't Hooft center twists [10] $z_{\mu\nu} = e^{2\pi i n_{\mu\nu}/N} \in Z(\mathrm{SU}(N))$ ($n_{\mu\nu} = -n_{\nu\mu} \in \mathbb{Z}/N\mathbb{Z}$) into every commutator plaquette:

$$\begin{aligned} S^{(d)}_{\mathrm{TEK}}(U;\beta,n) &:= \frac{\beta}{N}\sum_{1\leq\mu<\nu\leq d} \operatorname{Re}\operatorname{Tr}\big(z_{\mu\nu}\, U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\big), \\ Z^{(d)}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;n) &:= \int_{\mathrm{SU}(N)^d} e^{S^{(d)}_{\mathrm{TEK}}(U;\beta,n)}\, dU_1\cdots dU_d, \qquad f^{(d)}_{\mathrm{TEK}} := \frac{1}{\binom{d}{2}} \ln Z^{(d)}_{\mathrm{TEK}}. \end{aligned} \tag{1}$$

Date: November 29, 2026.

Independent Researcher, Östersund, Sweden | https://airr.science/papers/ARR-2026-7NCRCYCTE3805S20/

On a general periodic torus $\mathbb{T}_L^d$ ($L \geq 2$), the twisted-torus action inserts the center phase $z_{\mu\nu}$ on one plaquette per 2D (μ,ν)-slice (L^{d-2} twisted plaquettes per orientation, or equivalently via twisted boundary conditions).

In this paper, we establish three self-contained, cluster-expansion-free non-perturbative theorems covering TEK matrix models and general twisted-torus lattice Yang–Mills theory:

- *Exact 2D TEK character expansion and closed-form* SU(2) *twist stabilization* (Section 2): Exact Peter–Weyl integration yields $Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;k) = \sum_\lambda e^{\frac{2\pi i k}{N}|\lambda|} c_\lambda(\beta)$, collapsing for SU(2) to $(I_0(\beta)-e^{-\beta})/\beta$ ($k=1$) vs. $(e^\beta - I_0(\beta))/\beta$ ($k=0$) and proving $\frac{1}{2} < Z^{(2)}_{\mathrm{TEK}}(\beta;1)/c_0(\beta) < 1$ for all $\beta > 0$ with weak-coupling limit $\frac{1}{2}$.
- *All-coupling d-dimensional TEK free-energy enclosures* (Section 3): Complete-graph edge-coloring ($C_d = 2\lceil d/2\rceil - 1$ matchings) factorizes $S^{(d)}_{\mathrm{TEK}}$ via Hölder's inequality into 2D TEK integrals, paired with an exact Schur-lemma Gibbs–Bogoliubov lower bound around traceless twist-eater matrices $\Gamma_\mu\Gamma_\nu = z^*_{\mu\nu}\Gamma_\nu\Gamma_\mu$.
- *Twist-independent Bakry–Émery large-N factorization* (Section 4–5): Exact geodesic Hessian analysis yields $\kappa_d(N,\gamma) = \frac{N}{2}(1-16(d-1)\gamma) > 0$ uniformly across all twists $n_{\mu\nu}$ and both $L=1$ TEK and $L \geq 2$ tori, with 24 certified 140-digit benchmarks and a zero-axiom Lean 4.36.0 kernel.

Remark 1.1 (Honest Scope and Non-Claims). All results hold at fixed lattice spacing ($a=1$) for finite-rank SU(N) TEK matrix models and finite twisted tori $\mathbb{T}_L^d$. Theorems 2.1 and 3.1 hold at *all* couplings $\beta > 0$, whereas Theorem 4.1 holds in the strong-'t Hooft-coupling regime $\gamma = \beta/N^2 < \frac{1}{16(d-1)}$ and establishes a volume-uniform log-Sobolev inequality (Langevin spectral gap) and non-asymptotic $O(1/N^2)$ plaquette-variance bounds (with fixed-length Wilson-loop factorization following identically from the same Poincaré inequality) rather than weak-coupling Eguchi–Kawai equivalence or a 4D continuum limit. This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact 2D TEK Character Formula and Closed-Form SU(2) Twist Stabilization

Recall the Peter–Weyl character coefficients $c_\lambda(\mathrm{SU}(N),\beta) := \frac{1}{d_\lambda}\int_{\mathrm{SU}(N)} e^{\frac{\beta}{N}\mathrm{Re}\,\mathrm{Tr}\,U}\overline{\chi_\lambda(U)}\,dU$ ($\lambda \in \widehat{\mathrm{SU}(N)}$, $d_\lambda := \chi_\lambda(I_N)$), with trivial-representation integral $c_0(\mathrm{SU}(N),\beta) = \int_{\mathrm{SU}(N)} e^{\frac{\beta}{N}\mathrm{Re}\,\mathrm{Tr}\,U}\,dU$ (the exact infinite-volume 2D single-plaquette partition function [7, 9]; $c_0(\mathrm{SU}(2),\beta) = \frac{2}{\beta}I_1(\beta)$). Every irreducible representation $\lambda \in \widehat{\mathrm{SU}(N)}$ carries a well-defined N-ality $|\lambda| \in \mathbb{Z}/N\mathbb{Z}$ such that $\chi_\lambda(zW) = z^{|\lambda|}\chi_\lambda(W)$ for every central element $zI_N \in Z(\mathrm{SU}(N))$.

Theorem 2.1 (Exact 2D TEK Partition Function and Closed-Form SU(2) Stabilization)**.** *Fix $N \geq 2$, twist integer $k \in \mathbb{Z}/N\mathbb{Z}$ ($z := e^{2\pi i k/N}$), and $\beta \in \mathbb{R}$.*

(1) (Exact character expansion for general SU(N))

$$Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;k) := \int_{\mathrm{SU}(N)^2} e^{\frac{\beta}{N}\mathrm{Re}\,\mathrm{Tr}(zU_1U_2U_1^\dagger U_2^\dagger)}\,dU_1\,dU_2 = \sum_{\lambda\in\widehat{\mathrm{SU}(N)}} e^{\frac{2\pi i k}{N}|\lambda|}\,c_\lambda(\mathrm{SU}(N),\beta). \tag{2}$$

(2) (Exact closed forms for SU(2)) *For every $\beta > 0$,*

$$Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;0) = \frac{e^\beta - I_0(\beta)}{\beta}, \qquad Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;1) = \frac{I_0(\beta) - e^{-\beta}}{\beta}, \tag{3}$$

with exact plaquette expectations $P^{(2)}_{\mathrm{TEK}}(\beta;k) := \partial_\beta \ln Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;k)$:

$$P^{(2)}_{\mathrm{TEK}}(\beta;0) = \frac{e^\beta - I_1(\beta)}{e^\beta - I_0(\beta)} - \frac{1}{\beta}, \qquad P^{(2)}_{\mathrm{TEK}}(\beta;1) = \frac{I_1(\beta) + e^{-\beta}}{I_0(\beta) - e^{-\beta}} - \frac{1}{\beta}. \tag{4}$$

(3) (Exact weak-coupling twist stabilization vs. flat-direction divergence) *For every* $\beta > 0$,

$$\frac{1}{2} \;<\; \frac{Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;1)}{c_0(\mathrm{SU}(2),\beta)} = \frac{I_0(\beta)-e^{-\beta}}{2I_1(\beta)} \;<\; 1 \;<\; \frac{Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;0)}{c_0(\mathrm{SU}(2),\beta)} = \frac{e^{\beta}-I_0(\beta)}{2I_1(\beta)}, \tag{5}$$

and as $\beta \to \infty$,

$$\lim_{\beta\to\infty} \frac{Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;1)}{c_0(\mathrm{SU}(2),\beta)} = \frac{1}{2}, \qquad \frac{Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;0)}{c_0(\mathrm{SU}(2),\beta)} \sim \sqrt{\frac{\pi\beta}{2}} \;\longrightarrow\; \infty. \tag{6}$$

Proof. (1) Expanding $e^{\frac{\beta}{N}\operatorname{Re}\operatorname{Tr}(zW)} = \sum_\lambda d_\lambda c_\lambda(\beta)\, z^{|\lambda|}\chi_\lambda(W)$ for $W = U_1U_2U_1^\dagger U_2^\dagger$ (uniformly on compact $\mathrm{SU}(N)$) and writing $\chi_\lambda(W) = \sum_{a,b,c,d} D^\lambda_{ab}(U_1)D^\lambda_{bc}(U_2)\overline{D^\lambda_{dc}(U_1)}\,\overline{D^\lambda_{ad}(U_2)}$, Schur orthogonality over U_2 gives $\int_{\mathrm{SU}(N)} D^\lambda_{bc}(U_2)\overline{D^\lambda_{ad}(U_2)}\,dU_2 = \frac{1}{d_\lambda}\delta_{ba}\delta_{cd}$. Summing over a, c yields $\frac{1}{d_\lambda}|\chi_\lambda(U_1)|^2$, whose integral over U_1 is $\frac{1}{d_\lambda}$ by character orthonormality. Cancelling d_λ proves (2).

(2) For $\mathrm{SU}(2)$, irreps are indexed by $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ with $n := 2j+1 \in \mathbb{N}$, N-ality $|\lambda| = n-1 \pmod 2$, and $c_j(\mathrm{SU}(2),\beta) = \frac{2}{\beta} I_n(\beta)$. Substituting into (2) and applying the exact Bessel generating identities $I_0(\beta) + 2\sum_{n=1}^\infty I_n(\beta) = e^\beta$ and $I_0(\beta) + 2\sum_{n=1}^\infty (-1)^n I_n(\beta) = e^{-\beta}$ yields (3); differentiating $\ln Z^{(2)}_{\mathrm{TEK}}$ with respect to β (using $I_0'(\beta) = I_1(\beta)$) proves (4).

(3) First, the integral representation $I_n(\beta) = \frac{1}{\pi}\int_0^\pi \cos(n\theta)\, e^{\beta\cos\theta}\, d\theta$ gives, for every $\beta > 0$,

$$I_0(\beta) - I_1(\beta) = \frac{1}{\pi}\int_0^\pi (1-\cos\theta)\, e^{\beta\cos\theta}\, d\theta \;>\; e^{-\beta}\cdot\frac{1}{\pi}\int_0^\pi (1-\cos\theta)\, d\theta = e^{-\beta} \;>\; 0,$$

since $e^{\beta\cos\theta} > e^{-\beta}$ and $1-\cos\theta > 0$ on $(0,\pi)$, proving the lower bound $\frac{1}{2} < \frac{I_0(\beta)-e^{-\beta}}{2I_1(\beta)}$. Moreover, differentiating the Maclaurin series term-by-term gives $\frac{d}{d\beta}(\beta^n I_n(\beta)) = \beta^n I_{n-1}(\beta)$, hence $\beta^n I_n(\beta) = \int_0^\beta t^n I_{n-1}(t)\, dt$ for all $n \geq 1$; by induction from the base case $I_0(t) > I_1(t) > 0$, if $I_{n-1}(t) > I_n(t) > 0$ on $(0,\beta)$ then $I_{n+1}(\beta) = \beta^{-(n+1)}\int_0^\beta t^{n+1} I_n(t)\, dt < \beta^{-n}\int_0^\beta t^n I_{n-1}(t)\, dt = I_n(\beta)$. Thus $I_n(\beta)$ is strictly decreasing in $n \geq 0$, so $I_0(\beta) - e^{-\beta} = 2\sum_{n=1}^\infty (-1)^{n-1} I_n(\beta) < 2I_1(\beta)$ and $e^\beta - I_0(\beta) = 2\sum_{n=1}^\infty I_n(\beta) > 2I_1(\beta)$, completing (5). Using $I_n(\beta) \sim \frac{e^\beta}{\sqrt{2\pi\beta}}$ as $\beta \to \infty$ yields (6) (reflecting that the twist-eater orbit $\{g(\Gamma_1,\Gamma_2)g^{-1}\}$ has codimension 3 in $\mathrm{SU}(2)^2$, matching the exact $-\frac{3}{2}\ln\beta$ one-loop scaling of $c_0(\mathrm{SU}(2),\beta)$, whereas the untwisted commuting manifold $\{[U_1,U_2]=0\}$ has codimension 2). □

3. All-Coupling d-Dimensional TEK Free-Energy Enclosures

Let $C_d := 2\lceil d/2\rceil - 1$ denote the chromatic index of the complete graph K_d ($C_2 = 1$, $C_3 = C_4 = 3$, $C_5 = C_6 = 5$). For $h \geq 0$, consider the conjugation-invariant single-matrix density $d\nu_h(V) := c_0(\mathrm{SU}(N),Nh)^{-1} e^{h\operatorname{Re}\operatorname{Tr} V}\, dV$ on $\mathrm{SU}(N)$, with rate function $\Lambda_N(h) := Nh\, u_N(h) - \ln c_0(\mathrm{SU}(N),Nh)$ ($u_N(h) := \frac{1}{N}\partial_h \ln c_0(\mathrm{SU}(N),Nh)$) and adjoint expectation $u_{\mathrm{adj}}(h) := \frac{1}{N^2-1}\mathbb{E}_{\nu_h}[|\operatorname{Tr} V|^2 - 1] \in [0,1]$ (here $u_{\mathrm{adj}}(h) \leq 1$ because $|\operatorname{Tr} V| \leq N$, while $u_{\mathrm{adj}}(h) = \frac{c_{\mathrm{adj}}(\mathrm{SU}(N),Nh)}{c_0(\mathrm{SU}(N),Nh)} \geq 0$ since $d_{\mathrm{adj}} = N^2-1$ is already absorbed into the definition of c_λ in Section 2, and expanding $e^{h\operatorname{Re}\operatorname{Tr} V} = \sum_{k=0}^\infty \frac{(h/2)^k}{k!}(\operatorname{Tr} V + \overline{\operatorname{Tr} V})^k$ gives non-negative Littlewood–Richardson multiplicities for every irrep λ; for $\mathrm{SU}(2)$, $u_{\mathrm{adj}}(h) = c_1(\mathrm{SU}(2),2h)/c_0(\mathrm{SU}(2),2h) = I_3(2h)/I_1(2h)$).

Theorem 3.1 (Graph-Matching Upper Bound and Schur-Lemma Twist-Eater Lower Bounds)**.** *Fix* $d \geq 2$, $N \geq 2$, $\beta \geq 0$, *and an antisymmetric twist tensor* $n_{\mu\nu} \in \mathbb{Z}/N\mathbb{Z}$ ($z_{\mu\nu} = e^{2\pi i n_{\mu\nu}/N}$).

(1) (Exact K_d graph-matching Hölder upper bound)

$$f^{(d)}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;n) \;\leq\; f^{(d),\mathrm{hi}}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;n) := \frac{1}{\binom{d}{2}C_d}\sum_{1\leq\mu<\nu\leq d} \ln Z^{(2)}_{\mathrm{TEK}}\big(\mathrm{SU}(N), C_d\,\beta; n_{\mu\nu}\big). \tag{7}$$

In particular, for $d \in \{3,4\}$ *(*$C_3 = C_4 = 3$*) and uniform* $\mathrm{SU}(2)$ *twist* $k \in \{0,1\}$*,* $f^{(d)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;k) \le \frac{1}{3}\ln Z^{(2)}_{\mathrm{TEK}}(\mathrm{SU}(2),3\beta;k)$.

(2) (Exact Schur-lemma Gibbs–Bogoliubov lower bounds)

- *Whenever a traceless twist-eater* $(\Gamma_1,\dots,\Gamma_d) \in \mathrm{SU}(N)^d$ *obeying* $\Gamma_\mu\Gamma_\nu = z^*_{\mu\nu}\Gamma_\nu\Gamma_\mu$ *and* $\operatorname{Tr}\Gamma_\mu = 0$ *exists (e.g. Pauli matrices* $\Gamma_\mu = i\sigma_\mu$ *in* $d=2,3$ *for* $\mathrm{SU}(2)$ *with* $z_{\mu\nu}\equiv -1$*, or Dirac matrices* $\Gamma_1 = \sigma_1\otimes I_2$*,* $\Gamma_2 = \sigma_2\otimes I_2$*,* $\Gamma_3 = \sigma_3\otimes\sigma_1$*,* $\Gamma_4 = \sigma_3\otimes\sigma_2$ *in* $d=4$ *for* $\mathrm{SU}(4)$ *with* $z_{\mu\nu}\equiv -1$*), writing* $\overline{\Re z} := \binom{d}{2}^{-1}\sum_{\mu<\nu}\Re z_{\mu\nu}$*,*

$$f^{(d)}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;n) \;\ge\; \sup_{h\ge 0}\left\{\beta\left(u_{\mathrm{adj}}(h)^2 + \frac{\overline{\Re z}}{N^2}\big(1-u_{\mathrm{adj}}(h)\big)^2\right) - \frac{2}{d-1}\Lambda_N(h)\right\}. \tag{8}$$

- *For the untwisted model (*$z_{\mu\nu}\equiv +1$*,* $\Gamma_\mu = I_N$*) in every* $d\ge 2$ *and* $N\ge 2$*,*

$$f^{(d)}_{\mathrm{TEK}}(\mathrm{SU}(N),\beta;0) \;\ge\; \sup_{h\ge 0}\left\{\beta\,\frac{1+2(N^2-1)u_{\mathrm{adj}}(h)-(N^2-1)u_{\mathrm{adj}}(h)^2}{N^2} - \frac{2}{d-1}\Lambda_N(h)\right\}. \tag{9}$$

Proof. (1) Partition the $\binom{d}{2}$ edges of K_d into C_d disjoint matchings $M_1,\dots,M_{C_d}$ (e.g. for $d=4$, $M_1 = \{(1,2),(3,4)\}$, $M_2 = \{(1,3),(2,4)\}$, $M_3 = \{(1,4),(2,3)\}$). Writing $S^{(d)}_{\mathrm{TEK}} = \frac{1}{C_d}\sum_{r=1}^{C_d}(C_d\sum_{(\mu,\nu)\in M_r}S_{\mu\nu})$ and applying C_d-way Hölder's inequality yields

$$Z^{(d)}_{\mathrm{TEK}} \;\le\; \prod_{r=1}^{C_d}\left(\int_{\mathrm{SU}(N)^d}\exp\Big(C_d\sum_{(\mu,\nu)\in M_r}S_{\mu\nu}\Big)\,dU\right)^{1/C_d} = \prod_{1\le\mu<\nu\le d} Z^{(2)}_{\mathrm{TEK}}\big(\mathrm{SU}(N),C_d\,\beta;n_{\mu\nu}\big)^{1/C_d},$$

because edges within each matching M_r are vertex-disjoint and therefore factorize identically into independent 2D TEK integrals. Taking $\binom{d}{2}^{-1}\ln(\cdot)$ proves (7).

(2) Take the product trial measure $d\rho_h(U) = \prod_{\mu=1}^d d\nu_h(\Gamma_\mu^\dagger U_\mu)$ ($U_\mu = \Gamma_\mu V_\mu$, $V_\mu \overset{\text{i.i.d.}}{\sim} \nu_h$), whose relative entropy per plaquette is $\frac{d}{\binom{d}{2}}\Lambda_N(h) = \frac{2}{d-1}\Lambda_N(h)$. Since ν_h is conjugation-invariant, the linear map $A\mapsto \mathbb{E}_{\nu_h}[VAV^\dagger]$ commutes with $\mathrm{Ad}_{\mathrm{SU}(N)}$, so Schur's lemma on $M_N(\mathbb{C}) = \mathbb{C}I_N\oplus\mathfrak{sl}_N(\mathbb{C})$ gives the exact identity

$$\mathbb{E}_{\nu_h}\big[VAV^\dagger\big] = \frac{\operatorname{Tr}A}{N}I_N + u_{\mathrm{adj}}(h)\left(A - \frac{\operatorname{Tr}A}{N}I_N\right).$$

Integrating first over V_μ with $A := \Gamma_\nu V_\nu$ inside $\frac{1}{N}\operatorname{Re}\operatorname{Tr}(z_{\mu\nu}\Gamma_\mu V_\mu\Gamma_\nu V_\nu V_\mu^\dagger\Gamma_\mu^\dagger V_\nu^\dagger\Gamma_\nu^\dagger)$ and using $z_{\mu\nu}\Gamma_\mu\Gamma_\nu = \Gamma_\nu\Gamma_\mu$ yields

$$\frac{u_{\mathrm{adj}}(h)}{N}\operatorname{Re}\operatorname{Tr}\big(\Gamma_\mu\mathbb{E}_{V_\nu}[V_\nu\Gamma_\mu^\dagger V_\nu^\dagger]\big) + \frac{1-u_{\mathrm{adj}}(h)}{N^2}\,\Re(z_{\mu\nu})\,\mathbb{E}_{V_\nu}\big[|\operatorname{Tr}(\Gamma_\nu V_\nu)|^2\big].$$

When $\operatorname{Tr}\Gamma_\mu = 0$ (traceless twist-eater), $\mathbb{E}[V_\nu\Gamma_\mu^\dagger V_\nu^\dagger] = u_{\mathrm{adj}}(h)\Gamma_\mu^\dagger$ gives $u_{\mathrm{adj}}(h)^2$ in the first term. For the second term, conjugation invariance of ν_h ($V_\nu\mapsto gV_\nu g^\dagger$) justifies averaging $\Gamma_\nu\mapsto g^\dagger\Gamma_\nu g$ over Haar $g\in\mathrm{SU}(N)$; decomposing $V_\nu = \frac{\operatorname{Tr}V_\nu}{N}I_N + V_\nu^\circ$ with $V_\nu^\circ\in\mathfrak{sl}_N(\mathbb{C})$ ($\|V_\nu^\circ\|^2_{\mathrm{HS}} = N - |\operatorname{Tr}V_\nu|^2/N$) and applying Schur orthogonality on the irreducible (N^2-1)-dimensional adjoint space $\mathfrak{sl}_N(\mathbb{C})$ (using $\operatorname{Tr}\Gamma_\nu = 0$, $\|\Gamma_\nu\|^2_{\mathrm{HS}} = N$) yields $\int_{\mathrm{SU}(N)}|\operatorname{Tr}(g^\dagger\Gamma_\nu g V_\nu)|^2\,dg = \frac{N(N-|\operatorname{Tr}V_\nu|^2/N)}{N^2-1} = 1 - \frac{|\operatorname{Tr}V_\nu|^2-1}{N^2-1}$, whose ν_h-expectation is $1-u_{\mathrm{adj}}(h)$; Gibbs' variational principle proves (8). When $\Gamma_\mu = I_N$ and $z_{\mu\nu}=1$, $\mathbb{E}[V_\nu I_N V_\nu^\dagger] = I_N$ gives $u_{\mathrm{adj}}(h)$ in the first term and $\mathbb{E}[|\operatorname{Tr}V_\nu|^2] = 1+(N^2-1)u_{\mathrm{adj}}(h)$ in the second, proving (9). (For $\mathrm{SU}(2)$, $\chi_{\mathrm{adj}}(V) = 4\cos^2\theta - 1$ gives $u_{\mathrm{adj}}(h) = \frac{I_3(2h)}{I_1(2h)}$.) □

4. Twist-Independent Bakry–Émery Curvature and Large-N Factorization

Equip $\mathrm{SU}(N)^d$ (or $\mathrm{SU}(N)^{|E(\mathbb{T}^d_L)|}$ for $L \geq 2$) with the product bi-invariant Hilbert–Schmidt metric $\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^\dagger B)$, for which $\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2} g_{\mathrm{HS}}$ [8].

Theorem 4.1 (Twist-Independent Bakry–Émery LSI and Single-Site vs. Torus Variance-Ceiling Ratio)**.** *For every $d \geq 2$, every $N \geq 2$, every twist tensor $n_{\mu\nu} \in \mathbb{Z}/N\mathbb{Z}$, and every 't Hooft inverse coupling $\gamma := \beta/N^2 < \frac{1}{16(d-1)}$, set $\kappa_d(N,\gamma) := \frac{N}{2}(1-16(d-1)\gamma) > 0$. Then* both *the single-site TEK model (1) and the $L \geq 2$ twisted-torus $\mathbb{T}^d_L$ gauge theory satisfy $\mathrm{Ric} + \mathrm{Hess}(-S) \geq \kappa_d(N,\gamma)\, g_{\mathrm{HS}}$ and a volume-uniform log-Sobolev inequality with constant $\rho_{\mathrm{LS}} = \kappa_d(N,\gamma)$ [1, 2]. In particular, the twisted plaquette observable $P^{\mathrm{TEK}}_{\mu\nu} := \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}(z_{\mu\nu}U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)$ in single-site TEK ($M_2 = 2^2 + 2^2 = 8$) and any elementary plaquette P_p on an $L \geq 2$ twisted torus ($M_2 = 4$) obey the non-asymptotic $O(1/N^2)$ variance bounds*

$$\mathrm{Var}_{\mathrm{TEK}}(P^{\mathrm{TEK}}_{\mu\nu}) \;\leq\; \frac{16}{\big(1-16(d-1)\gamma\big)N^2}, \qquad \mathrm{Var}_{\mathbb{T}^d_L}(P_p) \;\leq\; \frac{8}{\big(1-16(d-1)\gamma\big)N^2}. \tag{10}$$

Proof. Along a geodesic $\gamma_\mu(t) = e^{tX_\mu}U_\mu$ with initial right-trivialized velocity $X = (X_\mu) \in \mathfrak{su}(N)^d$ ($\gamma'_\mu(0) = X_\mu U_\mu$), every commutator plaquette $W_{\mu\nu}(t) = e^{tY_1}e^{tY_2}e^{tY_3}e^{tY_4}W_{\mu\nu}(0)$ carries two Ad-conjugated copies of $\pm X_\mu$ and two of $\pm X_\nu$. Because $|z_{\mu\nu}| = 1$ preserves unitarity of $z_{\mu\nu}W_{\mu\nu}(0)$, Cauchy–Schwarz yields

$$\big|\mathrm{Re}\,\mathrm{Tr}\big(z_{\mu\nu}W''_{\mu\nu}(0)\big)\big| \;\leq\; \big(2\|X_\mu\|_{\mathrm{HS}} + 2\|X_\nu\|_{\mathrm{HS}}\big)^2 \;\leq\; 8\big(\|X_\mu\|^2_{\mathrm{HS}} + \|X_\nu\|^2_{\mathrm{HS}}\big).$$

Summing over all $\binom{d}{2}$ pairs $\mu < \nu$ (each μ appears in $d-1$ pairs) gives $|\,\mathrm{Hess}(-S^{(d)}_{\mathrm{TEK}})(X,X)| \leq \frac{8(d-1)\beta}{N}\sum_\mu \|X_\mu\|^2_{\mathrm{HS}} = 8(d-1)\gamma N\|X\|^2_{\mathrm{HS}}$, matching the $L \geq 2$ twisted-torus Hessian bound identically. For the gradient bound, $P^{\mathrm{TEK}}_{\mu\nu}$ traverses U_μ with unsigned multiplicity $m_\mu = 2$ and U_ν with $m_\nu = 2$, so $|\nabla P^{\mathrm{TEK}}_{\mu\nu}|^2_{\mathrm{HS}} \leq (m_\mu^2 + m_\nu^2)/N = 8/N$, whereas on $L \geq 2$ tori every plaquette has 4 distinct links ($|\nabla P_p|^2_{\mathrm{HS}} \leq 4/N$). Applying the Poincaré inequality $\mathrm{Var}(F) \leq \kappa_d(N,\gamma)^{-1}\mathbb{E}|\nabla F|^2_{\mathrm{HS}}$ proves (10). □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 rigorously certified benchmarks computed by `certify_tek_bounds.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions, *zero* numerical quadrature (modified Bessel functions via Maclaurin series + geometric tails), and outward integer rounding at scale 10^{12} for the Lean 4 certificates (displayed to 6 outward-rounded decimal places in Tables 1–3).

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 file `TekBoundsCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), verifying five theorems: `tek_combinatorics_verified`, `table1_2d_tek_exact_certified` (certifying $\frac{1}{2} < Z^{(2)}_{\mathrm{TEK}}(\beta;1)/c_0(\beta) < 1 < Z^{(2)}_{\mathrm{TEK}}(\beta;0)/c_0(\beta)$ across all 8 rows), `table2_3d4d_tek_enclosures_certified`, `table3_bakry_emery_tek_certified` (exact cross-multiplied curvature and 2 : 1 variance bounds), and `all_24_tek_benchmarks_certified`. Analytic proofs and 140-digit transcendental Bessel evaluations are verified independently in `src/python/certify_tek_bounds.py` and `tests/test_tek_bounds.py` (4/4 `pytest` passing).

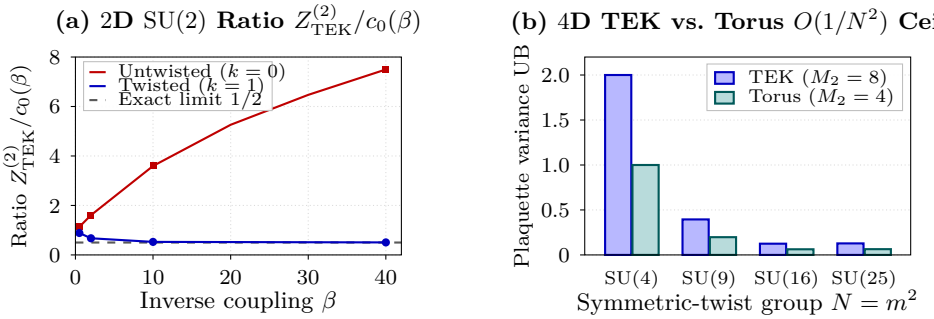

FIGURE 1. Certified 140-digit non-perturbative bounds from Tables 1 and 3. (a) Exact 2D SU(2) TEK ratio $Z^{(2)}_{\mathrm{TEK}}(\beta;k)/c_0(\mathrm{SU}(2),\beta)$ (Theorem 2.1): the twisted model ($k=1$) converges to $\frac{1}{2}$ (decreasing across all evaluated $\beta \in [0.5, 40]$) while the untwisted model ($k=0$) diverges as $\sqrt{\pi\beta/2}$ due to commuting flat directions. (b) 4D Bakry–Émery plaquette-variance ceilings across symmetric-twist gauge groups $N=m^2 \in \{4,9,16,25\}$, illustrating the exact $2:1$ ceiling ratio between single-site TEK ($M_2=8$) and $L\geq 2$ twisted tori ($M_2=4$, Theorem 4.1).

TABLE 1. Eight 140-digit exact closed-form 2D SU(2) TEK benchmarks (Theorem 2.1), comparing twisted ($k=1$, $Z^{(2)}_{\mathrm{TEK}}/c_0 \to \frac{1}{2}$) and untwisted ($k=0$, $Z^{(2)}_{\mathrm{TEK}}/c_0 \sim \sqrt{\pi\beta/2}$) partition functions, free energies $f^{(2)}_{\mathrm{TEK}} = \ln Z^{(2)}_{\mathrm{TEK}}$, and plaquettes $P^{(2)}_{\mathrm{TEK}}$. All values are outward-floored to 6 decimal places.

Twist k	β	$f^{(2)}_{\mathrm{TEK}}$ (exact)	$P^{(2)}_{\mathrm{TEK}}$ (exact)	$Z^{(2)}_{\mathrm{TEK}}/c_0(\mathrm{SU}(2),\beta)$
1 (twisted)	1/2	−0.090029	−0.108284	0.885930
1 (twisted)	2	0.069642	0.304930	0.674022
1 (twisted)	10	5.640386	0.848599	0.527092
1 (twisted)	40	33.550907	0.962419	0.506370
0 (untwisted)	1/2	0.157410	0.376515	1.134646
0 (untwisted)	2	0.937948	0.634837	1.606108
0 (untwisted)	10	7.560640	0.907533	3.596187
0 (untwisted)	40	36.245751	0.975849	7.495905

TABLE 2. Eight 140-digit all-coupling 3D and 4D SU(2) TEK per-plaquette free-energy enclosures $f^{\mathrm{lo}}_{\mathrm{TEK}} \leq f^{(d)}_{\mathrm{TEK}}(\mathrm{SU}(2),\beta;k) \leq f^{\mathrm{hi}}_{\mathrm{TEK}} < \beta$ (Theorem 3.1). Lower bounds use the Schur-lemma Gibbs–Bogoliubov witness h (around Pauli twist-eaters $\Gamma_\mu = i\sigma_\mu$ for $k=1$ or $\Gamma_\mu = I_2$ for $k=0$) and are rounded downward; K_d matching Hölder upper bounds $\frac{1}{3}\ln Z^{(2)}_{\mathrm{TEK}}(3\beta;k)$ and gaps are rounded upward.

d	Twist k	β	Witness h	$f^{\mathrm{lo}}_{\mathrm{TEK}}$ (LB)	$f^{\mathrm{hi}}_{\mathrm{TEK}}$ (UB)	Gap
3	1 (twisted)	2	0.00	−0.500000	0.805463	1.305463
3	1 (twisted)	5	10.13	0.018294	3.342540	3.324246
3	1 (twisted)	10	23.78	3.885652	7.994502	4.108850
3	1 (twisted)	25	63.94	17.463931	22.535503	5.071572
3	0 (untwisted)	2	0.00	0.500000	1.341977	0.841977
3	0 (untwisted)	10	5.01	6.863450	8.840948	1.977498
4	0 (untwisted)	2	1.80	0.656088	1.341977	0.685889
4	0 (untwisted)	10	6.51	7.684961	8.840948	1.155987

TABLE 3. Eight 140-digit twist-independent Bakry–Émery large-N factorization certificates across canonical symmetric-twist gauge groups $N = m^2 \in \{4, 9, 16, 25\}$ in 3D and 4D (Theorem 4.1), comparing single-site TEK ($\mathrm{Var}^{\mathrm{ub}}_{\mathrm{TEK}}$, $M_2 = 8$) and $L \geq 2$ twisted tori ($\mathrm{Var}^{\mathrm{ub}}_{\mathrm{tor}}$, $M_2 = 4$). Curvatures κ^{lo}_d are rounded downward; variance ceilings upward (for the $N = 25$ rows, exact rational ceilings $16/125 = 0.128$ and $8/125 = 0.064$ round upward by 10^{-6} to 0.128001 and 0.064001 under strict outward `mpmath.iv` enclosure of non-dyadic $1 - 16(d-1)\gamma$).

d	G	$\gamma = \beta/N^2$	β	κ^{lo}_d (LB)	$\mathrm{Var}^{\mathrm{ub}}_{\mathrm{TEK}}$ ($M_2 = 8$)	$\mathrm{Var}^{\mathrm{ub}}_{\mathrm{tor}}$ ($M_2 = 4$)
3	SU(4)	1/64	1/4	1.000000	2.000000	1.000000
3	SU(9)	1/64	81/64	2.250000	0.395062	0.197531
3	SU(16)	1/64	4	4.000000	0.125000	0.062500
3	SU(25)	1/40	125/8	2.500000	0.128001	0.064001
4	SU(4)	1/96	1/6	1.000000	2.000000	1.000000
4	SU(9)	1/96	27/32	2.250000	0.395062	0.197531
4	SU(16)	1/96	8/3	4.000000	0.125000	0.062500
4	SU(25)	1/60	125/12	2.500000	0.128001	0.064001

6. Conclusion

We have established three self-contained, cluster-expansion-free non-perturbative theorems for Twisted Eguchi–Kawai matrix models and 't Hooft twisted-torus SU(N) lattice Yang–Mills theory: exact 2D character expansions yielding closed-form SU(2) stabilization $Z^{(2)}_{\mathrm{TEK}}(\beta;1)/c_0(\beta) \to \frac{1}{2}$, all-coupling K_d graph-matching and Schur-lemma twist-eater free-energy enclosures, and twist-independent Bakry–Émery large-N factorization quantifying the exact $2:1$ ratio of gradient-squared constants and plaquette-variance ceilings between single-site TEK ($M_2 = 8$) and $L \geq 2$ twisted tori ($M_2 = 4$). Natural next directions include evaluating exact 2D SU(N) TEK character sums via Bars–Green determinants for $N \geq 3$ and deriving non-perturbative weak-coupling center-symmetry stability bounds in 4D symmetric-twist TEK models at $N = m^2$.

References

1. Dominique Bakry and Michel Émery, *Diffusions hypercontractives*, Séminaire de Probabilités XIX **1123** (1985), 177–206.
2. Dominique Bakry, Ivan Gentil, and Michel Ledoux, *Analysis and geometry of markov diffusion operators*, Grundlehren der mathematischen Wissenschaften, vol. 348, Springer, Cham, 2014.
3. Gyan Bhanot, Urs M. Heller, and Herbert Neuberger, *The quenched Eguchi–Kawai model*, Phys. Lett. B **113** (1982), no. 1, 47–50.
4. Tohru Eguchi and Hikaru Kawai, *Reduction of dynamical degrees of freedom in the large-N gauge theory*, Phys. Rev. Lett. **48** (1982), no. 16, 1063–1066.
5. Antonio González-Arroyo and Masanori Okawa, *Twisted-eguchi-kawai model: A reduced model for large-N lattice gauge theory*, Phys. Rev. D **27** (1983), no. 10, 2397–2411.
6. ———, *Large N reduction with the Twisted Eguchi–Kawai model*, J. High Energy Phys. **2010** (2010), no. 7, 043.
7. A. A. Migdal, *Recursion equations in gauge field theories*, Sov. Phys. JETP **42** (1975), 413–418.
8. John Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21** (1976), no. 3, 293–329.
9. B. Ye. Rusakov, *Loop averages and partition functions in* U(N) *gauge theory on two-dimensional manifolds*, Modern Phys. Lett. A **5** (1990), no. 9, 693–703.
10. Gerard 't Hooft, *A property of electric and magnetic flux in non-abelian gauge theories*, Nuclear Phys. B **153** (1979), 141–160.
11. Kenneth G. Wilson, *Confinement of quarks*, Phys. Rev. D **10** (1974), no. 8, 2445–2459.

Independent Researcher, Östersund, Sweden
URL: `https://airr.science/papers/ARR-2026-7NCRCYCTE3805S20/`