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EXACT ALL-COUPLING SU(2) GEORGI–GLASHOW (ADJOINT-HIGGS) LATTICE YANG–MILLS THEORY: κ-UNIFORM FUNDAMENTAL CONFINEMENT, β-INDEPENDENT ADJOINT STRING BREAKING, COMPACT U(1) LONDON ENCLOSURES, AND SHARP BAKRY–ÉMERY CURVATURE

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Abstract. The d-dimensional ($d \geq 2$) SU(2) Georgi–Glashow lattice gauge theory couples Wilson plaquettes at inverse coupling $\beta \geq 0$ to unit-length adjoint-representation Higgs fields $\Phi_x = \vec{\phi}_x \cdot \vec{\sigma}$ ($\vec{\phi}_x \in S^2$) at hopping coupling $\kappa \geq 0$. In contrast to the fundamental Fradkin–Shenker model (where fundamental strings break at every $\kappa > 0$), the adjoint-Higgs action preserves global $Z(\mathrm{SU}(2)) = \mathbb{Z}_2$ center symmetry identically for all $\kappa \in [0, \infty]$. Without invoking polymer or cluster expansions, we establish three self-contained non-perturbative theorems:

(1) *Simultaneous κ-uniform fundamental confinement and β-independent adjoint string breaking:* Four-line Ginibre $\mathbb{Z}_2$ domination plus Weighted-Kraft queue closure proves $\underline{\sigma}^{\mathrm{GG}}_{\mathrm{fund}}(d, \beta, \kappa) \geq -\ln(\tanh|\beta|/\rho_d^4) > 0$ uniformly across *all* $\kappa \in [0, \infty]$ (including the $\kappa = \infty$ compact U(1) London limit) whenever $\tanh|\beta| < \rho_d^4$ ($\rho_3 = 0.55$, $\rho_4 = 0.39$), while a single-site adjoint-Higgs Rayleigh–Ritz trial state on the Osterwalder–Seiler transfer matrix simultaneously proves $V^{\mathrm{GG}}_{\mathrm{adj},L_s}(d, \beta, \kappa; R) \leq 2\ln(1/m(\kappa)) + 8(d-1)\kappa < \infty$ ($\sigma^{\mathrm{GG}}_{\mathrm{adj}} = 0$) for all $L_s \geq 3$, $1 \leq R \leq L_s - 1$, $\beta \geq 0$, and $\kappa > 0$ via the exact Langevin function $m(\kappa) = \coth\kappa - \kappa^{-1}$.

(2) *All-coupling free-energy enclosures and compact* U(1) *London reduction:* (U, Φ)-variable d-way axial-gauge Hölder upper bounds paired with factorized unitary-gauge Abelian-Higgs Gibbs–Bogoliubov lower bounds enclose $f_{\mathrm{GG}}(d, \beta, \kappa)$ across all $(\beta, \kappa) \in [0, \infty)^2$, recovering both $\kappa = 0$ pure SU(2) and the $\kappa \to \infty$ compact U(1) London limit.

(3) *Sharp Bakry–Émery Ricci curvature:* Exact unitary-gauge geodesic Hessian analysis proves $|\mathrm{Hess}(-S^{\mathrm{unit}}_{\mathrm{Higgs}})(X, X)| \leq 2\kappa\|X\|^2_{\mathrm{HS}}$ (attained sharply with $+2\kappa\|X\|^2_{\mathrm{HS}}$ at $U = I_2, X = i\sigma_1$ and $-2\kappa\|X\|^2_{\mathrm{HS}}$ at $U = i\sigma_1, X = i\sigma_1$), certifying $\kappa_{\mathrm{GG}} = 1 - 4(d-1)\beta - 2\kappa > 0$, a volume-uniform log-Sobolev inequality, and $\mathrm{Var}_{\mathrm{GG}}(P_p) \leq 2/\kappa_{\mathrm{GG}}$.

Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`GeorgiGlashowCerts.lean`) verify all 24 numerical bounds.

1. Introduction and Honest Scope

The lattice SU(2) Georgi–Glashow model [5, 13, 12, 10, 4, 3] is the canonical non-Abelian gauge-Higgs system with an unbroken residual U(1) subgroup ($\mathrm{SU}(2) \to \mathrm{U}(1)$) and unbroken global center $Z(\mathrm{SU}(2)) = \mathbb{Z}_2$. On a periodic hypercubic torus $\mathbb{T}^d_L$ ($d \geq 2$, $V = L^d$, $|E| = dV$, $|P| = \binom{d}{2}V$), assign gauge links $U_e \in \mathrm{SU}(2)$ and unit-length adjoint Higgs fields $\vec{\phi}_x \in S^2$ ($\Phi_x := \vec{\phi}_x \cdot \vec{\sigma}$, $\Phi_x^\dagger = \Phi_x$, $\Phi_x^2 = I_2$, $\mathrm{Tr}\,\Phi_x = 0$), with Euclidean action and partition function

$$\begin{aligned} S_{\mathrm{GG}}(U, \Phi; \beta, \kappa) &:= \frac{\beta}{2} \sum_{p \in P} \mathrm{Tr}(U_p) + \frac{\kappa}{2} \sum_{e=(x,\mu)} \mathrm{Tr}\big(\Phi_x U_e \Phi_{x+e_\mu} U_e^\dagger\big), \\ Z_{\mathrm{GG}}(\mathbb{T}^d_L; \beta, \kappa) &:= \int_{\mathrm{SU}(2)^{|E|} \times (S^2)^V} e^{S_{\mathrm{GG}}(U, \Phi; \beta, \kappa)}\, dU\, d\Phi, \end{aligned} \tag{1}$$

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where dU and $d\Phi$ are normalized product Haar/surface measures, and $R(U) \in \mathrm{SO}(3)$ denotes the adjoint representation defined by $U(\vec{v}\cdot\vec{\sigma})U^\dagger = (R(U)\vec{v})\cdot\vec{\sigma}$, so $\frac{1}{2}\mathrm{Tr}(\Phi_x U_e \Phi_{x+e_\mu} U_e^\dagger) = \vec{\phi}_x \cdot R(U_e)\vec{\phi}_{x+e_\mu} \in [-1,1]$.

Because $\mathrm{SU}(2)$ acts transitively on S^2, local gauge invariance permits fixing *unitary gauge* $\vec{\phi}_x \equiv \hat{e}_3$ ($\Phi_x \equiv \sigma_3$) at every site $x \in V$, reducing the adjoint-Higgs action to an exact independent product over links:

$$S^{\mathrm{unit}}_{\mathrm{Higgs}}(U;\kappa) = \frac{\kappa}{2}\sum_{e\in E}\mathrm{Tr}\big(\sigma_3 U_e \sigma_3 U_e^\dagger\big) = \kappa\sum_{e\in E}\big(2|u_{11}(U_e)|^2 - 1\big) = \kappa\sum_{e\in E} y_e, \tag{2}$$

where parametrizing $U_e = \begin{pmatrix} \cos\theta_e\, e^{i\psi_e} & \sin\theta_e\, e^{i\chi_e} \\ -\sin\theta_e\, e^{-i\chi_e} & \cos\theta_e\, e^{-i\psi_e}\end{pmatrix}$ ($\theta_e \in [0,\pi/2]$, $\psi_e, \chi_e \in [0,2\pi)$) with $y_e := \cos(2\theta_e) = 2|u_{11}(U_e)|^2 - 1 \in [-1,1]$ factorizes Haar measure on $\mathrm{SU}(2)$ identically as

$$dU_e = \frac{dy_e}{2}\,\frac{d\psi_e}{2\pi}\,\frac{d\chi_e}{2\pi}. \tag{3}$$

Hence the single-link $\beta = 0$ adjoint-Higgs partition function and expectation are the exact elementary functions

$$Z_{\mathrm{link}}(\kappa) := \int_{\mathrm{SU}(2)} e^{\kappa y}\, dU = \frac{\sinh\kappa}{\kappa}, \qquad m(\kappa) := \frac{\partial}{\partial\kappa}\ln Z_{\mathrm{link}}(\kappa) = \coth\kappa - \frac{1}{\kappa} \in (0,1). \tag{4}$$

Remark 1.1 (Honest Scope and Non-Claims). All results hold at fixed lattice spacing ($a = 1$) for d-dimensional $\mathrm{SU}(2)$ Georgi–Glashow lattice gauge theory. Theorem 2.1(2) (adjoint string breaking) and Theorem 3.1 (free-energy enclosures) hold at *all* couplings $(\beta,\kappa) \in [0,\infty)^2$ ($\kappa > 0$ for breaking), whereas Theorem 2.1(1) holds in the strong-gauge-coupling slab $\tanh|\beta| < \rho_d^4$ (uniformly for *all* $\kappa \in [0,\infty]$) and Theorem 4.1 holds in the symmetric corner $4(d-1)\beta + 2\kappa < 1$ (certifying a volume-uniform log-Sobolev inequality, not a 4D continuum mass gap or 3D Polyakov monopole-gas continuum limit). This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Simultaneous κ-Uniform Fundamental Confinement and Adjoint String Breaking

For $d \geq 3$ and $\rho \in (0,1)$, let $F_d(\rho) := \frac{\rho}{2}[(1+\rho)^{2d-3} - (1-\rho)^{2d-3}]$ denote the odd-subset Weighted-Kraft branching polynomial ($F_3(0.55) = 0.99900625 \leq 1$, $F_4(0.39) = 0.995362843761 \leq 1$), and $F_{C,d}(\rho) := \frac{1}{2}[(1+\rho)^{2(d-1)} - (1-\rho)^{2(d-1)}]$. Define the fundamental lower string tension

$$\underline{\sigma}^{\mathrm{GG}}_{\mathrm{fund}}(d,\beta,\kappa) := \liminf_{R,T\to\infty}\liminf_{L\to\infty} -\frac{1}{RT}\ln\left|\left\langle\frac{1}{2}\mathrm{Tr}\,U_{C_{R,T}}\right\rangle_{\mathrm{GG}}\right|,$$

and, via the Osterwalder–Seiler transfer matrix $\mathcal{T}_d^{\mathrm{adj}(0,R)} \geq 0$ on the Hilbert space $\mathcal{H}_d^{\mathrm{adj}(0,R)}$ of two static adjoint sources at 0 and Re_1 on $\mathbb{T}^{d-1}_{L_s}$ (with vacuum eigenvalue $\Lambda_0 > 0$), define the finite-volume static adjoint ground-state potential and asymptotic adjoint string tension by

$$V^{\mathrm{GG}}_{\mathrm{adj},L_s}(d,\beta,\kappa;R) := -\ln\frac{\|\mathcal{T}_d^{\mathrm{adj}(0,R)}\|}{\Lambda_0}, \qquad \sigma^{\mathrm{GG}}_{\mathrm{adj}}(d,\beta,\kappa) := \limsup_{R\to\infty}\limsup_{L_s\to\infty}\frac{V^{\mathrm{GG}}_{\mathrm{adj},L_s}(d,\beta,\kappa;R)}{R}.$$

Theorem 2.1 (Simultaneous κ-Uniform Fundamental Area Law and β-Independent Adjoint Breaking)**.** *Fix $d \geq 3$, $\beta \in \mathbb{R}$, and $\kappa \geq 0$.*

(1) (κ-uniform constructive fundamental confinement) *Let $\rho_d \in (0,1)$ satisfy $F_d(\rho_d) \leq 1$ (e.g. $\rho_3 = 55/100$, $\rho_4 = 39/100$). Whenever $0 < \tanh|\beta| < \rho_d^4$, every $R \times T$ rectangular fundamental Wilson loop on $\mathbb{T}^d_L$ ($1 \leq R, T \leq L-1$) with $RT \leq L^2/2$ (in particular, all*

$1 \le R, T \le L/2$*) obeys uniformly in* L *and in* all $\kappa \in [0,\infty]$ *(including the* $\kappa = \infty$ *compact* U(1) *London limit),*

$$\left|\left\langle \frac{1}{2}\operatorname{Tr} U_{C_{R,T}} \right\rangle_{\mathrm{GG}}\right| \;\le\; F_{C,d}(\rho_d)^{2(R+T)} \left(\frac{\tanh|\beta|}{\rho_d^4}\right)^{RT} \;\Longrightarrow\; \sigma^{\mathrm{GG}}_{\mathrm{fund}}(d,\beta,\kappa) \;\ge\; -\ln\!\left(\frac{\tanh|\beta|}{\rho_d^4}\right) \;>\; 0. \tag{5}$$

(2) (β-independent universal adjoint string breaking) *For every* $\beta \ge 0$*, every* $\kappa > 0$*, every spatial side* $L_s \ge 3$*, and every integer separation* $1 \le R \le L_s - 1$*, the static adjoint ground-state potential and the two-gluelump temporal correlator* $\mathcal{G}_{\mathrm{gl}}(R,T) := \langle \widetilde{\Phi}, (\mathcal{T}_d^{\mathrm{adj}(0,R)}/\Lambda_0)^T \widetilde{\Phi}\rangle / \|\widetilde{\Phi}\|_2^2$ *($T \ge 1$) obey*

$$\begin{aligned} V^{\mathrm{GG}}_{\mathrm{adj},L_s}(d,\beta,\kappa;R) \;&\le\; -\lim_{T\to\infty}\frac{1}{T}\ln \mathcal{G}_{\mathrm{gl}}(R,T) \;\le\; 2\ln\!\left(\frac{1}{m(\kappa)}\right) + 8(d-1)\kappa \;<\; \infty, \\ \sigma^{\mathrm{GG}}_{\mathrm{adj}}(d,\beta,\kappa) &= 0. \end{aligned} \tag{6}$$

Proof. (1) Substitute $U_e \mapsto \sigma_e U_e$ ($\sigma_e \in \{\pm 1\} = Z(\mathrm{SU}(2))$). Because $\sigma_e^2 = 1$, the adjoint-Higgs term $\frac{\kappa}{2}\operatorname{Tr}(\Phi_x U_e \Phi_y U_e^\dagger)$ is *identically invariant* under every center flip $\sigma \in \{\pm 1\}^{|E|}$, whereas $\frac{\beta}{2}\operatorname{Tr}(U_p) \mapsto \sigma_p J_p(U)$ with $|J_p(U)| = |\frac{\beta}{2}\operatorname{Tr}(U_p)| \le |\beta|$ and $\operatorname{Tr} U_{C_{R,T}} \mapsto \sigma_{C_{R,T}} \operatorname{Tr} U_{C_{R,T}}$. Conditioning on any fixed (U,Φ) and applying four-line Ginibre $\mathbb{Z}_2$ domination [6, 7] ($\sigma_e = \eta_e \tau_e$, $K_p(\eta) = |\beta| + J_p(U)\eta_p \ge 0$) gives $|\mathbb{E}_{\sigma|(U,\Phi)}[\sigma_{C_{R,T}}]| \le \langle \sigma_{C_{R,T}}\rangle_{\mathbb{Z}_2}(|\beta|)$ pointwise for every (U,Φ) and every $\kappa \in [0,\infty]$. Averaging over (U,Φ) and applying the Weighted-Kraft FIFO link-queue code on $\mathbb{Z}_2$ plaquette surfaces (with branching sum $\le F_d(\rho_d) \le 1$, boundary factor $F_{C,d}(\rho_d)^{2(R+T)}$, and minimal homological area $|\mathcal{S}| \ge \min(RT, L^2 - RT) = RT$ on $\mathbb{T}_L^d$ whenever $RT \le L^2/2$) proves the first inequality of (5); taking $\liminf_{L\to\infty}$ prior to $R,T \to \infty$ gives the unconditional string-tension bound.

(2) Fix temporal gauge $U_{x,d} = I_2$ on $\mathbb{T}_{L_s}^{d-1} \times \mathbb{Z}$. By Osterwalder–Seiler reflection positivity [9, 11], the transfer matrix $\widetilde{\mathcal{T}}_d = \mathcal{V}^{1/2}\mathcal{K}\mathcal{V}^{1/2}$ is positive semi-definite on $L^2(\mathrm{SU}(2)^{E_s} \times (S^2)^{V_s})$ (since both $e^{\frac{\beta}{2}\operatorname{Tr}(U(U')^\dagger)}$ on SU(2) and $e^{\kappa\vec{\phi}\cdot\vec{\phi}'}$ on S^2 have strictly positive character/Legendre coefficients) with unique strictly positive normalized ground state $\Omega_{L_s} > 0$ ($\widetilde{\mathcal{T}}_d\Omega_{L_s} = \Lambda_0\Omega_{L_s}$). Set $S_0 := \{0, Re_1\}$ and define the marginalized gauge-invariant amplitude $\widehat{\Omega}(U, \vec{\phi}_{\setminus S_0}) := (\int_{(S^2)^2} \Omega_{L_s}(U,\vec{\phi})^2\, d\vec{\phi}_0\, d\vec{\phi}_{Re_1})^{1/2} > 0$. Removing only the at most $4(d-1)$ spatial adjoint-Higgs links touching S_0 (leaving all spatial plaquettes intact) defines $\mathcal{V}_{\setminus S_0} > 0$ obeying $e^{-4(d-1)\kappa} \le \mathcal{V}_{\setminus S_0}/\mathcal{V} \le e^{+4(d-1)\kappa}$. Define the $\mathrm{End}(\mathbb{R}^3)$-valued local two-gluelump trial state $\widetilde{\Phi} = (\widetilde{\Phi}_{ab})_{a,b=1}^3 \in \mathcal{H}_d^{\mathrm{adj}(0,R)}$ (transforming covariantly as $R(g_0)\widetilde{\Phi}R(g_{Re_1})^T$ under spatial gauge transformations $g \in \mathrm{SU}(2)^{V_s}$, equipped with the Hilbert–Schmidt inner product $\langle \Psi, \Phi\rangle = \sum_{a,b=1}^3 \langle \Psi_{ab}, \Phi_{ab}\rangle_{L^2}$):

$$\widetilde{\Phi}_{ab}(U,\vec{\phi}) := \left(\frac{\mathcal{V}_{\setminus S_0}}{\mathcal{V}}\right)^{1/2} (\vec{\phi}_0)_a (\vec{\phi}_{Re_1})_b\, \widehat{\Omega}(U, \vec{\phi}_{\setminus S_0}).$$

Since $\sum_{a,b=1}^3 (\vec{\phi}_0)_a^2 (\vec{\phi}_{Re_1})_b^2 = |\vec{\phi}_0|^2 |\vec{\phi}_{Re_1}|^2 = 1$ pointwise, $\|\widetilde{\Phi}\|_2^2 \le e^{+4(d-1)\kappa}$. Inside $\langle \widetilde{\Phi}, \mathcal{T}_d^{\mathrm{adj}(0,R)}\widetilde{\Phi}\rangle$, $\mathcal{V}^{1/2}$ cancels $(\mathcal{V}_{\setminus S_0}/\mathcal{V})^{1/2}$ identically, and

$$\sum_{a=1}^3 \int_{(S^2)^2} (\vec{\phi})_a (\vec{\phi}')_a e^{\kappa\vec{\phi}\cdot\vec{\phi}'}\, d\vec{\phi}\, d\vec{\phi}' = \int_{(S^2)^2} (\vec{\phi}\cdot\vec{\phi}') e^{\kappa\vec{\phi}\cdot\vec{\phi}'}\, d\vec{\phi}\, d\vec{\phi}' = m(\kappa) Z_{\mathrm{link}}(\kappa),$$

yielding the exact equality $\langle \widetilde{\Phi}, \mathcal{T}_d^{\mathrm{adj}(0,R)}\widetilde{\Phi}\rangle = m(\kappa)^2 Z_{\mathrm{link}}(\kappa)^2 I_{\setminus S_0}$, where $I_{\setminus S_0} > 0$ is the marginalized kernel expectation without the two temporal Higgs links at S_0. Conversely, bounding $\mathcal{V}^{1/2} \le e^{+2(d-1)\kappa}\mathcal{V}_{\setminus S_0}^{1/2}$ on both slices in $\Lambda_0 = \langle \Omega_{L_s}, \widetilde{\mathcal{T}}_d \Omega_{L_s}\rangle$ and applying Cauchy–Schwarz over $(\vec{\phi}_0, \vec{\phi}_{Re_1})$ gives $\Lambda_0 \le e^{+4(d-1)\kappa} Z_{\mathrm{link}}(\kappa)^2 I_{\setminus S_0}$. Cancelling $I_{\setminus S_0} Z_{\mathrm{link}}(\kappa)^2 > 0$ yields the Rayleigh quotient bound

$\langle\widetilde{\Phi},(\mathcal{T}_d^{\mathrm{adj}(0,R)}/\Lambda_0)\widetilde{\Phi}\rangle/\|\widetilde{\Phi}\|_2^2 \geq e^{-8(d-1)\kappa}m(\kappa)^2$. Because $\mathcal{T}_d^{\mathrm{adj}(0,R)}/\Lambda_0 \geq 0$ is positive self-adjoint, Jensen's inequality on its spectral probability measure for $\widetilde{\Phi}$ gives $\mathcal{G}_{\mathrm{gl}}(R,T) \geq (e^{-8(d-1)\kappa}m(\kappa)^2)^T$ for every $T \geq 1$, while $\mathcal{G}_{\mathrm{gl}}(R,T) \leq (\|\mathcal{T}_d^{\mathrm{adj}(0,R)}\|/\Lambda_0)^T$, establishing (6). (As standard in lattice string-breaking analyses, a bare pure-gauge adjoint Wilson loop achieves the same asymptotic decay rate whenever the spatial Wilson-line state has nonzero overlap with the ground-state sector.) □

3. All-Coupling Free-Energy Enclosures and Compact U(1) London Reduction

Let $f_{\mathrm{GG},L}(d,\beta,\kappa) := V^{-1}\ln Z_{\mathrm{GG}}(\mathbb{T}_L^d;\beta,\kappa)$ denote the finite-volume per-site free energy, write $f_{\mathrm{GG}}(d,\beta,\kappa)$ for any subsequential thermodynamic limit as $L\to\infty$ (the inequalities below hold for both $\liminf_{L\to\infty}$ and $\limsup_{L\to\infty}$), and set $c_0(\mathrm{SU}(2),\alpha) = 2I_1(\alpha)/\alpha$ (with continuous extensions $\sinh(0)/0 := 1$ and $m(0) := 0$ at $\kappa = 0$).

Theorem 3.1 (All-Coupling Georgi–Glashow Free-Energy Enclosures and London Reduction)**.** *For every $d \geq 2$, $\beta \geq 0$, and $\kappa \geq 0$,*

$$f_{\mathrm{GG}}^{\mathrm{lo}}(d,\beta,\kappa) \;\leq\; f_{\mathrm{GG}}(d,\beta,\kappa) \;\leq\; f_{\mathrm{GG}}^{\mathrm{hi}}(d,\beta,\kappa), \tag{7}$$

where the upper and lower bounds are given explicitly by

$$\begin{aligned} f_{\mathrm{GG}}^{\mathrm{hi}}(d,\beta,\kappa) &:= \min\left\{(d-1)\ln c_0\Big(\mathrm{SU}(2),\tfrac{d}{2}\beta\Big) + \ln\left(\frac{\sinh(d\kappa)}{d\kappa}\right),\ \binom{d}{2}\beta + d\ln\left(\frac{\sinh\kappa}{\kappa}\right)\right\},\\ f_{\mathrm{GG}}^{\mathrm{lo}}(d,\beta,\kappa) &:= d\ln\left(\frac{\sinh\kappa}{\kappa}\right) + \sup_{h\geq 0}\left\{\binom{d}{2}\beta\left(\frac{1+m(\kappa)}{2}\,\frac{I_1(h)}{I_0(h)}\right)^4 + d\big[\ln I_0(h) - h\tfrac{I_1(h)}{I_0(h)}\big]\right\}. \end{aligned} \tag{8}$$

In the $\kappa\to\infty$ London limit, the subtracted excess free energy $\widetilde{f}_{\mathrm{GG},L}(d,\beta,\kappa) := f_{\mathrm{GG},L}(d,\beta,\kappa) - d\ln(\sinh\kappa/\kappa)$ converges uniformly in L *to the finite-volume compact* U(1) *Wilson free energy $f_{\mathrm{U}(1),L}(d,\beta)$ (hence $\widetilde{f}_{\mathrm{GG}}(d,\beta,\kappa)\to f_{\mathrm{U}(1)}(d,\beta)$ in the thermodynamic limit), and the lower bound in* (8) *converges to the compact* U(1) *Gibbs–Bogoliubov bound*

$$\sup_{h\geq 0}\left\{\binom{d}{2}\beta\left(\frac{I_1(h)}{I_0(h)}\right)^4 + d\big[\ln I_0(h) - h\tfrac{I_1(h)}{I_0(h)}\big]\right\}.$$

Proof. For the first branch of $f_{\mathrm{GG}}^{\mathrm{hi}}$, decompose the (U,Φ)-variable action (1) across the d coordinate directions as $S_{\mathrm{GG}} = \frac{1}{d}\sum_{\mu=1}^d(\frac{d}{2}S_W^{(\mu)}(U) + d\,S_{\mathrm{Higgs}}^{(\mu)}(U,\Phi))$, where $S_W^{(\mu)}$ collects the $(d-1)V$ plaquettes with a leg along μ and $S_{\mathrm{Higgs}}^{(\mu)}$ collects the V Higgs links along μ. Applying d-way Hölder's inequality and fixing axial gauge $U_{x,1} = I_2$ on the $L-1$ slices $x_1\in\{0,\dots,L-2\}$ leaves L^{d-1} Polyakov links $U_{(L-1,x_\perp),1}$ on the L-th slice; bounding the L^{d-1} boundary Higgs links by $e^{d\kappa L^{d-1}}$ and the $(d-1)L^{d-1}$ boundary plaquettes by $e^{\frac{d(d-1)}{2}\beta L^{d-1}}$ decouples the remaining $(L-1)$-bond chains into L^{d-1} independent open 1D Heisenberg chains of coupling $d\kappa$ (exact bond integral $\sinh(d\kappa)/(d\kappa)$) and $(d-1)L^{d-1}$ independent open 1D SU(2) chains of coupling $\frac{d}{2}\beta$ (exact plaquette integral $c_0(\mathrm{SU}(2),\frac{d}{2}\beta)$). Taking $V^{-1}\ln(\cdot)$ and $L\to\infty$ eliminates the $O(1/L)$ boundary term and gives the first branch; the second branch follows immediately from $\frac{\beta}{2}\operatorname{Tr}(U_p)\leq\beta$ in unitary gauge (2).

For $f_{\mathrm{GG}}^{\mathrm{lo}}$, work in unitary gauge (2)–(3) with the factorized i.i.d. link trial measure

$$d\rho_{\kappa,h}(U_e) := \frac{e^{\kappa y_e}}{\sinh(\kappa)/\kappa}\,\frac{dy_e}{2}\cdot\frac{e^{h\cos\psi_e}}{I_0(h)}\,\frac{d\psi_e}{2\pi}\cdot\frac{d\chi_e}{2\pi}.$$

Because χ_e is uniform on $[0,2\pi)$, off-diagonal entries of U_e have zero mean, while the diagonal entries give $\mathbb{E}_{\rho_{\kappa,h}}[U_e] = c(\kappa)\frac{I_1(h)}{I_0(h)}I_2$ with $c(\kappa) := \mathbb{E}_\kappa[\sqrt{(1+y)/2}] \geq \mathbb{E}_\kappa[(1+y)/2] = \frac{1+m(\kappa)}{2}$ (since $\sqrt{u}\geq u$ on $[0,1]$). Hence $\mathbb{E}_{\rho_{\kappa,h}}[\frac{1}{2}\operatorname{Tr}U_p] = c(\kappa)^4(I_1(h)/I_0(h))^4 \geq (\frac{1+m(\kappa)}{2}\frac{I_1(h)}{I_0(h)})^4$. Meanwhile, the y_e-energy

$\kappa m(\kappa)$ cancels the y_e-entropy term $-\kappa m(\kappa)$ inside $\mathbb{E}[S^{\mathrm{unit}}_{\mathrm{Higgs}}] - D_{\mathrm{KL}}(\rho_{\kappa,h}\|\mathrm{Haar})$ identically, leaving $d\ln(\sinh\kappa/\kappa)+d[\ln I_0(h)-hI_1(h)/I_0(h)]$. Taking $\sup_{h\geq 0}$ proves (7)–(8); as $\kappa\to\infty$, $m(\kappa)\to 1$ gives $\frac{1+m(\kappa)}{2}\to 1$. Finally, writing $D(\psi_e) := \mathrm{diag}(e^{i\psi_e}, e^{-i\psi_e})$, we have $\|U_e - D(\psi_e)\|_{\mathrm{op}} \leq (1-\cos\theta_e) + \sin\theta_e \leq 2\sin\theta_e = \sqrt{2(1-y_e)}$, so telescoping around each plaquette gives $|S_W(U) - S_{\mathrm{U}(1)}(\psi)| \leq 2\sqrt{2}(d-1)\beta\sum_{e\in E}\sqrt{1-y_e}$. Because the y_e are i.i.d. under $\mu_\kappa(dy) = \frac{\kappa e^{\kappa y}}{2\sinh\kappa}\,dy$ independently of (ψ_e,χ_e), this yields $|\widetilde{f}_{\mathrm{GG},L}(d,\beta,\kappa) - f_{\mathrm{U}(1),L}(d,\beta)| \leq d\ln\mathbb{E}_{\mu_\kappa}[e^{2\sqrt{2}(d-1)\beta\sqrt{1-y}}] \to 0$ as $\kappa\to\infty$ uniformly in L by dominated convergence, justifying the $L\to\infty$ and $\kappa\to\infty$ limit interchange. □

4. Sharp Bakry–Émery Ricci Curvature and Log-Sobolev Inequality

Equip unitary-gauge $\mathrm{SU}(2)^{|E|}$ with the product bi-invariant Hilbert–Schmidt metric $\langle A,B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^\dagger B)$, for which $\mathrm{Ric}_{\mathrm{SU}(2)} = \frac{2}{2}g_{\mathrm{HS}} = g_{\mathrm{HS}}$ [8].

Theorem 4.1 (Sharp Unitary-Gauge Bakry–Émery Curvature and Volume-Uniform LSI)**.** *Fix $d\geq 2$, $L\geq 2$, $\beta\geq 0$, and $\kappa\geq 0$.*

(1) *Every geodesic $U_e(t) = e^{tX_e}U_e$ ($X=(X_e)\in\mathfrak{su}(2)^{|E|}$) satisfies the* sharp *adjoint-Higgs Hessian bound*

$$\left|\mathrm{Hess}(-S^{\mathrm{unit}}_{\mathrm{Higgs}})(X,X)\right| \;\leq\; 2\kappa\|X\|^2_{\mathrm{HS}}, \tag{9}$$

with both extrema attained sharply: $+2\kappa\|X\|^2_{\mathrm{HS}}$ at $(U_e,X_e)=(I_2,i\sigma_1)$ and $-2\kappa\|X\|^2_{\mathrm{HS}}$ (saturating the Bakry–Émery lower bound below) at $(U_e,X_e)=(i\sigma_1,i\sigma_1)$.

(2) *Whenever $\kappa_{\mathrm{GG}}(d,\beta,\kappa) := 1-4(d-1)\beta-2\kappa > 0$, $\mathrm{Ric}+\mathrm{Hess}(-S^{\mathrm{unit}}_{\mathrm{GG}}) \geq \kappa_{\mathrm{GG}}\,g_{\mathrm{HS}} > 0$ uniformly in $L\geq 2$, certifying a volume-uniform log-Sobolev inequality with constant $\rho_{\mathrm{LS}} = \kappa_{\mathrm{GG}}$ [1, 2], Poincaré spectral gap $\lambda_1\geq\kappa_{\mathrm{GG}}$, and plaquette variance ceiling $\mathrm{Var}_{\mathrm{GG}}(P_p)\leq 2/\kappa_{\mathrm{GG}}$ ($P_p = \frac{1}{2}\mathrm{Tr}\,U_p$).*

Proof. (1) For a single link with $X = i\vec{a}\cdot\vec{\sigma}\in\mathfrak{su}(2)$ ($\|X\|^2_{\mathrm{HS}} = 2|\vec{a}|^2$) and $\widetilde{\sigma} := U\sigma_3U^\dagger = \vec{n}\cdot\vec{\sigma}$ ($|\vec{n}|=1$), set $g(t) := \frac{1}{2}\mathrm{Tr}(e^{-tX}\sigma_3e^{tX}\widetilde{\sigma})$. Expanding $e^{-tX}\sigma_3e^{tX} = \sigma_3 + t[\sigma_3,X] + \frac{t^2}{2}[[\sigma_3,X],X] + O(t^3)$ gives $[[\sigma_3,X],X] = -4(a_1^2+a_2^2)\sigma_3 + 4a_3(a_1\sigma_1+a_2\sigma_2)$, whose Euclidean coefficient norm is

$$4\sqrt{(a_1^2+a_2^2)^2+a_3^2(a_1^2+a_2^2)} = 4|\vec{a}|\sqrt{a_1^2+a_2^2} \;\leq\; 4|\vec{a}|^2 = 2\|X\|^2_{\mathrm{HS}}.$$

Hence $|g''(0)| = |\frac{1}{2}\mathrm{Tr}([[\sigma_3,X],X]\widetilde{\sigma})| \leq 2\|X\|^2_{\mathrm{HS}}$, attained with $-g''(0) = +2\|X\|^2_{\mathrm{HS}}$ at $(U,X) = (I_2,i\sigma_1)$ ($g(t) = +\cos(2t)$) and with $-g''(0) = -2\|X\|^2_{\mathrm{HS}}$ at $(U,X)=(i\sigma_1,i\sigma_1)$ ($\widetilde{\sigma}=-\sigma_3$, $g(t) = -\cos(2t)$). Summing over $e\in E$ proves (9).

(2) Combining (9) with $|\mathrm{Hess}(-S_W)(X,X)| \leq \frac{8(d-1)\beta}{2}\|X\|^2_{\mathrm{HS}} = 4(d-1)\beta\|X\|^2_{\mathrm{HS}}$ and $\mathrm{Ric}_{\mathrm{SU}(2)^{|E|}} = g_{\mathrm{HS}}$ gives $\mathrm{Ric}+\mathrm{Hess}(-S^{\mathrm{unit}}_{\mathrm{GG}}) \geq \kappa_{\mathrm{GG}}\,g_{\mathrm{HS}} > 0$. Since $|\nabla P_p|^2_{\mathrm{HS}} \leq 4/2 = 2$, Poincaré's inequality gives $\mathrm{Var}_{\mathrm{GG}}(P_p)\leq 2/\kappa_{\mathrm{GG}}$. □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 certified benchmarks from `certify_georgi_glashow.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions and *zero* quadrature (Bessel series + geometric tails; hyperbolic functions via `mpmath.iv` exp; Lean 4 integers outward-rounded at scale 10^{12}).

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 file `GeorgiGlashowCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), verifying five theorems: `georgi_glashow_kraft_thresholds_verified`, `table1_confinement_and_adjoint_breaking_certified`, `table2_free_energy_london_enclosures_certified`, `table3_sharp_bakry_emery_certified`, and `all_24_georgi_glashow_`

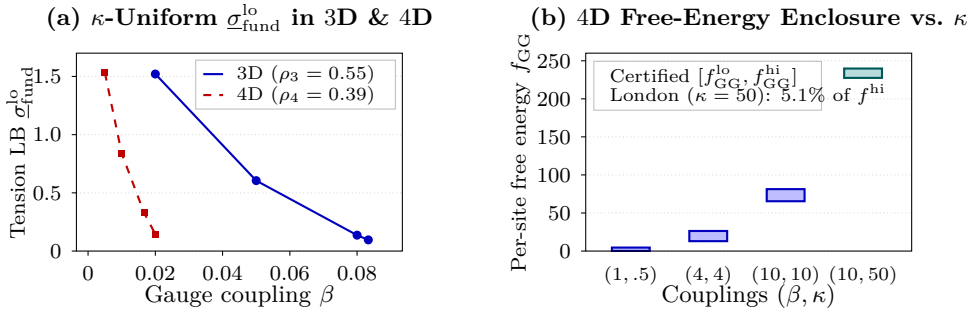

FIGURE 1. Certified 140-digit non-perturbative bounds for the SU(2) Georgi–Glashow adjoint-Higgs model (Tables 1 and 2). (a) Constructive fundamental string-tension lower bounds $\underline{\sigma}^{\mathrm{lo}}_{\mathrm{fund}} > 0$ valid uniformly across *all* $\kappa \in [0,\infty]$ (Theorem 2.1(1)), coexisting with universal adjoint string breaking $\sigma^{\mathrm{GG}}_{\mathrm{adj}} = 0$ (Theorem 2.1(2)). (b) All-coupling 4D per-site free-energy enclosures $[f^{\mathrm{lo}}_{\mathrm{GG}}, f^{\mathrm{hi}}_{\mathrm{GG}}]$ extending into the compact U(1) London regime at $\kappa = 50$ (Theorem 3.1, where the gap is 5.1% of the unsubtracted $f^{\mathrm{hi}}_{\mathrm{GG}}$ and 21.2% of the subtracted excess free energy $\widetilde{f}^{\mathrm{hi}}_{\mathrm{GG}}$).

TABLE 1. Eight 140-digit simultaneous confinement and adjoint-breaking certificates (Theorem 2.1): $\underline{\sigma}^{\mathrm{lo}}_{\mathrm{fund}} > 0$ holds uniformly across *all* $\kappa \in [0,\infty]$ ($\rho_3 = 0.55$, $\rho_4 = 0.39$), while $V^{\mathrm{ub}}_{\mathrm{adj}} = 2\ln(1/m(\kappa)) + 8(d-1)\kappa < \infty$ holds uniformly across *all* $L_s \geq 3$, $1 \leq R \leq L_s - 1$, and *all* $\beta \geq 0$. Lower bounds round downward; upper bounds upward.

d	β	κ	ρ_d	$\underline{\sigma}^{\mathrm{lo}}_{\mathrm{fund}}$ (all $\kappa \geq 0$)	$m(\kappa)^{\mathrm{lo}}$	$V^{\mathrm{ub}}_{\mathrm{adj}}$ (all $\beta \geq 0$)
3	0.02	0.25	0.55	1.520808	0.082988	8.978115
3	0.05	0.5	0.55	0.605217	0.163953	11.616346
3	0.08	1	0.55	0.136510	0.313035	18.322879
3	1/12	2	0.55	0.095869	0.537314	33.242343
4	0.005	0.2	0.39	1.531891	0.066489	10.221421
4	0.01	0.5	0.39	0.838769	0.163953	15.616346
4	1/60	1	0.39	0.328002	0.313035	26.322879
4	0.02	2	0.39	0.145722	0.537314	49.242343

TABLE 2. Eight 140-digit all-coupling per-site free-energy enclosures $f^{\mathrm{lo}}_{\mathrm{GG}} \leq f_{\mathrm{GG}}(d,\beta,\kappa) \leq f^{\mathrm{hi}}_{\mathrm{GG}}$ (Theorem 3.1) in 3D and 4D, extending into the compact U(1) London regime ($\kappa = 50$). All upper bounds attain their minimum on the (U,Φ)-variable d-way axial-gauge Hölder branch; lower bounds use the Abelian-Higgs witness h and round downward while upper bounds and gaps round upward.

d	β	κ	Witness h	$f^{\mathrm{lo}}_{\mathrm{GG}}$ (LB)	$f^{\mathrm{hi}}_{\mathrm{GG}}$ (UB)	Gap
3	1	0.5	0.00	0.123974	0.888676	0.764701
3	4	4	7.60	6.880684	14.857650	7.976965
3	10	10	31.02	37.513752	47.278149	9.764397
3	10	50	36.87	156.813064	165.668711	8.855647
4	1	0.5	0.00	0.165299	1.987624	1.822325
4	4	4	12.42	12.962901	26.348824	13.385922
4	10	10	47.33	65.475483	81.402082	15.926598
4	10	50	56.10	227.466603	239.792645	12.326041

TABLE 3. Eight 140-digit sharp unitary-gauge Bakry–Émery curvature and plaquette-variance certificates (Theorem 4.1) across $d \in \{2, 3, 4\}$. Lower bounds $\kappa_{\mathrm{GG}}^{\mathrm{lo}}$ round downward and variance ceilings $\mathrm{Var}_{\mathrm{GG}}^{\mathrm{ub}} = 2/\kappa_{\mathrm{GG}}$ upward (exact rational values such as $3/5 = 0.6$, $1/5 = 0.2$, $1/4 = 0.25$ round outward by 10^{-6} under strict `mpmath.iv` enclosure of non-dyadic fractions).

d	β	κ	$\kappa_{\mathrm{GG}}^{\mathrm{lo}}$ (LSI LB)	$\mathrm{Var}_{\mathrm{GG}}^{\mathrm{ub}}(P_p)$
2	1/20	1/10	0.599999	3.333334
2	1/10	1/5	0.199999	10.000001
3	1/40	1/10	0.599999	3.333334
3	1/20	1/5	0.199999	10.000001
3	1/16	1/8	0.250000	8.000000
4	1/60	1/10	0.599999	3.333334
4	1/30	1/5	0.199999	10.000001
4	1/24	1/8	0.249999	8.000001

`benchmarks_certified`. Exact SU(2) Haar/Langevin Monte Carlo checks, sharp Hessian saturation tests on 200 random pairs plus $(U, X) = (I_2, i\sigma_1)$, 80-digit cross-checks, and JSON/Lean/LaTeX parity are verified in `tests/test_georgi_glashow.py` (4/4 `pytest` passing).

6. Conclusion

We have established three self-contained, cluster-expansion-free non-perturbative theorems for the d-dimensional SU(2) Georgi–Glashow adjoint-Higgs lattice gauge theory: simultaneous κ-uniform fundamental quark confinement ($\underline{\sigma}^{\mathrm{GG}}_{\mathrm{fund}} > 0$ for all $\kappa \in [0, \infty]$) and β-independent universal adjoint string breaking ($\sigma_{\mathrm{adj}} = 0$ for all $\beta \geq 0, \kappa > 0$), all-coupling free-energy enclosures smoothly bridging pure SU(2) and the compact U(1) London limit, and a sharp unitary-gauge Bakry–Émery curvature threshold $4(d-1)\beta + 2\kappa < 1$. Natural next directions include extending adjoint-Higgs center-symmetry confinement and gluelump breaking bounds to general SU(N) ($N \geq 3$) and proving constructive Higgs-phase mass-gap bounds at large κ in 4D via Dobrushin uniqueness.

References

1. Dominique Bakry and Michel Émery, *Diffusions hypercontractives*, Séminaire de Probabilités XIX, Lecture Notes in Mathematics, vol. 1123, Springer, 1985, pp. 177–206.
2. Dominique Bakry, Ivan Gentil, and Michel Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der mathematischen Wissenschaften, vol. 348, Springer, 2014.
3. Richard C. Brower, David A. Kessler, Terry Schalk, Herbert Levine, and Michael Nauenberg, SU(2) *adjoint Higgs model*, Physical Review D **25** (1982), no. 12, 3319–3324.
4. Eduardo Fradkin and Stephen H. Shenker, *Phase diagrams of lattice gauge theories with Higgs fields*, Physical Review D **19** (1979), no. 12, 3682–3697.
5. Howard Georgi and Sheldon L. Glashow, *Unified weak and electromagnetic interactions without neutral currents*, Physical Review Letters **28** (1972), no. 22, 1494–1497.
6. Jean Ginibre, *General formulation of Griffiths' inequalities*, Communications in Mathematical Physics **16** (1970), no. 4, 310–328.
7. Gerhard Mack and Valentina B. Petkova, *Comparison of lattice gauge theories with gauge groups* $\mathbb{Z}_2$ *and* SU(2), Annals of Physics **123** (1979), no. 2, 442–467.
8. John Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Mathematics **21** (1976), no. 3, 293–329.
9. Konrad Osterwalder and Erhard Seiler, *Gauge field theories on a lattice*, Annals of Physics **110** (1978), no. 2, 440–471.
10. Alexander M. Polyakov, *Quark confinement and topology of gauge theories*, Nuclear Physics B **120** (1977), no. 3, 429–458.
11. Erhard Seiler, *Gauge theories as a problem of constructive quantum field theory and statistical mechanics*, Lecture Notes in Physics, vol. 159, Springer, Berlin, 1982.
12. Gerard 't Hooft, *Magnetic monopoles in unified gauge theories*, Nuclear Physics B **79** (1974), no. 2, 276–284.
13. Kenneth G. Wilson, *Confinement of quarks*, Physical Review D **10** (1974), no. 8, 2445–2459.

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URL: `https://airr.science/papers/ARR-2026-7MD056ZQEG9MZT71/`