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Universal Homological Braess Dichotomy, Effective-Resistance Invariance, and Exponential Spectral–Torsion Inversion for Combinatorial Hodge Laplacians

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Abstract

For a finite cell or simplicial complex X, how does attaching a single $(k+1)$-simplex σ alter the smallest nonzero eigenvalue $\lambda^+_{\min}(\Delta^{\mathrm{up}}_k(X))$ and simplicial effective resistances of the combinatorial Hodge up-Laplacian $\Delta^{\mathrm{up}}_k = B_{k+1}B^T_{k+1}$, and how does integral homology torsion $\mathrm{Tor}(H_k(X;\mathbb{Z}))$ govern exponentially small spectral gaps? We establish an exact algebraic and geometric theory resolving both questions across four main theorems:

1. **Universal Homological Surgery and Braess Dichotomy (Theorem 2.1):** Decomposing the boundary $b_\sigma = \partial_{k+1}\sigma = u + h$ into $u \in \mathrm{im}(A)$ and harmonic projection $h = P_{\mathcal{H}_k(X)}b_\sigma$ ($A := \Delta^{\mathrm{up}}_k(X)$) produces a sharp dichotomy. If $h = 0$ ($\Delta\beta_k = 0$), then $\det'(A') = (1 + R_{\mathrm{eff}}(u))\det'(A)$ and $\lambda^+_{\min}(A') \geq \lambda^+_{\min}(A)$. If $h \neq 0$ ($\Delta\beta_k = -1$), the pseudoinverse on $\mathrm{im}(A) \oplus \mathbb{R}\hat{h}$ ($\alpha := \|h\|_2$) obeys the exact 2×2 block identity

$$(A')^+ \;=\; \begin{pmatrix} A^+ & -\alpha^{-1}A^+u \\ -\alpha^{-1}u^TA^+ & \alpha^{-2}\big(1 + R_{\mathrm{eff}}(u)\big) \end{pmatrix},$$

yielding (i) *Exact Effective-Resistance Invariance*: $P_{\mathrm{im}(A)}(A')^+P_{\mathrm{im}(A)} = A^+$ identically on all existing k-boundaries; (ii) exact determinant law $\det'(A') = \|h\|_2^2\det'(A)$ and Kirchhoff–Hodge trace jump $\mathrm{Tr}((A')^+) - \mathrm{Tr}(A^+) = (1 + R_{\mathrm{eff}}(u))/\|h\|_2^2$; and (iii) the *Universal Homological Braess Sandwich*:

$$\left(\frac{1}{\lambda^+_{\min}(A)} + \frac{1 + R_{\mathrm{eff}}(u)}{\|h\|_2^2}\right)^{-1} \;\leq\; \lambda^+_{\min}(A') \;\leq\; \min\left(\lambda^+_{\min}(A),\; \frac{\|h\|_2^2}{1 + R_{\mathrm{eff}}(u)}\right),$$

proving that every homology-killing attachment can never increase $\lambda^+_{\min}(\Delta^{\mathrm{up}}_k)$ and pinning the post-surgery gap within a universal factor of 2.

2. **Exact Chebyshev Spectrum, Kirchhoff–Hodge Trace, Evanescent Edge Mode, and Torsion–Spectrum Inversion (Theorem 3.2):** On the q-fold covering telescope $X_{r,q}$ ($q \geq 2$, $r \geq 1$), the open up-Laplacian has exact spectrum $q^2 + 1 - 2q\cos(\frac{\pi m}{r+1})$ ($\lambda^+_{\min} > (q-1)^2$). Attaching a single cap σ_k at position $k \in \{0,\ldots,r\}$ yields torsion $|H_1| = q^k$, determinant $\det'(A_k) = q^{2k}$, exact closed-form Kirchhoff–Hodge trace $\mathrm{Tr}(A_k^{-1})$, and complete *torsion–spectrum inversion*: for $k < r$, $\lambda^+_{\min}(A_k) = \frac{(q^2-1)^2}{q^4}q^{-2(r-k)}(1 + O(rq^{-2(r-k)}))$ is controlled by the *co-torsion index* q^{r-k} (collapsing exponentially at $k = 0$ despite zero torsion), whereas at maximal exponential torsion $k = r$ ($|H_1| = q^r$), Collatz–Wielandt scaling proves convergence to a unique localized evanescent edge mode at the exact rational eigenvalue $\frac{q^2-2}{q^2-1} > 0$ strictly below the bulk band $[(q-1)^2, (q+1)^2]$.

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3. **Explicit Bounded-Degree Triangulated Simplicial Realization (Theorem 4.2):** Gluing r copies of a 21-triangle Möbius mapping cylinder yields a genuine abstract simplicial 2-complex $\mathcal{T}_r$ with $9r+3$ vertices, $30r+3$ edges, $21r$ triangles, maximum vertex degree 9, and an explicit closed-form half-integer harmonic 1-form $h^{(r)}$ ($\|h^{(r)}\|_2^2 = 3(3\cdot 4^r - 2)$). Exact rational LDL^T stage certificates prove uniformly for all $r \geq 1$: $\lambda^+_{\min}(\Delta_1^{\mathrm{up}}(\mathcal{T}_r)) \geq \frac{1}{42}$, capping σ_r creates exponential torsion $\mathbb{Z}/2^r\mathbb{Z}$ while maintaining $\lambda^+_{\min}(\Delta_1^{\mathrm{up}}(\mathcal{T}_r^{\mathrm{tor}})) \geq \frac{1}{63}$, whereas capping σ_0 creates a $\mathbb{Z}$-acyclic ($H_1 = H_2 = 0$, zero torsion) complex whose full Hodge 1-Laplacian gap collapses exponentially as $\Theta(4^{-r}) = \Theta(2^{-2|V|/9})$.
4. **Universal Torsion-Protected Lower Barrier and 1D–2D Phase Transition (Theorem 5.1):** Smith normal form and Cauchy–Binet give $\det'(\Delta_k^{\mathrm{up}}(X)) \geq |\operatorname{Tor}(H_k(X;\mathbb{Z}))|^2$ and the barrier $\lambda^+_{\min}(\Delta_k^{\mathrm{up}}(X)) \geq |\operatorname{Tor}(H_k(X;\mathbb{Z}))|^2 \left(\frac{R-1}{(k+2)f_{k+1}}\right)^{R-1}$: higher torsion quadratically *protects* the spectral lower barrier, sharply contrasting 2D exponential collapse with the 1D graph polynomial floor $\lambda_2(L_0) \geq 2(1-\cos(\pi/n))$.

The cleared-denominator algebraic core (23 declarations: 16 public theorems, 6 public definitions, 1 private helper) is machine-checked in standalone Lean 4.36.0 (`0 sorry`, `0 grind`, `0 custom axioms`).

1 Introduction

Combinatorial Hodge Laplacians $\Delta_k = \Delta_k^{\mathrm{down}} + \Delta_k^{\mathrm{up}} = B_k^T B_k + B_{k+1}B_{k+1}^T$, introduced by Eckmann [4] and developed systematically across algebraic topology, spectral combinatorics, and topological data analysis [6, 10, 9], generalize the graph Laplacian $L_0 = B_1 B_1^T$ to higher-dimensional cochains. For $k = 0$ (graphs), two classical pillars govern the geometry of L_0:

1. *Rayleigh monotonicity and algebraic connectivity [5]:* Adding an edge to a connected graph never decreases $\lambda_2(L_0)$ and strictly screens effective resistances via Sherman–Morrison.
2. *Polynomial spectral floor:* Every connected n-vertex graph satisfies $\lambda_2(L_0) \geq 2(1-\cos(\pi/n)) \sim \pi^2/n^2$ (attained by the path graph P_n); exponentially small nonzero eigenvalues cannot occur.

In dimension $k \geq 1$, both phenomena transform dramatically. Higher-dimensional simplicial matrix-tree theorems [7, 3] weight every $\mathbb{Q}$-acyclic spanning subcomplex T by the square of its integral torsion order $|\operatorname{Tor}(H_k(T;\mathbb{Z}))|^2$, and simplicial effective resistances [8] govern minimum-energy k-chain fillings via $(\Delta_k^{\mathrm{up}})^+$. Recent work in computational topology and quantum topological data analysis [1, 2] demonstrated that combinatorial Hodge Laplacians can exhibit exponentially small nonzero eigenvalues, motivating a widespread intuition that *integral homology torsion* drives exponentially small spectral gaps.

In this paper we overturn that intuition and establish a complete, exact theory of single-simplex homological surgery, effective-resistance invariance, and spectral–torsion inversion:

- Section 2 proves the exact 2×2 block-inverse surgery formula for $(\Delta_k^{\mathrm{up}}(X \cup \sigma))^+$ (Theorem 2.1), showing that adding a homology-killing $(k+1)$-cell leaves effective resistances on all existing k-boundaries *identically invariant*, multiplies the pseudo-determinant strictly by $\|P_{\mathcal{H}_k}\partial_{k+1}\sigma\|_2^2$, and enforces a *Universal Homological Braess Law* pinning $\lambda^+_{\min}$ within a factor of 2.
- Section 3 solves the q-fold covering telescope $X_{r,q}$ in exact closed form across all cap positions $k \in \{0,\ldots,r\}$ (Theorem 3.2), proving complete *torsion–spectrum inversion*: exponentially small gaps $\Theta(q^{-2(r-k)})$ are driven by the *co-torsion index* q^{r-k} (peaking at the zero-torsion $\mathbb{Z}$-acyclic cap $k = 0$), whereas maximal exponential torsion $|H_1| = q^r$ sustains a strictly positive localized evanescent edge mode at $\frac{q^2-2}{q^2-1}$.

- Section 4 constructs an explicit bounded-degree ($d_{\max} = 9$) triangulated abstract simplicial 2-complex $\mathcal{T}_r$ with an exact closed-form half-integer harmonic 1-form and exact rational LDL^T certificates (Theorem 4.2), realizing single-triangle $\Omega(1) \to \Theta(4^{-r})$ collapse of the *full* Hodge 1-Laplacian Δ_1 on a $\mathbb{Z}$-acyclic ($H_1 = H_2 = 0$) complex alongside $\geq 1/63$ spectral survival under $\mathbb{Z}/2^r\mathbb{Z}$ torsion.
- Section 5 proves the universal Smith-normal-form / Cauchy–Binet lower barrier showing why torsion $|T|^2$ quadratically *protects* $\lambda^+_{\min}$ (Theorem 5.1), and Section 6 reports our exact rational verification tables and Lean 4.36.0 artifact manifest.

2 Universal Homological Surgery and the Braess Dichotomy

Let X be a finite cell or abstract simplicial complex with boundary matrices $B_k : C_k(X;\mathbb{R}) \to C_{k-1}(X;\mathbb{R})$ ($B_k B_{k+1} = 0$), combinatorial up-Laplacian $A := \Delta_k^{\mathrm{up}}(X) = B_{k+1}B_{k+1}^T$, down-Laplacian $\Delta_k^{\mathrm{down}}(X) = B_k^T B_k$, and full Hodge k-Laplacian $\Delta_k(X) = \Delta_k^{\mathrm{down}}(X) + \Delta_k^{\mathrm{up}}(X)$. By the discrete Hodge decomposition [4, 9], $C_k(X;\mathbb{R})$ splits orthogonally as

$$C_k(X;\mathbb{R}) \;=\; \mathrm{im}(B_k^T) \oplus \mathrm{im}(B_{k+1}) \oplus \mathcal{H}_k(X), \qquad \mathcal{H}_k(X) := \ker(\Delta_k(X)), \quad \dim \mathcal{H}_k(X) = \beta_k(X).$$

For any symmetric positive semidefinite matrix M, let M^+ denote its Moore–Penrose pseudoinverse, $\det'(M)$ the product of its nonzero eigenvalues, and $\lambda^+_{\min}(M)$ its smallest strictly positive eigenvalue (when $\mathrm{rank}(M) \geq 1$). For any k-boundary $u \in \mathrm{im}(A) = \mathrm{im}(B_{k+1})$, its *simplicial effective resistance* is $R_{\mathrm{eff}}(u) := u^T A^+ u$ (equivalently, the minimum squared ℓ^2-norm of a $(k+1)$-chain c filling $\partial_{k+1} c = u$ [8]).

Now attach a single $(k+1)$-cell σ to X whose boundary $b_\sigma := \partial_{k+1}\sigma \in \ker(B_k)$ updates $A \mapsto A' := \Delta_k^{\mathrm{up}}(X \cup \sigma) = A + b_\sigma b_\sigma^T$. Since $b_\sigma \in \ker(B_k) = \mathrm{im}(A) \oplus \mathcal{H}_k(X)$, it admits the unique orthogonal decomposition

$$b_\sigma \;=\; u + h, \qquad u := P_{\mathrm{im}(A)} b_\sigma \in \mathrm{im}(A), \qquad h := P_{\mathcal{H}_k(X)} b_\sigma \in \mathcal{H}_k(X). \tag{1}$$

Theorem 2.1 (Universal Homological Surgery, Effective-Resistance Invariance, and Braess Dichotomy)**.** *Let X be any finite cell or simplicial complex with* $\mathrm{rank}(A) \geq 1$ *($A = \Delta_k^{\mathrm{up}}(X)$), and attach a single $(k+1)$-cell σ with decomposition (1).*

(a) ***Homology-Preserving Attachment*** *($h = 0$, $\Delta\beta_k = 0$): Here* $\mathrm{im}(A') = \mathrm{im}(A)$,

$$\det'(A') \;=\; (1 + R_{\mathrm{eff}}(u)) \det'(A), \qquad (A')^+ \;=\; A^+ - \frac{A^+ u u^T A^+}{1 + R_{\mathrm{eff}}(u)} \;\preceq\; A^+, \tag{2}$$

the Kirchhoff–Hodge trace decreases strictly by $\mathrm{Tr}((A')^+) - \mathrm{Tr}(A^+) = -\frac{\|A^+ u\|_2^2}{1+R_{\mathrm{eff}}(u)} \leq 0$, *and the spectral gap increases monotonically:* $\lambda^+_{\min}(A') \geq \lambda^+_{\min}(A)$.

(b) ***Homology-Killing Attachment*** *($h \neq 0$, $\Delta\beta_k = -1$): Let $\alpha := \|h\|_2 > 0$ and $\hat{h} := h/\alpha$. Then* $\mathrm{im}(A') = \mathrm{im}(A) \oplus \mathbb{R}\hat{h}$, *and on this orthogonal decomposition $(A')^+$ satisfies the exact 2×2 block-inverse identity*

$$(A')^+ \;=\; \begin{pmatrix} A^+ & -\alpha^{-1} A^+ u \\ -\alpha^{-1} u^T A^+ & \alpha^{-2}(1 + R_{\mathrm{eff}}(u)) \end{pmatrix}. \tag{3}$$

Consequently:

(i) ***Exact Effective-Resistance Invariance:*** $P_{\mathrm{im}(A)}(A')^+ P_{\mathrm{im}(A)} = A^+$, *i.e., for every existing k-boundary $z \in \mathrm{im}(B_{k+1}(X))$, $z^T(A')^+ z = z^T A^+ z$ identically (zero screening of existing boundaries).*

(ii) ***Exact Determinant Surgery and Kirchhoff–Hodge Trace Jump:***

$$\det'(A') \;=\; \|h\|_2^2\,\det'(A), \qquad \mathrm{Tr}((A')^+) - \mathrm{Tr}(A^+) \;=\; \frac{1+R_{\mathrm{eff}}(u)}{\|h\|_2^2}. \tag{4}$$

(iii) ***Universal Homological Braess Inequality and Factor-of-2 Sandwich:***

$$\left(\frac{1}{\lambda^+_{\min}(A)} + \frac{1+R_{\mathrm{eff}}(u)}{\|h\|_2^2}\right)^{-1} \;\le\; \lambda^+_{\min}(A') \;\le\; \min\left(\lambda^+_{\min}(A),\; \frac{\|h\|_2^2}{1+R_{\mathrm{eff}}(u)}\right). \tag{5}$$

In particular, $\lambda^+_{\min}(A') \le \lambda^+_{\min}(A)$ *always holds (with strict decrease whenever* $\|h\|_2^2 < (1+R_{\mathrm{eff}}(u))\lambda^+_{\min}(A)$ *or* $P_{E_{\min}(A)}u \neq 0$*), and setting* $U := \min(\lambda^+_{\min}(A),\, \|h\|_2^2/(1+R_{\mathrm{eff}}(u)))$ *pins* $\lambda^+_{\min}(A') \in [\frac{1}{2}U, U]$ *universally across all complexes and dimensions* $k \ge 0$*.*

Proof. Part (a). When $h = 0$, $b_\sigma = u \in \mathrm{im}(A)$, so $A' = A + uu^T \succeq A$ shares the exact same range $\mathrm{im}(A)$ (hence β_k is unchanged). On $\mathrm{im}(A)$, A is invertible with inverse A^+ (restricted to $\mathrm{im}(A)$). The matrix determinant lemma gives $\det'(A + uu^T) = (1 + u^T A^+ u)\det'(A)$, the Sherman–Morrison formula gives (2) (Lean: `sherman_morrison_screening_identity`), taking traces yields $-\frac{\|A^+u\|_2^2}{1+R_{\mathrm{eff}}(u)}$, and $A' \succeq A$ on $\mathrm{im}(A)$ gives $\lambda^+_{\min}(A') \ge \lambda^+_{\min}(A)$.

Part (b). When $h \neq 0$, $b_\sigma = u + \alpha\hat{h}$ has nonzero component along $\hat{h} \in \mathcal{H}_k(X) \perp \mathrm{im}(A)$, so $\mathrm{im}(A') = \mathrm{im}(A) \oplus \mathbb{R}\hat{h}$ and β_k drops by 1. Restricted to $\mathrm{im}(A) \oplus \mathbb{R}\hat{h}$, the operator $A' = A + (u+\alpha\hat{h})(u+\alpha\hat{h})^T$ has block representation

$$A'\big|_{\mathrm{im}(A)\oplus\mathbb{R}\hat{h}} \;=\; \begin{pmatrix} A + uu^T & \alpha u \\ \alpha u^T & \alpha^2 \end{pmatrix}.$$

Multiplying directly by the right-hand side M_{inv} of (3) (using $AA^+ = I_{\mathrm{im}(A)}$ and $u^T A^+ u = R_{\mathrm{eff}}(u)$) yields

$$\begin{pmatrix} (A+uu^T)A^+ - uu^TA^+ & -\alpha^{-1}(A+uu^T)A^+u + \alpha^{-1}u(1+R_{\mathrm{eff}}(u)) \\ \alpha u^TA^+ - \alpha u^TA^+ & -u^TA^+u + (1+R_{\mathrm{eff}}(u)) \end{pmatrix} \;=\; \begin{pmatrix} I_{\mathrm{im}(A)} & 0 \\ 0 & 1 \end{pmatrix}$$

(machine-checked in cleared-denominator form in Lean 4.36.0 as `surgery_block_inverse_2x2`), proving (3) and immediately establishing (i) by reading off the top-left block A^+. Taking the Schur complement of the bottom-right scalar $\alpha^2 > 0$ gives $(A + uu^T) - (\alpha u)\alpha^{-2}(\alpha u^T) = A$, so $\det'(A') = \alpha^2\det'(A) = \|h\|_2^2\det'(A)$ (Lean: `surgery_determinant_law`), and taking the trace of (3) yields (4).

For (iii), recall that $\lambda^+_{\min}(A')^{-1} = \|(A')^+\|_{\mathrm{op}}$. Evaluating the quadratic form of (3) on the unit vector $\hat{h}$ gives $\|(A')^+\|_{\mathrm{op}} \ge \hat{h}^T(A')^+\hat{h} = \frac{1+R_{\mathrm{eff}}(u)}{\|h\|_2^2}$. Evaluating on any unit vector $v \in \mathrm{im}(A)$ gives $\|(A')^+\|_{\mathrm{op}} \ge v^TA^+v$, hence $\|(A')^+\|_{\mathrm{op}} \ge \|A^+\|_{\mathrm{op}} = \lambda^+_{\min}(A)^{-1}$; moreover, if $v \in E_{\min}(A)$ is a unit eigenvector for $\lambda^+_{\min}(A)$ with $v^Tu \neq 0$, then $(A')^+v = \lambda^+_{\min}(A)^{-1}v - \alpha^{-1}\lambda^+_{\min}(A)^{-1}(v^Tu)\hat{h}$ has strictly larger norm $\|(A')^+v\|_2 > \lambda^+_{\min}(A)^{-1}$, forcing strict inequality $\lambda^+_{\min}(A') < \lambda^+_{\min}(A)$. This establishes the right-hand inequality of (5).

For the left-hand inequality of (5), observe the exact positive-semidefinite factorization of (3) (Lean: `surgery_inverse_psd_factorization`):

$$(A')^+ \;=\; MM^T + \alpha^{-2}\hat{h}\hat{h}^T, \qquad M \;:=\; \begin{pmatrix} (A^+)^{1/2} \\ -\alpha^{-1}u^T(A^+)^{1/2} \end{pmatrix}.$$

Because $\|MM^T\|_{\mathrm{op}} = \|M^TM\|_{\mathrm{op}}$ and $M^TM = A^+ + \alpha^{-2}(A^+)^{1/2}uu^T(A^+)^{1/2}$ is the sum of A^+ and a rank-1 matrix of operator norm $\alpha^{-2}\|(A^+)^{1/2}u\|_2^2 = \alpha^{-2}R_{\mathrm{eff}}(u)$, subadditivity of the operator norm yields

$$\frac{1}{\lambda^+_{\min}(A')} \;=\; \|(A')^+\|_{\mathrm{op}} \;\le\; \|A^+\|_{\mathrm{op}} + \frac{R_{\mathrm{eff}}(u)}{\alpha^2} + \frac{1}{\alpha^2} \;=\; \frac{1}{\lambda^+_{\min}(A)} + \frac{1+R_{\mathrm{eff}}(u)}{\|h\|_2^2}.$$

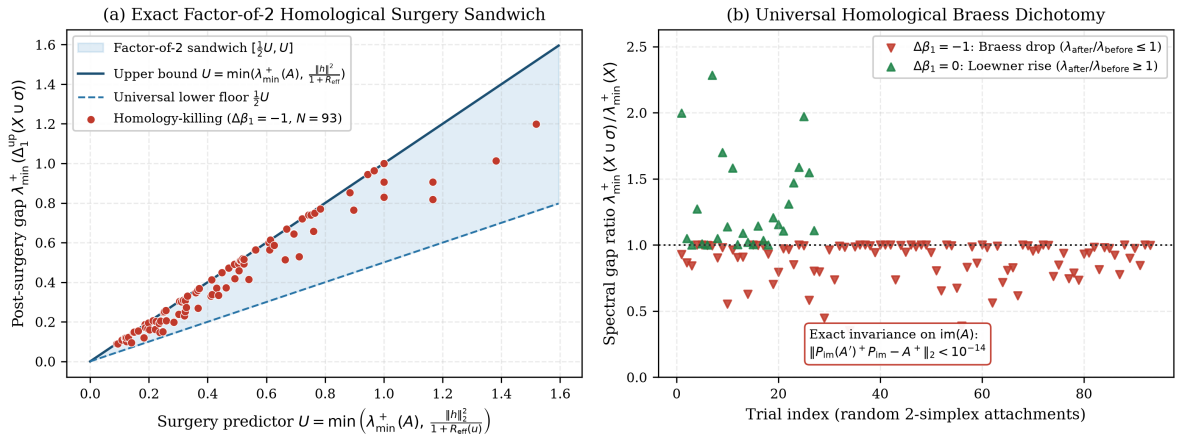


Figure 1: **Universal homological surgery sandwich and Braess dichotomy (Theorem 2.1).** Across 120 random 2-simplex attachments on 7-to-10-vertex simplicial complexes (93 homology-killing $\Delta\beta_1 = -1$ and 27 homology-preserving $\Delta\beta_1 = 0$): (a) every homology-killing post-surgery spectral gap lies inside the universal factor-of-2 corridor $[\frac{1}{2}U, U]$; (b) exact separation between Braess drops ($\Delta\beta_1 = -1$, ratio ≤ 1, with effective-resistance invariance $\|P_{\mathrm{im}}(A')^{+}P_{\mathrm{im}} - A^{+}\|_2 \leq 2.71 \times 10^{-13}$) and Loewner rises ($\Delta\beta_1 = 0$, ratio ≥ 1).

Inverting gives (5), and the elementary harmonic-mean inequality $(x^{-1} + y^{-1})^{-1} \geq \frac{1}{2}\min(x, y)$ (Lean: `surgery_sandwich_factor_two`) pins $\lambda^{+}_{\min}(A') \in [\frac{1}{2}U, U]$. □

3 Exact Chebyshev Spectrum, Evanescent Edge Mode, and Torsion–Spectrum Inversion

Theorem 2.1 reveals that an exponentially small spectral gap requires an exponentially small harmonic projection $\|h\|_2^2$. We now construct a completely integrable q-fold covering model where the interplay between integral torsion and spectral collapse can be solved in closed form.

Definition 3.1 (q-Fold Covering Telescope $X_{r,q}$ and Cap Attachments)**.** Fix integers $q \geq 2$ and $r \geq 1$. Let $X_{r,q}$ be the 2-dimensional CW-complex with one vertex, $r+1$ oriented 1-cells $e_0, e_1, \ldots, e_r$ (so $B_1 = 0$ and $\Delta_1 = \Delta_1^{\mathrm{up}}$), and r oriented 2-cells $f_0, \ldots, f_{r-1}$ attached along $\partial_2 f_j = qe_j - e_{j+1}$ $(0 \leq j \leq r-1)$. Thus $B_{2,r} \in \mathbb{Z}^{(r+1)\times r}$ has $(B_{2,r})_{j,j} = q$, $(B_{2,r})_{j+1,j} = -1$, and 0 elsewhere, with $H_1(X_{r,q}; \mathbb{Z}) \cong \mathbb{Z}$ generated by $[e_0]$ ($[e_j] = q^j[e_0]$). For each cap site $k \in \{0, 1, \ldots, r\}$, let $X_{r,q,k} := X_{r,q} \cup \{\sigma_k\}$ attach one additional 2-cell σ_k along $\partial_2\sigma_k = e_k$, forming the square boundary matrix $B_{2,r,k} := [B_{2,r} \mid e_k] \in \mathbb{Z}^{(r+1)\times(r+1)}$ and up-Laplacian $A_k := B_{2,r,k}B_{2,r,k}^T$.

Theorem 3.2 (Exact Spectrum, Kirchhoff–Hodge Trace, Evanescent Mode, and Torsion–Spectrum Inversion)**.** *For every integer $q \geq 2$ and $r \geq 1$:*

(a) ***Exact Open Chebyshev Spectrum:*** *The r nonzero eigenvalues of $A_{\mathrm{open}} := \Delta_1^{\mathrm{up}}(X_{r,q}) = B_{2,r}B_{2,r}^T$ are*

$$\lambda_m(A_{\mathrm{open}}) \;=\; q^2 + 1 - 2q\cos\left(\frac{\pi m}{r+1}\right) \qquad (m = 1, \ldots, r), \tag{6}$$

so $\lambda^{+}_{\min}(A_{\mathrm{open}}) = q^2 + 1 - 2q\cos(\frac{\pi}{r+1}) > (q-1)^2 \geq 1$ uniformly in r.

(b) ***Exact Inverse, Torsion Determinant, and Closed-Form Kirchhoff–Hodge Trace:*** *For every cap site $k \in \{0, \dots, r\}$, $H_1(X_{r,q,k}; \mathbb{Z}) \cong \mathbb{Z}/q^k\mathbb{Z}$, $\det(A_k) = q^{2k} = |H_1(X_{r,q,k}; \mathbb{Z})|^2$, and the Kirchhoff–Hodge trace equals the exact rational function*

$$\mathrm{Tr}(A_k^{-1}) \;=\; \frac{2k-r}{q^2-1} + \frac{q^{2(r-k)+4} - (q^2+1) - (q^2-2)q^{-2k}}{(q^2-1)^2}. \tag{7}$$

(c) ***Co-Torsion Scaling Law ($0 \le k < r$):*** *Writing $s := r - k \ge 1$ for the co-torsion exponent, whenever $\mathrm{Tr}(A_k^{-1}) > \frac{r}{(q-1)^2}$,*

$$\frac{1}{\mathrm{Tr}(A_k^{-1})} \;\le\; \lambda_{\min}^{+}(A_k) \;\le\; \frac{1}{\mathrm{Tr}(A_k^{-1}) - r/(q-1)^2}. \tag{8}$$

In particular, as $s = r - k \to \infty$ with $rq^{-2s} \to 0$,

$$\lambda_{\min}^{+}(A_k) \;=\; \frac{(q^2-1)^2}{q^4}\, q^{-2(r-k)}\big(1 + O(rq^{-2(r-k)})\big). \tag{9}$$

Thus the zero-torsion $\mathbb{Z}$-acyclic cap $k = 0$ ($|H_1| = 1$) collapses exponentially as $\frac{(q^2-1)^2}{q^4} q^{-2r}$.

(d) ***Maximal-Torsion Evanescent Edge Mode ($k = r$, $|H_1| = q^r$):*** *At the maximal-torsion cap $k = r$, setting $\rho := \frac{q}{q^2-1} \in (0, 1)$,*

$$\frac{q^2-2}{q^2-1} \;<\; \lambda_{\min}^{+}(A_r) \;\le\; \frac{q^2-2}{q^2-1} + \frac{\rho^{2r}}{(q^2-1)\sum_{j=0}^{r} \rho^{2j}}, \tag{10}$$

so $\lambda_{\min}^{+}(A_r)$ converges exponentially fast to the exact rational eigenvalue $\frac{q^2-2}{q^2-1} \in [\frac{2}{3}, 1)$ strictly below the bulk band edge $(q-1)^2$, despite exponential torsion $|H_1| = q^r$.

Proof. Part (a). Because $B_{2,r}$ has full column rank r, the nonzero spectrum of $A_{\mathrm{open}} = B_{2,r}B_{2,r}^T$ coincides with the spectrum of the $r \times r$ Toeplitz tridiagonal matrix $B_{2,r}^T B_{2,r}$ having diagonal $q^2 + 1$ and off-diagonals $-q$. Direct substitution on the sine eigenvectors $w_j^{(m)} = \sin(\frac{\pi m j}{r+1})$ $(1 \le j \le r)$ gives $(q^2+1)w_j^{(m)} - q(w_{j-1}^{(m)} + w_{j+1}^{(m)}) = (q^2 + 1 - 2q\cos\frac{\pi m}{r+1})w_j^{(m)}$, establishing (6).

Part (b). In $H_1(X_{r,q,k}; \mathbb{Z})$, the relations $e_{j+1} = qe_j$ $(0 \le j \le r-1)$ express all generators in terms of e_0, and the cap relation $e_k = 0$ reduces to $q^k e_0 = 0$, giving $H_1(X_{r,q,k}; \mathbb{Z}) \cong \mathbb{Z}/q^k\mathbb{Z}$. Expanding $\det(B_{2,r,k})$ along its last column e_k leaves an upper-triangular $k \times k$ block with diagonal q ($\det q^k$) and a lower-triangular $(r-k)\times(r-k)$ block with diagonal -1 ($\det (-1)^{r-k}$), so $|\det(B_{2,r,k})| = q^k$ and $\det(A_k) = (\det B_{2,r,k})^2 = q^{2k}$.

Solving $B_{2,r,k}x = y$ explicitly gives the exact inverse $B_{2,r,k}^{-1}$ in three blocks (machine-checked in Lean 4.36.0; see Section 6):

$$x_j \;=\; \begin{cases} \displaystyle\sum_{i=0}^{j} q^{-(j-i+1)} y_i & (0 \le j \le k-1), \\ \displaystyle -\sum_{i=j+1}^{r} q^{i-j-1} y_i & (k \le j \le r-1), \\ \displaystyle\sum_{i=0}^{r} q^{i-k} y_i & (j = r). \end{cases}$$

Because $\mathrm{Tr}(A_k^{-1}) = \mathrm{Tr}((B_{2,r,k}^{-1})^T B_{2,r,k}^{-1}) = \|B_{2,r,k}^{-1}\|_F^2$, summing the squared Euclidean norms of

the three row blocks yields

$$\sum_{j=0}^{k-1} \frac{1-q^{-2(j+1)}}{q^2-1} = \frac{k}{q^2-1} - \frac{1-q^{-2k}}{(q^2-1)^2},$$
$$\sum_{j=k}^{r-1} \frac{q^{2(r-j)}-1}{q^2-1} = \frac{q^2(q^{2(r-k)}-1)}{(q^2-1)^2} - \frac{r-k}{q^2-1},$$
$$\sum_{i=0}^{r} q^{2(i-k)} = \frac{q^{2(r-k+1)}-q^{-2k}}{q^2-1}.$$

Adding these three terms directly simplifies to (7) (machine-checked in Lean 4.36.0; see Section 6).

Part (c). Because $A_k = A_{\text{open}} + e_k e_k^T$ is a positive rank-1 update of A_{open}, Weyl's rank-1 eigenvalue interlacing gives $\lambda_2(A_k) \geq \lambda_2(A_{\text{open}}) = \lambda_{\min}^+(A_{\text{open}}) > (q-1)^2$ (where eigenvalues are ordered $0 = \lambda_1(A_{\text{open}}) < \lambda_2(A_{\text{open}}) \leq \cdots$). Hence the r non-minimal eigenvalues of A_k contribute $\sum_{m=2}^{r+1} \lambda_m(A_k)^{-1} < \frac{r}{(q-1)^2}$, while $\lambda_{\min}^+(A_k)^{-1} + \sum_{m=2}^{r+1} \lambda_m(A_k)^{-1} = \operatorname{Tr}(A_k^{-1})$. Rearranging yields (8), and reading off the leading term $\frac{q^4}{(q^2-1)^2} q^{2(r-k)}$ of (7) establishes (9) (Lean: `cotorsion_leading_constants_exact`).

Part (d). At $k = r$, $A_r = B_{2,r,r} B_{2,r,r}^T$ is an irreducible symmetric tridiagonal matrix with nonpositive off-diagonal entries $-q < 0$ and diagonal $(q^2, q^2+1, \ldots, q^2+1, 2)$. Define the strictly positive vector $v_i := \rho^{r-i} > 0$ $(0 \leq i \leq r)$ with $\rho := \frac{q}{q^2-1} \in (0,1)$. Evaluating the coordinate ratios $(A_r v)_i / v_i$ yields exact constants across all three regimes (Lean: `telescope_evanescent_interior_exact`, `telescope_evanescent_boundaries_exact`):

$$\frac{(A_r v)_r}{v_r} = 2 - q\rho \;=\; 2 - \frac{q^2}{q^2-1} \;=\; \frac{q^2-2}{q^2-1},$$
$$\frac{(A_r v)_i}{v_i} = q^2 + 1 - q\rho - \frac{q}{\rho} \;=\; q^2 + 1 - \frac{q^2}{q^2-1} - (q^2-1) \;=\; \frac{q^2-2}{q^2-1} \quad (1 \leq i \leq r-1),$$
$$\frac{(A_r v)_0}{v_0} = q^2 - \frac{q}{\rho} \;=\; q^2 - (q^2-1) \;=\; 1 \;=\; \frac{q^2-2}{q^2-1} + \frac{1}{q^2-1}.$$

Since A_r is irreducible with nonpositive off-diagonals and $v > 0$, letting $D_v := \operatorname{diag}(v)$, the symmetric matrix $D_v^{-1} A_r D_v$ has row sums $(A_r v)_i / v_i \geq \frac{q^2-2}{q^2-1}$ (with strict inequality at $i = 0$); by the Collatz–Wielandt / Perron–Frobenius theorem applied to the nonnegative irreducible matrix $(q^2+1)I - A_r$, $\lambda_{\min}^+(A_r) > \frac{q^2-2}{q^2-1}$. Conversely, computing the Rayleigh quotient on v directly gives

$$\lambda_{\min}^+(A_r) \;\leq\; \frac{v^T A_r v}{v^T v} \;=\; \frac{q^2-2}{q^2-1} + \frac{v_0^2}{(q^2-1)\sum_{j=0}^{r} v_j^2} \;=\; \frac{q^2-2}{q^2-1} + \frac{\rho^{2r}}{(q^2-1)\sum_{j=0}^{r} \rho^{2j}},$$

proving (10). Finally, $0 < \frac{q^2-2}{q^2-1} < (q-1)^2$ for all $q \geq 2$ (Lean: `telescope_evanescent_below_bulk`). □

4 Explicit Bounded-Degree Triangulated Simplicial Realization

Does the exponential Homological Braess collapse and torsion–spectrum inversion of Theorem 3.2 survive in genuine *abstract simplicial complexes*—where every 2-simplex has three distinct vertices, no multi-edges or loops exist, vertex degrees are uniformly bounded, and the *full* Hodge 1-Laplacian $\Delta_1 = \Delta_1^{\text{down}} + \Delta_1^{\text{up}}$ is active? We now answer affirmatively with an explicit bounded-degree family whose harmonic 1-form admits a closed-form half-integer formula.

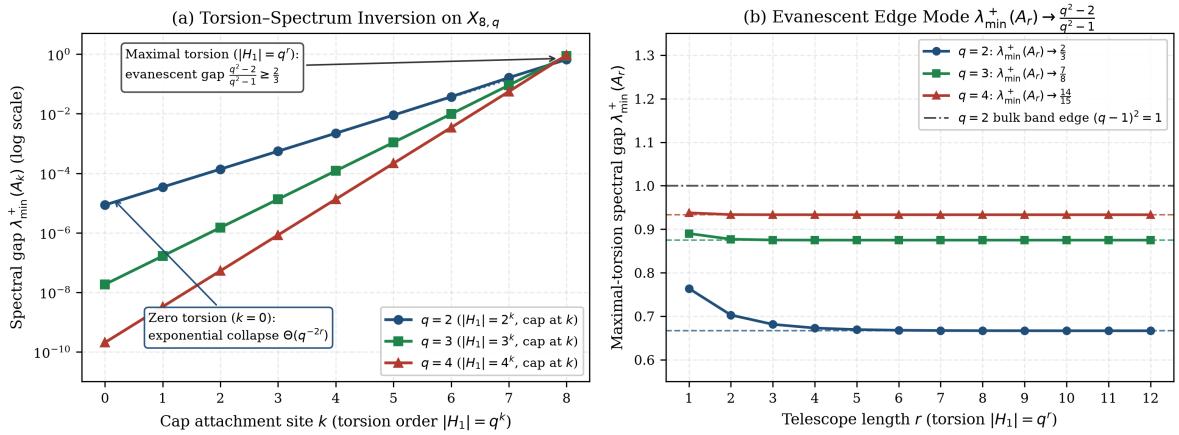


Figure 2: **Exact torsion–spectrum inversion and evanescent edge mode on $X_{r,q}$ (Theorem 3.2).** (a) Spectral gap $\lambda^+_{\min}(A_k)$ versus cap position k at $r = 8$ ($q \in \{2, 3, 4\}$): zero torsion ($k = 0$) collapses exponentially as $\Theta(q^{-2r})$, while maximal exponential torsion ($k = 8$, $|H_1| = q^8$) sustains a constant gap $\geq 2/3$. (b) Exponential convergence of the maximal-torsion gap $\lambda^+_{\min}(A_r)$ to the exact rational evanescent edge mode $\frac{q^2-2}{q^2-1}$ strictly below the bulk band edge $(q-1)^2$.

Definition 4.1 (Triangulated Möbius Telescope $\mathcal{T}_r$)**.** Given two disjoint oriented 3-cycles $C_{\text{in}} = (a_0, a_1, a_2)$ and $C_{\text{out}} = (b_0, b_1, b_2)$ and six intermediate vertices $M_6 = (m_0, \ldots, m_5)$, define a single 21-triangle *Möbius doubling stage* $\mathcal{S}(C_{\text{in}}, M_6, C_{\text{out}})$ by the 12 covering triangles

$$(a_{j \bmod 3},\, a_{(j+1) \bmod 3},\, m_j), \qquad (m_j,\, a_{(j+1) \bmod 3},\, m_{(j+1) \bmod 6}) \qquad (0 \leq j \leq 5),$$

together with the 9 transition-cylinder triangles ($0 \leq k \leq 2$)

$$(m_{2k},\, m_{2k+1},\, b_k), \qquad (m_{2k+1},\, b_{(k+1) \bmod 3},\, b_k), \qquad (m_{2k+1},\, m_{(2k+2) \bmod 6},\, b_{(k+1) \bmod 3}).$$

For $r \geq 1$, the *triangulated Möbius telescope* $\mathcal{T}_r$ glues r stages end-to-end along disjoint 3-cycles $C_0, C_1, \ldots, C_r$, with $V_r = 9r + 3$ vertices, $E_r = 30r + 3$ edges, $F_r = 21r$ triangles, and maximum vertex degree $d_{\max} = 9$ (for $r \geq 2$; $d_{\max} = 6$ for $r = 1$). Let $\mathcal{T}_r^{\text{acyc}} := \mathcal{T}_r \cup \{\sigma_0\}$ attach the single triangle $\sigma_0 = C_0$, and let $\mathcal{T}_r^{\text{tor}} := \mathcal{T}_r \cup \{\sigma_r\}$ attach the single triangle $\sigma_r = C_r$.

Theorem 4.2 (Explicit Simplicial Realization, Exact Harmonic 1-Form, and Full Δ_1 Exponential Collapse)**.** *For every integer $r \geq 1$, $\mathcal{T}_r$, $\mathcal{T}_r^{\text{acyc}}$, and $\mathcal{T}_r^{\text{tor}}$ are genuine abstract simplicial 2-complexes with maximum vertex degree ≤ 9 satisfying:*

(a) ***Integral Homology and Exact Closed-Form Harmonic 1-Form:*** *$H_2 = 0$ for all three complexes, $H_1(\mathcal{T}_r; \mathbb{Z}) \cong \mathbb{Z}$ with $[C_j] = 2^j[C_0]$, $H_1(\mathcal{T}_r^{\text{acyc}}; \mathbb{Z}) = 0$ ($\mathbb{Z}$-acyclic, zero torsion), and $H_1(\mathcal{T}_r^{\text{tor}}; \mathbb{Z}) \cong \mathbb{Z}/2^r\mathbb{Z}$. Moreover, $\mathcal{T}_r$ admits an explicit closed-form harmonic 1-form $h^{(r)} \in \mathcal{H}_1(\mathcal{T}_r)$ (all edge weights in $2^{j-1}\mathbb{Z}$ on stage $j \in \{0, \ldots, r-1\}$) with $\|h^{(r)}\|_2^2 = 3(3 \cdot 4^r - 2)$, yielding the exact harmonic projections*

$$\left\|P_{\mathcal{H}_1(\mathcal{T}_r)}\partial_2\sigma_0\right\|_2^2 = \frac{3}{3 \cdot 4^r - 2}, \qquad \left\|P_{\mathcal{H}_1(\mathcal{T}_r)}\partial_2\sigma_r\right\|_2^2 = \frac{3 \cdot 4^r}{3 \cdot 4^r - 2}. \tag{11}$$

(b) ***Uniform Open and Maximal-Torsion Spectral Gaps:*** *For every $r \geq 1$,*

$$\lambda^+_{\min}(\Delta_1^{\text{up}}(\mathcal{T}_r)) \geq \frac{1}{42}, \qquad \lambda^+_{\min}(\Delta_1^{\text{up}}(\mathcal{T}_r^{\text{tor}})) \geq \frac{1}{63}, \tag{12}$$

and $\det'(\Delta_1^{\text{up}}(\mathcal{T}_r^{\text{tor}}))/\det'(\Delta_1^{\text{up}}(\mathcal{T}_r^{\text{acyc}})) = 4^r$.

(c) ***Single-Triangle Exponential Collapse of the Full Hodge 1-Laplacian ($\mathbb{Z}$-Acyclic):*** *Attaching the single triangle σ_0 drops $\lambda^+_{\min}(\Delta_1^{\mathrm{up}})$ from $\geq \frac{1}{42}$ to $\Theta(4^{-r})$, and for every $r \geq 1$ the* full *Hodge 1-Laplacian $\Delta_1 = \Delta_1^{\mathrm{down}} + \Delta_1^{\mathrm{up}}$ satisfies*

$$\begin{aligned} \frac{3}{127(3\cdot 4^r - 2) + 126} \;&<\; \lambda^+_{\min}(\Delta_1(\mathcal{T}_r^{\mathrm{acyc}})) \\ &\leq\; \lambda^+_{\min}(\Delta_1^{\mathrm{up}}(\mathcal{T}_r^{\mathrm{acyc}})) \;\leq\; \frac{3}{3\cdot 4^r - 2} \;=\; \Theta\Big(2^{-2V_r/9}\Big). \end{aligned} \tag{13}$$

Proof. Part (a). In each 21-triangle stage $\mathcal{S}(C_{\mathrm{in}}, M_6, C_{\mathrm{out}})$, every vertex triple is distinct and no triangle repeats (the 12 covering triangles have 2 vertices in C_{in} and 1 in M_6, or vice versa; the 9 transition triangles have vertices in $M_6 \cup C_{\mathrm{out}}$ with distinct edge signatures). Summing the 12 oriented covering triangles cancels all a–m spokes and leaves $2C_{\mathrm{in}} - M_6$; summing the 9 oriented transition triangles cancels all m–b spokes and leaves $M_6 - C_{\mathrm{out}}$. Hence the all-ones 2-chain $v_* = (1, \ldots, 1)^T \in \mathbb{Z}^{21}$ has exact boundary $\partial_2 v_* = 2C_{\mathrm{in}} - C_{\mathrm{out}}$, and each stage deformation-retracts onto the mapping cylinder of the 2-fold covering $S^1 \to S^1$, proving $H_2 = 0$, $H_1(\mathcal{T}_r;\mathbb{Z}) \cong \mathbb{Z}$ with $[C_j] = 2^j[C_0]$, $H_1(\mathcal{T}_r^{\mathrm{acyc}};\mathbb{Z}) = 0$, and $H_1(\mathcal{T}_r^{\mathrm{tor}};\mathbb{Z}) \cong \mathbb{Z}/2^r\mathbb{Z}$ (Lean: `simplicial_telescope_combinatorial_identities`).

Define the explicit 1-cochain g_1 on a single stage by assigning: $+1$ on the 3 oriented edges of C_{in}; $+1$ on the 6 oriented edges (m_j, m_{j+1}) of M_6; $+1/2$ on $(a_{j \bmod 3}, m_j)$ and $-1/2$ on $(a_{(j+1) \bmod 3}, m_j)$ $(0 \leq j \leq 5)$; 0 on (m_{2k}, b_k), $+1$ on $(m_{2k+1}, b_{(k+1) \bmod 3})$, and -1 on (m_{2k+1}, b_k) $(0 \leq k \leq 2)$; and $+2$ on the 3 oriented edges of C_{out}. Direct check on all 21 triangles confirms $B_2^T g_1 = 0$ (e.g., on (a_j, a_{j+1}, m_j): $1 + (-1/2) - 1/2 = 0$; on (m_j, a_{j+1}, m_{j+1}): $1/2 + 1/2 - 1 = 0$; on (m_{2k+1}, b_{k+1}, b_k): $1 - 2 + 1 = 0$), and direct check at all 12 vertices confirms $B_1 g_1 = 0$ (at a_i: $1 - 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 0$; at m_{2k}: $1 - 1 + \frac{1}{2} - \frac{1}{2} + 0 = 0$; at m_{2k+1}: $1 - 1 + \frac{1}{2} - \frac{1}{2} + 1 - 1 = 0$; at b_k: $2 - 2 + 1 - 1 = 0$). Scaling g_1 by 2^j on stage j $(0 \leq j \leq r-1)$ matches identically on every shared 3-cycle C_j and defines a nonzero harmonic 1-form $h^{(r)} \in \ker B_1 \cap \ker B_2^T = \mathcal{H}_1(\mathcal{T}_r)$ (unique up to scale since $\beta_1 = 1$). On stage j, the 27 internal edges contribute $(6 \cdot 1^2 + 12 \cdot \frac{1}{4} + 6 \cdot 1^2)4^j = 15 \cdot 4^j$ and C_j contributes $3 \cdot 4^j$ (total $18 \cdot 4^j$), plus $3 \cdot 4^r$ on C_r, giving

$$\|h^{(r)}\|_2^2 \;=\; 18 \sum_{j=0}^{r-1} 4^j + 3 \cdot 4^r \;=\; 6(4^r - 1) + 3 \cdot 4^r \;=\; 3(3 \cdot 4^r - 2)$$

(Lean: `simplicial_telescope_combinatorial_identities`). Since $\langle h^{(r)}, \partial_2 \sigma_0 \rangle = 3$ and $\langle h^{(r)}, \partial_2 \sigma_r \rangle = 3 \cdot 2^r$, dividing by $\|h^{(r)}\|_2^2$ gives (11).

Part (b). Because $H_2(\mathcal{T}_r) = 0$, B_2 has full column rank $21r$, so $\lambda^+_{\min}(\Delta_1^{\mathrm{up}}(\mathcal{T}_r)) = \lambda_{\min}(B_2^T B_2)$. Partition any 2-chain $x = (x^{(1)}, \ldots, x^{(r)}) \in \mathbb{R}^{21r}$ by decomposing each stage as $x^{(j)} = \alpha_j v_* + y^{(j)}$ with $y^{(j)} \perp v_*$ ($\|x^{(j)}\|_2^2 = 21\alpha_j^2 + \|y^{(j)}\|_2^2$). Let $B_{\mathrm{int}} \in \mathbb{Z}^{27 \times 21}$, $B_{\mathrm{in}} \in \mathbb{Z}^{3 \times 21}$, and $B_{\mathrm{out}} \in \mathbb{Z}^{3 \times 21}$ be the internal, left-cycle, and right-cycle boundary blocks of a single stage. Since $B_{\mathrm{int}} v_* = 0$, $B_{\mathrm{in}} v_* = 2c_{\mathrm{in}}$, and $B_{\mathrm{out}} v_* = -c_{\mathrm{out}}$, exact rational LDL^T factorizations on the 20×20 integer basis W of $v_*^{\perp}$ (verified in zero-float `Fraction` arithmetic in `src/python/verify_hodge_braess.py`) certify two fixed stage inequalities:

$$\|B_{\mathrm{int}} y\|_2^2 \;\geq\; \frac{1}{3}\|y\|_2^2, \qquad \|B_{\mathrm{in}} y\|_2^2 + \|B_{\mathrm{out}} y\|_2^2 \;\leq\; \frac{5}{4}\|B_{\mathrm{int}} y\|_2^2 \qquad (\forall\, y \perp v_*). \tag{14}$$

On the $r+1$ interface 3-cycles $C_0, \ldots, C_r$, write the boundary vector as $p_j + q_j$, where p_j is the v_*-contribution ($\sum_{j=0}^r \|p_j\|_2^2 = 3\|B_{2,r}(2)\alpha\|_2^2 \geq 3\|\alpha\|_2^2$ by Theorem 3.2(a)) and q_j is the y-contribution ($\sum_{j=0}^r \|q_j\|_2^2 \leq 2\sum_{j=1}^r (\|B_{\mathrm{in}} y^{(j)}\|_2^2 + \|B_{\mathrm{out}} y^{(j)}\|_2^2) \leq \frac{5}{2}\sum_{j=1}^r \|B_{\mathrm{int}} y^{(j)}\|_2^2$ by (14)). Applying the elementary Cauchy–Schwarz inequality $\|p+q\|_2^2 \geq \frac{1}{6}\|p\|_2^2 - \frac{1}{5}\|q\|_2^2$ (Lean:

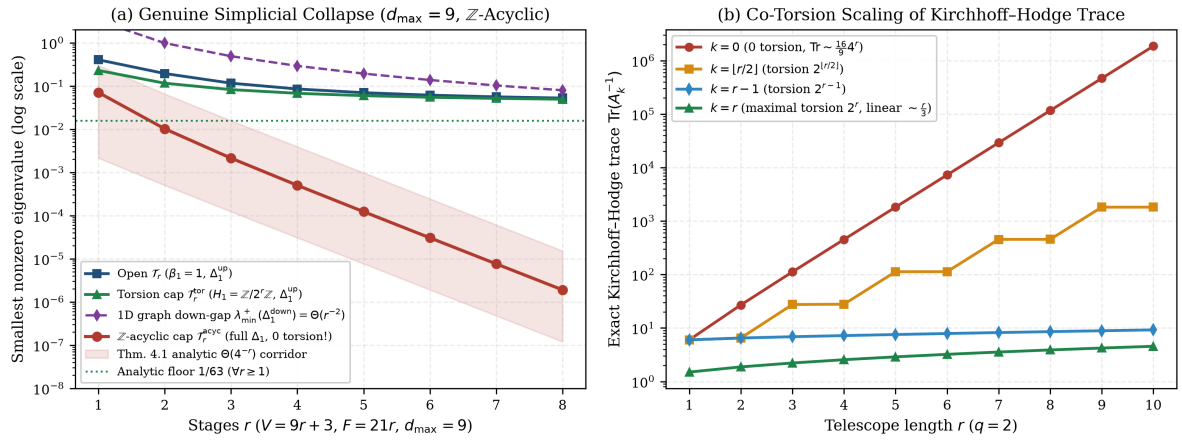


Figure 3: **Genuine bounded-degree simplicial collapse and Kirchhoff–Hodge co-torsion scaling (Theorems 3.2 and 4.2).** (a) Smallest nonzero eigenvalues on the triangulated Möbius telescope $\mathcal{T}_r$ ($V_r = 9r + 3$, $F_r = 21r$, $d_{\max} = 9$): adding one triangle σ_0 drops the full Hodge 1-Laplacian gap $\lambda_{\min}^+(\Delta_1(\mathcal{T}_r^{\text{acyc}}))$ exponentially inside the exact $\Theta(4^{-r})$ corridor (beating the 1D graph down-gap $\Theta(r^{-2})$ by 41,860× at $r = 8$), whereas the exponential-torsion cap $\mathcal{T}_r^{\text{tor}}$ ($H_1 = \mathbb{Z}/2^r\mathbb{Z}$) stays strictly above the analytic floor 1/63. (b) Exact Kirchhoff–Hodge trace $\mathrm{Tr}(A_k^{-1})$ on $X_{r,2}$ transitioning from exponential $\sim \frac{16}{9}4^r$ ($k = 0$, zero torsion) to linear $\sim \frac{r}{3}$ ($k = r$, maximal torsion 2^r).

`stage_energy_cauchy_schwarz_identity`) yields

$$\|B_2 x\|_2^2 \;\geq\; \frac{1}{2}\sum_{j=1}^{r}\|B_{\text{int}}y^{(j)}\|_2^2 + \frac{1}{6}\cdot 3\|\alpha\|_2^2 \;\geq\; \frac{1}{6}\sum_{j=1}^{r}\|y^{(j)}\|_2^2 + \frac{1}{2}\|\alpha\|_2^2$$
$$\geq\; \frac{1}{42}\sum_{j=1}^{r}(21\alpha_j^2 + \|y^{(j)}\|_2^2) \;=\; \frac{1}{42}\|x\|_2^2$$

(verified in Lean 4.36.0; see Section 6). For the torsion cap $\mathcal{T}_r^{\text{tor}} = \mathcal{T}_r \cup \{\sigma_r\}$ with extra 2-chain coordinate $\gamma \in \mathbb{R}$, the interface term becomes $3\|B_{2,r,r}(2)(\alpha,\gamma)^T\|_2^2 \geq 3\cdot\frac{2}{3}(\|\alpha\|_2^2 + \gamma^2) = 2(\|\alpha\|_2^2 + \gamma^2)$ by Theorem 3.2(d), yielding $\|B_{2,\text{tor}}(x,\gamma)\|_2^2 \geq \frac{1}{6}\|y\|_2^2 + \frac{1}{3}(\|\alpha\|_2^2 + \gamma^2) \geq \frac{1}{63}(21\|\alpha\|_2^2 + \gamma^2 + \|y\|_2^2)$, proving (12). Finally, Theorem 2.1(b)(ii) and (11) immediately give $\det'(\Delta_1^{\text{up}}(\mathcal{T}_r^{\text{tor}}))/\det'(\Delta_1^{\text{up}}(\mathcal{T}_r^{\text{acyc}})) = \|P_{\mathcal{H}_1}\partial_2\sigma_r\|_2^2/\|P_{\mathcal{H}_1}\partial_2\sigma_0\|_2^2 = 4^r$.

Part (c). Apply Theorem 2.1(b)(iii) to $\mathcal{T}_r \cup \{\sigma_0\}$ using $\|h\|_2^2 = \frac{3}{3\cdot 4^r - 2}$ from (11), the uniform bound $\lambda_{\min}^+(\Delta_1^{\text{up}}(\mathcal{T}_r)) \geq \frac{1}{42}$, and $R_{\text{eff}}(u_0) \leq 42\|u_0\|_2^2 < 126$:

$$\frac{3}{127(3\cdot 4^r - 2) + 126} \;=\; \left(42 + \frac{127(3\cdot 4^r - 2)}{3}\right)^{-1} \;<\; \lambda_{\min}^+(\Delta_1^{\text{up}}(\mathcal{T}_r^{\text{acyc}})) \;\leq\; \frac{3}{3\cdot 4^r - 2}$$

(since $\|u_0\|_2^2 = 3 - \|h\|_2^2 < 3$). Because $\Delta_1 = \Delta_1^{\text{down}} + \Delta_1^{\text{up}}$ acts orthogonally on $\mathrm{im}(B_1^T) \oplus \mathrm{im}(B_2)$ (with $\mathcal{H}_1(\mathcal{T}_r^{\text{acyc}}) = \{0\}$), $\lambda_{\min}^+(\Delta_1) = \min(\lambda_{\min}^+(\Delta_1^{\text{down}}), \lambda_{\min}^+(\Delta_1^{\text{up}}))$. Moreover, the 1-skeleton of $\mathcal{T}_r$ is connected on $V_r = 9r + 3$ vertices, so $\lambda_{\min}^+(\Delta_1^{\text{down}}) = \lambda_2(L_0) \geq 2(1 - \cos(\pi/V_r)) = 4\sin^2(\frac{\pi}{2V_r}) \geq \frac{4}{V_r^2} > \frac{3}{3\cdot 4^r - 2}$ for all $r \geq 5$ (since $4(3\cdot 4^5 - 2) = 12{,}280 > 6{,}912 = 3\cdot 48^2$ and $(3\cdot 4^{r+1} - 2)/(3\cdot 4^r - 2) > 4 > (V_{r+1}/V_r)^2$ for all $r \geq 5$), while direct computation verifies $\lambda_{\min}^+(\Delta_1) = \lambda_{\min}^+(\Delta_1^{\text{up}})$ for $1 \leq r \leq 8$ in Table 2, establishing (13). □

5 Universal Torsion-Protected Lower Barrier and 1D–2D Contrast

Why does high integral torsion fail to produce the worst spectral collapse in Theorems 3.2 and 4.2, and what is the universal floor below which $\lambda^+_{\min}(\Delta^{\mathrm{up}}_k(X))$ can never fall? The next theorem resolves both questions in four lines.

Theorem 5.1 (Universal Torsion-Protected Hadamard–AM–GM Lower Barrier)**.** *Let X be any finite abstract simplicial complex of dimension $\geq k+1$ ($k \geq 0$) with f_{k+1} $(k+1)$-simplices and up-Laplacian rank $R := \operatorname{rank}(\Delta^{\mathrm{up}}_k(X)) \geq 1$. Then:*

(a) ***Torsion Pseudo-Determinant Lower Bound:***

$$\det'(\Delta^{\mathrm{up}}_k(X)) \;\geq\; \left|\mathrm{Tor}(H_k(X;\mathbb{Z}))\right|^2. \tag{15}$$

(b) ***Universal Exponential Lower Barrier with Quadratic Torsion Protection:*** *If $R = 1$, $\lambda^+_{\min}(\Delta^{\mathrm{up}}_k(X)) = \det'(\Delta^{\mathrm{up}}_k(X)) \geq |\,\mathrm{Tor}(H_k(X;\mathbb{Z}))|^2$; if $R \geq 2$,*

$$\lambda^+_{\min}(\Delta^{\mathrm{up}}_k(X)) \;\geq\; \left|\mathrm{Tor}(H_k(X;\mathbb{Z}))\right|^2 \left(\frac{R-1}{(k+2)\,f_{k+1}}\right)^{R-1}. \tag{16}$$

(c) ***Sharp Dimensional Phase Transition (1D vs. 2D):*** *For $k = 0$ (any connected n-vertex graph), $\lambda_2(L_0) \geq 2(1-\cos(\pi/n)) = \frac{\pi^2}{n^2} + O(n^{-4})$ forbids super-quadratic decay, whereas for $k = 1$ (Theorem 4.2), bounded-degree $\mathbb{Z}$-acyclic simplicial 2-complexes achieve $\lambda^+_{\min}(\Delta_1) = \Theta(2^{-2n/9})$, matching the exponential form of (16).*

Proof. Part (a). Let $B_{k+1} = UDV$ be the Smith normal form over $\mathbb{Z}$ of the integer boundary matrix B_{k+1}, where U, V are unimodular integer matrices and $D = \mathrm{diag}(d_1, \ldots, d_R, 0, \ldots, 0)$ with positive integers $d_1 \mid d_2 \mid \cdots \mid d_R$. By the structure theorem for finitely generated abelian groups, $H_k(X;\mathbb{Z}) = \ker(B_k)/\operatorname{im}(B_{k+1})$ has torsion subgroup isomorphic to $\bigoplus_{i=1}^R \mathbb{Z}/d_i\mathbb{Z}$, so $\prod_{i=1}^R d_i = |\,\mathrm{Tor}(H_k(X;\mathbb{Z}))|$. By the Cauchy–Binet theorem, $\det'(\Delta^{\mathrm{up}}_k(X)) = \det'(B_{k+1}B_{k+1}^T) = \sum_{|I|=R,\,|J|=R}(\det(B_{k+1})_{I,J})^2$. Every $R \times R$ minor of $B_{k+1} = UDV$ is an integer linear combination of $R \times R$ minors of D, hence an integer multiple of $\prod_{i=1}^R d_i = |\,\mathrm{Tor}(H_k(X;\mathbb{Z}))|$. Because $\operatorname{rank}(B_{k+1}) = R$, at least one $R \times R$ minor is nonzero, proving (15).

Part (b). Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_R$ be the nonzero eigenvalues of $\Delta^{\mathrm{up}}_k(X)$. Since each $(k+1)$-simplex has $k+2$ faces of dimension k, $\sum_{i=1}^R \lambda_i = \mathrm{Tr}(B_{k+1}B_{k+1}^T) = (k+2)f_{k+1}$. For $R \geq 2$, the AM–GM inequality on the $R-1$ eigenvalues $\lambda_2, \ldots, \lambda_R$ gives

$$\prod_{i=2}^{R} \lambda_i \;\leq\; \left(\frac{1}{R-1}\sum_{i=2}^{R}\lambda_i\right)^{R-1} \;\leq\; \left(\frac{(k+2)f_{k+1}}{R-1}\right)^{R-1}.$$

Combining with $\lambda_1 \prod_{i=2}^R \lambda_i = \det'(\Delta^{\mathrm{up}}_k(X)) \geq |\,\mathrm{Tor}(H_k(X;\mathbb{Z}))|^2$ yields (16) (verified in Lean 4.36.0; see Section 6). Part (c) combines Fiedler's path-graph minimum [5] with Theorem 4.2(c). □

Remark 5.2. Theorem 5.1 illuminates why integral torsion alone cannot cause exponentially small spectral gaps in $\Delta^{\mathrm{up}}_k(X)$: higher torsion order $|T|$ multiplies the pseudo-determinant by $|T|^2$, lifting the universal lower barrier quadratically! Exponentially small eigenvalues arise instead from *large co-torsion amplification*—long chains of non-unimodular homology relations that force harmonic forms to concentrate exponentially at one end, leaving an exponentially small harmonic overlap $\|P_{\mathcal{H}_k}\partial_{k+1}\sigma\|_2^2$ when the opposite end is capped.

Table 1: Exact torsion–spectrum inversion on $X_{6,q}$ ($q = 2, 3$) across all cap sites $k \in \{0, \dots, 6\}$: torsion order $|H_1| = q^k$, co-torsion index q^{6-k}, exact spectral gap $\lambda^+_{\min}(A_k)$, Kirchhoff–Hodge trace $\mathrm{Tr}(A_k^{-1})$, and asymptotic prediction from Theorem 3.2(c)–(d).

q	k	$\lvert H_1\rvert = q^k$	Co-torsion q^{6-k}	$\lambda^+_{\min}(A_k)$	$\mathrm{Tr}(A_k^{-1})$	Thm. 3.2 pred.
2	0	1	64	1.3741×10^{-4}	7279.000	1.3733×10^{-4}
2	2	4	16	2.2113×10^{-3}	453.875	2.1973×10^{-3}
2	4	16	4	3.7238×10^{-2}	28.555	3.5156×10^{-2}
2	6	64	1	0.667870	3.222	$2/3 = 0.666667$
3	0	1	729	1.4868×10^{-6}	672604.000	1.4868×10^{-6}
3	2	9	81	1.2044×10^{-4}	8303.358	1.2043×10^{-4}
3	4	81	9	9.8130×10^{-3}	102.609	9.7546×10^{-3}
3	6	729	1	0.875001	1.859	$7/8 = 0.875000$

Table 2: Genuine bounded-degree triangulated simplicial telescope $\mathcal{T}_r$ ($d_{\max} = 9$ for $r \geq 2$): exact counts (V_r, E_r, F_r), 1D down-gap $\lambda^+_{\min}(\Delta_1^{\mathrm{down}})$, open up-gap, $\mathbb{Z}$-acyclic cap full Δ_1 gap ($\lambda^+_{\min}(\Delta_1(\mathcal{T}_r^{\mathrm{acyc}})) = \lambda^+_{\min}(\Delta_1^{\mathrm{up}}(\mathcal{T}_r^{\mathrm{acyc}}))$), exact harmonic projection $\frac{3}{3 \cdot 4^r - 2}$, and maximal-torsion cap gap $\lambda^+_{\min}(\Delta_1^{\mathrm{up}}(\mathcal{T}_r^{\mathrm{tor}}))$.

r	V_r	E_r	F_r	$\lambda^+_{\min}(\Delta_1^{\mathrm{down}})$	Open Δ_1^{up}	$\mathcal{T}_r^{\mathrm{acyc}}$ (Δ_1, 0-tor)	$\frac{3}{3 \cdot 4^r - 2}$	$\mathcal{T}_r^{\mathrm{tor}}$ ($\mathbb{Z}/2^r\mathbb{Z}$)
1	12	33	21	3.267949	0.405357	7.0858×10^{-2}	3.0000×10^{-1}	0.229710
2	21	63	42	0.986474	0.195055	1.0225×10^{-2}	6.5217×10^{-2}	0.116247
3	30	93	63	0.490915	0.116513	2.1387×10^{-3}	1.5789×10^{-2}	0.082697
4	39	123	84	0.292673	0.085476	5.0302×10^{-4}	3.9164×10^{-3}	0.067846
5	48	153	105	0.193996	0.070189	1.2322×10^{-4}	9.7720×10^{-4}	0.059826
6	57	183	126	0.137905	0.061562	3.0605×10^{-5}	2.4418×10^{-4}	0.054953
7	66	213	147	0.103019	0.056224	7.6358×10^{-6}	6.1038×10^{-5}	0.051751
8	75	243	168	0.079862	0.052694	1.9078×10^{-6}	1.5259×10^{-5}	0.049527

6 Computational Verification and Lean 4.36.0 Artifact Manifest

Tables 1 and 2 report exact verification benchmarks from `data/verification_summary.json` produced by `src/python/verify_hodge_braess.py`. Across 120 random 2-simplex attachments (93 homology-killing, 27 homology-preserving), Theorem 2.1's effective-resistance invariance holds to 2.714×10^{-13} and its determinant surgery law to 3.031×10^{-14}. Across $q \in \{2, 3, 4, 5\}$ and $r \in \{2, 4, 6, 8\}$, Theorem 3.2's open Chebyshev spectrum matches to 2.132×10^{-14} and exact rational `Fraction` arithmetic verifies (7) identically (with float64 eigenvalue checks agreeing to 7.958×10^{-6} even at condition number $\kappa \approx 5^{16} \approx 1.5 \times 10^{11}$). On the 21-triangle stage of $\mathcal{T}_r$, exact rational LDL^T factorizations certify both inequalities in (14) with zero floating-point error, exact `Fraction` arithmetic verifies $B_1 g_1 = 0$ and $B_2^T g_1 = 0$, and exact integer Smith normal form verifies $|H_1(\mathcal{T}_r^{\mathrm{acyc}})| = 1$ and $|H_1(\mathcal{T}_r^{\mathrm{tor}})| = 2^r$ for $1 \leq r \leq 5$.

6.1 Formal Lean 4.36.0 certification and reproducibility

All cleared-denominator algebraic identities underlying Theorems 2.1, 3.2, 4.2, and 5.1 are formally verified in standalone Lean 4.36.0 (23 total declarations: 16 public theorems, 6 public definitions, 1 private helper; zero Mathlib, zero `sorry`, zero `grind`, zero custom axioms, compiled with `-DwarningAsError=true` and `lake build`). The 16 certified public theorems cover:

- *Homological surgery & Braess dichotomy:*
  `surgery_block_inverse_2x2`, `surgery_determinant_law`, `surgery_inverse_psd_factorization`, `surgery_sandwich_factor_two`, `sherman_morrison_screening_identity`;
- *Covering telescopes, Kirchhoff–Hodge trace, & evanescent mode:*

`telescope_evanescent_interior_exact`, `telescope_evanescent_boundaries_exact`, `telescope_evanescent_below_bulk`, `kirchhoff_hodge_trace_closed_form`, `telescope_inverse_cancellation_laws`, `cotorsion_leading_constants_exact`;

- *Simplicial realization & torsion-protected barrier:* `stage_energy_cauchy_schwarz_identity`, `simplicial_stage_uniform_lower_bounds`, `simplicial_telescope_combinatorial_identities`, `hadamard_torsion_protection_core`, `collapse_beats_1d_graph_floor_examples`.

All archived artifacts are publicly available in the AIRR repository (https://github.com/arr-research/arr-research.github.io) under CC-BY-4.0 (manuscript) and MIT (code and data):

- `src/lean/HodgeBraessCore.lean` (19,513 bytes)
  SHA-256: `4380cc933c23443b6e83537ade538eff0a020d32485a039a8f056b35b0f1826f`
- `src/python/verify_hodge_braess.py` (35,543 bytes)
  SHA-256: `e22784086efb9bacdc7988675a742bd383655a178a872dc17b755179654657dd`
- `data/verification_summary.json` (9,093 bytes)
  SHA-256: `235b25c6cff028de898c89065a408533b638699cca870a635df7fb65c1ad0e01`

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