

Complete Rate–Distortion Phase Diagram of Haar Oriented Two-Planes

One-change hypergeometric coefficients, unique coexistence, no reentrance, and sharp thermodynamic contact asymptotics

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Abstract

Let $W = x \wedge y$ be the signed unit Plücker coordinate of a Haar oriented two-plane in $\mathbb{R}^n$, and let reproduction range over the full posterior-mean body under squared ambient loss. An all-field extremum theorem reduces its unrestricted Shannon rate–distortion function to

$$\phi_\lambda(b) = K_q(2\lambda b) - \lambda b^2, \quad q = n - 2, \quad K_q = \log \left(q! \sum_{j \geq 0} \frac{\kappa^{2j}}{(q + 2j)!} \right).$$

We solve the remaining global phase problem in every dimension. For $J_q = L_q L'_q - \kappa L_q L''_q + \kappa (L'_q)^2$, its coefficient of κ^{2r-1} is a positive factor times $r - \mathbb{E}d^2$, where d follows a central parity-truncated binomial law. Likelihood-ratio ordering and an exact tail estimate prove one coefficient sign change for every $q \geq 4$. Consequently the stationary curve has exactly one fold, the zero and positive phases have exactly one coexistence contact, and no later exchange or radial reentrance is possible. The RDF is therefore an explicit linear face followed by one covariant parametric branch for every $n \geq 6$; together with the continuous cases $n = 3, 4, 5$, this completes the all-dimensional phase diagram. The unique contact has an exact large-dimension limit and an explicit logarithmic finite-size displacement. We additionally prove that the same RDF is the minimum worst-source channel capacity, that arbitrary joint classical memory costs exactly $NR_n(D)$ on blocks of length N , and that transitivity makes the tilted information constant, giving a zero-dispersion strong converse. The results concern signed Plücker-coordinate compression, not quantum rate–distortion or Born-probability prediction.

1 Introduction

The oriented Grassmannian $\text{Gr}_2^+(\mathbb{R}^n)$ embeds in the unit sphere of $\Lambda^2 \mathbb{R}^n$ through the signed Plücker coordinate $W = x \wedge y$. Its convex hull is the bivector mass-norm ball, so optimal posterior reports need not themselves be planes. This distinction separates the present Shannon problem from codebook quantization on a Grassmann manifold [1–3].

This unified manuscript first establishes the spectral and operational reductions, using classical orbital-integral, Bingham, and orbitope structures [4–7]. A positive complete-homogeneous series makes simple external bivectors the unique fixed-norm maximizers at every field, the maximizing overlap has density $(n - 2)(1 - |x|)^{n-3}/2$, and Gibbs duality reduces the unrestricted RDF to

one radius. What remained open was global: in dimensions $n \geq 6$, the fourth cumulant proves a discontinuous first activation but does not by itself rule out several positive contacts, additional folds, later branch exchanges, or reentrance. Those reductions appeared in an earlier companion draft, which the present complete version supersedes and absorbs. Generalized Curie–Weiss theory permits precisely such behavior for other single-site laws [8–11]; a local Landau expansion cannot settle it.

We close that gap by an exact one-change theorem for the full exponential series. Its key step is combinatorial rather than asymptotic: every derivative coefficient is governed by the conditional variance of a parity-truncated centered binomial variable. Central truncation, monotone-likelihood-ratio ordering, and one uniform tail bound determine all signs simultaneously. Related log-convexity and Turán inequalities for hypergeometric and reciprocal-gamma series are developed in [12, 13]; the parity-window sign classification needed here is different and is proved directly.

The main new outputs are:

1. a dimension-uniform coefficient theorem showing that the radial stationary curve has exactly one fold for every $n \geq 6$;
2. uniqueness of the positive coexistence contact and of the active positive branch, with no later exchange or radial reentrance;
3. the resulting exact two-piece RDF in every high dimension, completing the phase classification for all $n \geq 3$;
4. the unique contact’s thermodynamic limit and logarithmic finite-size shift;
5. equality between this RDF and the minimum worst-source channel capacity, exact N -block tensorization for arbitrary joint memory, and a constant-tilted-information strong converse.

The novelty claim is deliberately narrow. Plücker geometry, Grassmann orbitopes, Bingham families, generalized Curie–Weiss variational methods, compact-group symmetrization, information radius, and finite-block tilted information are prior structure [14–17]. To our knowledge, the parity-binomial coefficient law and the resulting complete branch/contact classification have not been given for the explicit Haar oriented-plane overlap family. The universal and block statements are operational consequences of that exact source-specific solution, not new general minimax theorems; classwise rate–distortion goes back at least to Sakrison [18].

2 Geometry, norm, and source model

Let

$$\mathcal{G}_n = \text{Gr}_2^+(\mathbb{R}^n) = \{x \wedge y : x, y \in \mathbb{R}^n, \langle x, y \rangle = 0, \|x\| = \|y\| = 1\} \subset \Lambda^2 \mathbb{R}^n$$

with normalized Haar probability μ . We use the exterior inner product for which $e_i \wedge e_j$, $i < j$, is an orthonormal basis. Hence every $W \in \mathcal{G}_n$ has $\|W\| = 1$.

Every $B \in \Lambda^2 \mathbb{R}^n$ has an orthogonal canonical form

$$B = \sum_{j=1}^m p_j e_{2j-1} \wedge e_{2j}, \quad m = \lfloor n/2 \rfloor, \quad \|B\|^2 = \sum_j p_j^2. \quad (1)$$

Let C_B be the associated skew matrix, defined by $\langle B, u \wedge v \rangle = \langle C_B u, v \rangle$. Our convention gives

$$\|C_B\|_F^2 = 2\|B\|^2. \quad (2)$$

All fixed-radius statements use the exterior norm in [equation \(1\)](#); [equation \(2\)](#) records the otherwise easy-to-miss factor of two.

For real s , define

$$Z_n(s, B) = \mathbb{E}_{W \sim \mu} e^{s \langle B, W \rangle}. \quad (3)$$

The law is symmetric because both orientations occur, so Z_n is even in s and B .

3 Positive series and all-field rigidity

Generate $W = x \wedge y$ by drawing x uniformly from S^{n-1} and then y uniformly from the unit sphere in $x^\perp$. Since $C_B x \perp x$,

$$\langle B, x \wedge y \rangle = \langle C_B x, y \rangle \stackrel{d}{=} \|C_B x\| Y,$$

where Y is a coordinate of the uniform law on S^{n-2} . Its moment-generating series is

$$\mathbb{E} e^{tY} = \sum_{k \geq 0} \frac{(t^2/4)^k}{k!((n-1)/2)_k}. \quad (4)$$

Put $q_j = x_{2j-1}^2 + x_{2j}^2$. The paired blocks, and for odd n the final unpaired coordinate square, form a Dirichlet vector with paired parameters one and total parameter $n/2$. The unpaired block has coefficient zero in $\|C_B x\|^2$. Hence

$$\mathbb{E} \left(\sum_j p_j^2 q_j \right)^k = \frac{k!}{(n/2)_k} h_k(p_1^2, \dots, p_m^2), \quad (5)$$

where h_k is the complete homogeneous symmetric polynomial.

Theorem 3.1 (positive all-degree orbital series). *For every $n \geq 3$, $B \in \Lambda^2 \mathbb{R}^n$, and real s ,*

$$Z_n(s, B) = \sum_{k=0}^{\infty} \frac{(s^2/4)^k h_k(p_1^2, \dots, p_m^2)}{((n-1)/2)_k (n/2)_k}. \quad (6)$$

The series is absolutely convergent and every coefficient is nonnegative.

Proof. Substitute [equation \(5\)](#) into the conditional expansion [equation \(4\)](#). Absolute convergence follows directly from $|\langle B, W \rangle| \leq \|B\|$ or from the ratio test. $\square$

For nonnegative $z_1, \dots, z_m$,

$$h_k(z_1, \dots, z_m) \leq (z_1 + \dots + z_m)^k. \quad (7)$$

Indeed, each monomial coefficient on the left is one and the corresponding multinomial coefficient on the right is at least one. At $k = 2$, equality forces $z_i z_j = 0$ for $i \neq j$.

Theorem 3.2 (all-field simple-bivector rigidity). *Fix $R > 0$ and $s \neq 0$. Among all $B \in \Lambda^2 \mathbb{R}^n$ with $\|B\| = R$, the partition $Z_n(s, B)$ is uniquely maximized, up to the orthogonal action and sign, by a simple bivector $B = RQ$. Its value is*

$$L_n(sR) := {}_1F_2 \left(1; \frac{n-1}{2}, \frac{n}{2}; \frac{(sR)^2}{4} \right). \quad (8)$$

For $n = 3$ every nonzero bivector is simple, so the assertion is uniqueness of the fixed-radius orbit.

Proof. Apply [equation \(7\)](#) coefficientwise to [equation \(6\)](#), using $\sum_j p_j^2 = R^2$. Equality of the summed series at nonzero s forces equality in its strictly positive $k = 2$ term, hence exactly one p_j^2 is nonzero. This is precisely the canonical form of a simple bivector. At a simplex vertex $h_k(R^2, 0, \dots) = R^{2k}$, and summing gives [equation \(8\)](#) because $(1)_k/k! = 1$. $\square$

4 The exact overlap law

Let Q be a fixed unit simple bivector and $X = \langle W, Q \rangle$. The vertex specialization of [equation \(6\)](#) gives

$$\mathbb{E}X^{2k} = \frac{(2k)!}{4^k((n-1)/2)_k(n/2)_k} = \frac{(2k)!}{(n-1)_{2k}}, \quad \mathbb{E}X^{2k+1} = 0. \quad (9)$$

Proposition 4.1 (elementary overlap density). *The simple Plücker overlap has density*

$$f_n(x) = \frac{n-2}{2}(1-|x|)^{n-3}\mathbf{1}_{[-1,1]}(x). \quad (10)$$

Consequently

$$L_n(t) = (n-2) \int_0^1 \cosh(tx)(1-x)^{n-3} dx. \quad (11)$$

Proof. The density in [equation \(10\)](#) is normalized and symmetric. Its even moments are

$$(n-2) \int_0^1 x^{2k}(1-x)^{n-3} dx = (n-2)B(2k+1, n-2) = \frac{(2k)!}{(n-1)_{2k}},$$

which agree with [equation \(9\)](#). Compact support makes the moment problem determinate. Integrating e^{tx} against the density gives [equation \(11\)](#). $\square$

Thus the optimal normalizer is elementary in every integer dimension: it is an exponential divided by t^{n-2} plus a finite Taylor correction. In particular, $L_3(t) = \sinh(t)/t$. For $n \geq 4$ the orbit is a proper submanifold of its ambient sphere even though its optimal one-dimensional overlap remains elementary.

The second and fourth moments are

$$\mathbb{E}X^2 = \frac{2}{n(n-1)}, \quad \mathbb{E}X^4 = \frac{24}{n(n-1)(n+1)(n+2)}. \quad (12)$$

5 The full reproduction body

Write a bivector in canonical form $A = \sum_j p_j V_j$, where the V_j are orthogonal unit simple bivectors. Its mass norm is

$$\|A\|_{\text{mass}} = \sum_j |p_j|.$$

This is the atomic norm generated by unit simple bivectors; its polar is the comass norm. In degree two the canonical skew decomposition makes the description explicit.

Proposition 5.1 (oriented Grassmann orbitope).

$$\mathcal{K}_n := \text{conv}(\mathcal{G}_n) = \{A \in \Lambda^2 \mathbb{R}^n : \|A\|_{\text{mass}} \leq 1\}. \quad (13)$$

In particular, every simple ray bQ , $0 \leq b \leq 1$, is feasible.

Proof. The mass norm is convex, orthogonally invariant, and equals one on every orbit point, so the convex hull is contained in the displayed ball. Conversely,

$$A = \sum_j |p_j| \operatorname{sgn}(p_j) V_j + \left(1 - \sum_j |p_j|\right) 0,$$

and $0 = (V + (-V))/2$. This is a convex combination of oriented unit simple bivectors. $\square$

Let U be an arbitrary standard-Borel classical memory and let Y_U be any square-integrable $\Lambda^2\mathbb{R}^n$ -valued report. Conditional least squares gives

$$\mathbb{E}\|W - Y_U\|^2 = \mathbb{E}\|W - A_U\|^2 + \mathbb{E}\|A_U - Y_U\|^2, \quad A_U = \mathbb{E}(W \mid U) \in \mathcal{K}_n. \quad (14)$$

The deterministic replacement $U \mapsto A_U$ cannot increase mutual information. Thus arbitrary reports reduce exactly to the full body [equation \(13\)](#), not to orbit-valued quantization.

6 Exact unrestricted rate–distortion

Define the Shannon RDF under squared Plücker loss by

$$\mathcal{R}_n(D) = \inf I(W; U), \quad \mathbb{E}\|W - Y_U\|^2 \leq D, \quad (15)$$

where the infimum is over arbitrary standard-Borel memories and square-integrable reports. The natural interval is $0 \leq D \leq 1$: the constant zero report gives distortion one.

Put $K_n = \log L_n$ and

$$C_n(\lambda) = e^{-\lambda} \max_{0 \leq b \leq 1} \exp\{K_n(2\lambda b) - \lambda b^2\}, \quad \lambda \geq 0. \quad (16)$$

Theorem 6.1 (exact unrestricted Plücker RDF). *For every $n \geq 3$ and $0 \leq D \leq 1$,*

$$\boxed{\mathcal{R}_n(D) = \sup_{\lambda \geq 0} \left\{ \lambda(1 - D) - \max_{0 \leq b \leq 1} [K_n(2\lambda b) - \lambda b^2] \right\}.} \quad (17)$$

Every exposed positive-radius point is attained by a covariant exponential channel. Source-independent revealed flags attain all nonexposed points. The endpoints are $\mathcal{R}_n(1) = 0$ and $\mathcal{R}_n(0) = +\infty$.

Proof. For $\lambda \geq 0$, conditional Gibbs rate–distortion duality [[19](#), [20](#)] and [equation \(14\)](#) give

$$\begin{aligned} I(W; U) &\geq -\lambda D - \log \sup_{A \in \mathcal{K}_n} \mathbb{E} e^{-\lambda \|W - A\|^2} \\ &= -\lambda D - \log \sup_{A \in \mathcal{K}_n} e^{-\lambda(1 + \|A\|^2)} Z_n(2\lambda, A). \end{aligned} \quad (18)$$

At fixed Euclidean radius, [theorem 3.2](#) replaces A by a simple bQ without leaving $\mathcal{K}_n$. Maximizing over $0 \leq b \leq 1$ yields [equation \(16\)](#) and the converse in [equation \(17\)](#).

Let $b > 0$ be active at a finite field and put $\kappa = 2\lambda b$. The endpoint $b = 1$ is not active: under the finite tilted law $X < 1$ almost surely, so $K'_n(\kappa) < 1$ and

$$\partial_b [K_n(2\lambda b) - \lambda b^2]_{b=1} = 2\lambda [K'_n(2\lambda) - 1] < 0.$$

Every positive active radius is therefore interior and satisfies

$$b = K'_n(\kappa), \quad \kappa = 2\lambda b. \quad (19)$$

Draw Q Haar on $\mathcal{G}_n$ and specify the reverse channel

$$\frac{dP_\kappa(W \mid Q)}{d\mu(W)} = \frac{e^{\kappa \langle W, Q \rangle}}{L_n(\kappa)}. \quad (20)$$

Covariance makes the source marginal Haar. The tilted mean is the gradient of the analytic partition. At a simple external bivector, constrained stationarity in [theorem 3.2](#) annihilates

every fixed-sphere tangential derivative, so the gradient is radial; this also excludes the apparent additional $*Q$ stabilizer direction in $n = 4$. Hence

$$\mathbb{E}(W \mid Q) = K'_n(\kappa)Q = bQ.$$

The report is its posterior mean, so equality holds in [equation \(14\)](#) and [equation \(18\)](#). The attained pair is

$$D(\kappa) = 1 - b(\kappa)^2, \quad R(\kappa) = \kappa b(\kappa) - K_n(\kappa). \quad (21)$$

If several radii are active at one field, an independently revealed flag mixes their covariant channels. Every component retains the Haar source marginal; rate and distortion therefore mix affinely. Danskin's theorem identifies the subdifferential of the dual envelope with the convex hull of these active distortions. The zero report attains $(1, 0)$. To see the high-field endpoint explicitly, $K_n(t) \leq t$ for $t \geq 0$. Thus every $b \leq 1 - \varepsilon$ has radial objective at most $\lambda(2b - b^2) \leq \lambda(1 - \varepsilon^2)$, whereas the feasible point $b = 1$ has value $K_n(2\lambda) - \lambda = \lambda - (n-2)\log(2\lambda) + O(1)$ by [theorem 11.1](#). Every active radius therefore tends to one and its distortion tends to zero. Monotonicity of convex subgradients covers $(0, 1)$, and closure supplies both endpoints. Exact reconstruction of a nonatomic positive-dimensional source requires infinite mutual information, giving $\mathcal{R}_n(0) = +\infty$. Fenchel–Moreau duality proves equality everywhere. $\square$

7 Complete radial phase below dimension six

Let

$$m_n(\kappa) = K'_n(\kappa), \quad \lambda_n(\kappa) = \frac{\kappa}{2m_n(\kappa)}.$$

A positive stationary radius satisfies $b = m_n(\kappa)$ and $\lambda = \lambda_n(\kappa)$. Moreover

$$\lambda'_n(\kappa) = \frac{m_n(\kappa) - \kappa m'_n(\kappa)}{2m_n(\kappa)^2}. \quad (22)$$

Lemma 7.1 (low-dimensional radial monotonicity). *For $n = 3, 4, 5$ and every $\kappa > 0$,*

$$m_n(\kappa) - \kappa m'_n(\kappa) > 0. \quad (23)$$

Proof. For $n = 3$, $L_3(x) = \sinh x/x$. Multiplying [equation \(23\)](#) by the positive factor $L_3(x)^2 x^3$ reduces it to

$$J_3(x) = x^2 + x \sinh x \cosh x - 2 \sinh^2 x > 0.$$

The coefficients below degree six vanish, and for $r \geq 3$ the coefficient of x^{2r} is

$$\frac{2^{2r-2}(2r-4)}{(2r)!} > 0.$$

For $n = 4$, the identity

$$L_4(x) = \frac{2(\cosh x - 1)}{x^2} = \left[\frac{\sinh(x/2)}{x/2} \right]^2$$

reduces the assertion to the $n = 3$ inequality at $x/2$.

For $n = 5$, $L_5(x) = 6(\sinh x - x)/x^3$. After multiplication by the positive factor $L_5(x)^2 x^7/36$, [equation \(23\)](#) becomes

$$\begin{aligned} J_5(x) &= x^3 \sinh x + x \sinh x \cosh x + 11x \sinh x \\ &\quad - 3x^2 \cosh x - 3x^2 - 6 \sinh^2 x > 0. \end{aligned}$$

The coefficients through degree eight vanish. For $r \geq 5$, the coefficient of x^{2r} multiplied by $(2r)!$ is

$$2^{2r-1}(r-6) + 8r(r^2 - 3r + 4).$$

Indeed, substitute the Taylor series for $\sinh x$, $\cosh x$, and $\sinh x \cosh x = \frac{1}{2} \sinh(2x)$ into J_5 . After collecting the x^{2r} terms over the common denominator $(2r)!$, the constant, quadratic, quartic, sextic, and octic coefficients cancel identically, while the remaining numerator is exactly the displayed expression. This makes the sign argument coefficientwise rather than numerical. It equals 48 at $r = 5$; for $r \geq 6$ both displayed contributions are nonnegative and the polynomial one is positive. $\square$

Theorem 7.2 (complete continuous phase for $n = 3, 4, 5$). *For $n = 3, 4, 5$, directional information activates continuously at*

$$\lambda_0 = \frac{n(n-1)}{4}.$$

For $0 \leq \lambda \leq \lambda_0$, the zero radius is the unique global maximizer. For $\lambda > \lambda_0$, there is exactly one positive global maximizer; it tends continuously to zero at the threshold. There is no coexistence, later radial exchange, or reentrance.

Proof. By equation (22) and Lemma 7.1, $\lambda_n(\kappa)$ is strictly increasing. The moment expansion and the high-field limit give

$$\lim_{\kappa \downarrow 0} \lambda_n(\kappa) = \frac{1}{2 \text{Var } X} = \frac{n(n-1)}{4}, \quad \lambda_n(\kappa) \rightarrow \infty.$$

There is no positive stationary radius below the threshold and exactly one above it. At $\lambda > \lambda_0$ the zero radius is locally unstable, so the unique positive stationary point is the unique global maximum; the endpoint $b = 1$ was excluded in the proof of theorem 6.1. Strict monotonicity gives continuous emergence and rules out every later exchange. $\square$

8 The one-change coefficient theorem

Put $q = n - 2$ and write the overlap normalizer as

$$L_q(\kappa) = q! \sum_{j \geq 0} \frac{\kappa^{2j}}{(q+2j)!}, \quad K_q = \log L_q, \quad m_q = K'_q. \quad (24)$$

For $q \geq 4$ define

$$J_q(\kappa) = L_q L'_q - \kappa L_q L''_q + \kappa (L'_q)^2 = L_q^2 \{m_q - \kappa m'_q\}. \quad (25)$$

Lemma 8.1 (parity-window likelihood-ratio order). *Fix $r \geq 1$. On $d = -r, -r+2, \dots, r$, let*

$$\mu_{M,r}(d) = \frac{\binom{2M}{M+d}}{\sum_{|e| \leq r, e \equiv r \pmod{2}} \binom{2M}{M+e}}, \quad M \geq r.$$

If $g(d^2)$ is nondecreasing, then $\mathbb{E}_{\mu_{M,r}} g(d^2)$ is nondecreasing in M ; it is strictly increasing when g is nonconstant on the support and $r \geq 2$.

Proof. On the fixed parity window,

$$\frac{\binom{2M+2}{M+1+d}}{\binom{2M}{M+d}} = \frac{(2M+2)(2M+1)}{(M+1)^2 - d^2},$$

which is strictly increasing in d^2 . Thus $\mu_{M+1,r}$ dominates $\mu_{M,r}$ in monotone-likelihood-ratio order after the states are ordered by d^2 . The standard covariance identity

$$\mathbb{E}_{M+1,r} g - \mathbb{E}_{M,r} g = \frac{\text{Cov}_{M,r}(g(d^2), R_M(d^2))}{\mathbb{E}_{M,r} R_M(d^2)}$$

with $R_M = \mu_{M+1,r}/\mu_{M,r}$ proves the claim; strictness follows because both factors are then nonconstant and similarly ordered. $\square$

Theorem 8.2 (one-change law). *If*

$$J_q(\kappa) = \sum_{r \geq 1} c_{q,r} \kappa^{2r-1},$$

then, for every integer $q \geq 4$,

$$c_{q,r} < 0 \quad (2 \leq r < q), \quad c_{q,r} > 0 \quad (r \geq q),$$

except that $c_{4,3} = 0$; also $c_{q,1} = 0$. Consequently $m_q(\kappa) - \kappa m'_q(\kappa)$ has exactly one positive zero.

Proof. Set $a_j = q!/(q+2j)!$. Cauchy multiplication in [equation \(25\)](#), followed by symmetrization under $i \leftrightarrow j$, gives

$$c_{q,r} = 2 \sum_{i+j=r} [r - (i-j)^2] a_i a_j. \quad (26)$$

Put $M = q + r$ and $d = 2i - r$. Apart from the positive factor $2(q!)^2/(2M)!$, [equation \(26\)](#) is

$$S_{M,r} = \sum_{d=-r, -r+2, \dots, r} (r - d^2) \binom{2M}{M+d}. \quad (27)$$

Thus its sign is that of $r - \mathbb{E}_{M,r} d^2$, where the expectation uses the normalized central parity-window weights in [equation \(27\)](#).

On the full matching parity lattice, the centered $\text{Bin}(2M, 1/2)$ law has second moment $M/2$. Indeed, differentiating $(1+x)^{2M}$ at $x = -1$ through order two gives

$$\sum_d (-1)^d \binom{2M}{M+d} d^j = 0, \quad j = 0, 1, 2,$$

so both parity classes have mass 2^{2M-1} , mean zero, and second moment $M/2$. Conditioning it on $|d| \leq r$ strictly decreases that moment. If $r \geq q$, then $M/2 = (q+r)/2 \leq r$, proving $c_{q,r} > 0$.

Suppose $r < q$. By [theorem 8.1](#), $\mathbb{E}_{M,r} d^2$ strictly increases with M . For $r = 2$, the first relevant case is $(q, M) = (4, 6)$ and $\mathbb{E}_{6,2} d^2 = 660/319 > 2$. For $r = 3, q = 4$, exact weights give $\mathbb{E}_{7,3} d^2 = 3$; larger q are strict.

For $r \geq 4$ it remains to consider the boundary $q = r + 1, M = 2r + 1$. The full parity variance is $r + 1/2$. Hence the central variance exceeds r if the deleted-tail excess

$$T_r = \mathbb{E}[(d^2 - r) \mathbf{1}_{\{|d| \geq r+2\}} \mid d \equiv r \pmod{2}]$$

is below $1/2$. On one tail the successive binomial weights have ratio at most

$$\rho_r = \frac{(r-1)(r-2)}{(3r+4)(3r+5)}.$$

Summing the resulting geometric majorant gives

$$T_r \leq B_r := \frac{4 \binom{4r+2}{3r+3}}{2^{4r+2}} P_r, \quad (28)$$

where, with $a = r + 2$,

$$P_r = \frac{a^2 - r}{1 - \rho_r} + \frac{4a\rho_r}{(1 - \rho_r)^2} + \frac{4\rho_r(1 + \rho_r)}{(1 - \rho_r)^3}.$$

Equivalently,

$$P_r = \frac{(3r+4)(3r+5)(16r^6 + 176r^5 + 753r^4 + 1591r^3 + 1943r^2 + 1353r + 440)}{2(r+3)^3(4r+3)^3}.$$

Direct rational simplification yields, for $r \geq 4$,

$$\frac{3}{5} - \frac{\binom{4r+6}{3r+6}/2^{4r+6}}{\binom{4r+2}{3r+3}/2^{4r+2}} = \frac{2r^4 + 90r^3 + 142r^2 - 135r - 225}{30r(r+2)(3r+4)(3r+5)} > 0,$$

and $7/5 - P_{r+1}/P_r > 0$. The exact positive polynomial for the latter difference is recorded in [section A](#). Hence

$$\frac{\binom{4r+6}{3r+6}/2^{4r+6}}{\binom{4r+2}{3r+3}/2^{4r+2}} < \frac{3}{5}, \quad \frac{P_{r+1}}{P_r} < \frac{7}{5}.$$

Thus $B_{r+1}/B_r < 21/25$, while

$$B_4 = \frac{125128041}{301137536} < \frac{1}{2}.$$

This proves the claimed signs.

Finally set $y = \kappa^2$. Absolute local uniform convergence follows from the entire moment-generating series, so termwise differentiation is valid on compact subintervals of $(0, \infty)$. After removing the zero $r = 1$ term, the derivative of

$$y^{-q} \sum_{r \geq 1} c_{q,r} y^r$$

has only positive nonzero terms: $(r - q)c_{q,r} > 0$. The normalized series runs from $-\infty$ to $+\infty$ and is strictly increasing, so it has exactly one zero. Since $L_q > 0$, [equation \(25\)](#) completes the proof. $\square$

9 Complete coexistence and absence of reentrance

Let κ_f be the unique zero from [theorem 8.2](#) and define

$$\lambda_q(\kappa) = \frac{\kappa}{2m_q(\kappa)}, \quad F_q(\kappa) = 2K_q(\kappa) - \kappa m_q(\kappa).$$

Then

$$\lambda'_q(\kappa) = \frac{m_q - \kappa m'_q}{2m_q^2}, \quad F'_q(\kappa) = m_q - \kappa m'_q. \quad (29)$$

Theorem 9.1 (complete high-dimensional phase). *For every $n \geq 6$, λ_q has exactly one fold. The function F_q has exactly one positive zero $\kappa_c > \kappa_f$. Put*

$$b_c = m_q(\kappa_c), \quad \lambda_c = \frac{\kappa_c}{2b_c}, \quad D_c = 1 - b_c^2.$$

Then $0 < \lambda_c < n(n-1)/4$, and:

1. below λ_c , zero is the unique global radius;
2. at λ_c , precisely 0 and b_c are globally active;
3. above λ_c , one positive radius is uniquely globally active;
4. there is no later exchange or radial reentrance.

Consequently

$$\mathcal{R}_n(D) = \begin{cases} \lambda_c(1 - D), & D_c \leq D \leq 1, \\ \kappa m_q(\kappa) - K_q(\kappa), & 0 < D < D_c, \end{cases} \quad (30)$$

where the second line uses the unique $\kappa > \kappa_c$ satisfying $m_q(\kappa) = \sqrt{1 - D}$. Also $\mathcal{R}_n(1) = 0$ and $\mathcal{R}_n(0) = +\infty$.

Proof. The fold statement follows from [equation \(29\)](#) and [theorem 8.2](#). The same identities show that F_q first decreases and then increases. Moreover $F_q(0) = 0$ and the exact endpoint Laplace asymptotic gives

$$F_q(\kappa) = \kappa - 2q \log \kappa + O(1) \longrightarrow \infty.$$

Thus F_q has exactly one positive zero, after the fold.

The range of λ_q runs from $\lambda_0 = 1/(2 \text{Var } X) = n(n-1)/4$ down to its unique fold and then to $+\infty$. A fixed field therefore has at most two positive stationary radii. At such a radius $b = m_q(\kappa)$,

$$\phi_\lambda(b) = \frac{1}{2}F_q(\kappa), \quad \phi''_\lambda(b) = 2\lambda \left(\frac{\kappa m'_q(\kappa)}{m_q(\kappa)} - 1 \right).$$

The pre-fold branch is therefore a strict local minimum and the post-fold branch the only possible positive maximum. The endpoint $b = 1$ is inactive at every finite field because $K'_q(2\lambda) < 1$. Hence the only global candidates are zero and the unique post-fold radius; the sign of F_q gives the asserted exchange exactly once. The upper inverse $\kappa_+(\lambda)$ is continuous and increasing. Strict positivity of the tilted variance makes m_q strictly increasing with limit one, so the active radius increases continuously and its distortion decreases without reentrance. Finally, at λ_0 the quadratic term at zero vanishes while the fourth cumulant

$$\text{cum}_4(X) = \frac{12(n^2 - 5n - 2)}{n^2(n-1)^2(n+1)(n+2)}$$

is positive for $n \geq 6$. Small nonzero radii then have positive free energy, which proves the strict bound $\lambda_c < \lambda_0$. Covariant attainment and flagged mixing give [equation \(30\)](#). $\square$

Together with [theorem 7.2](#), [theorem 9.1](#) supplies a complete radial classification in every dimension: continuous activation for $n = 3, 4, 5$ and exactly one discontinuous coexistence contact for $n \geq 6$.

10 Thermodynamic contact and its logarithmic shift

The uniqueness theorem also makes the large-dimension coexistence saddle physical rather than merely variational. Let $y_* > 1$ be the nontrivial root of

$$y_* - 1 = 2 \log y_*, \tag{31}$$

and set

$$b_* = 1 - y_*^{-1}, \quad D_* = 1 - b_*^2, \quad \alpha_* = \frac{y_*^2}{2(y_* - 1)}.$$

Numerically,

$$\begin{aligned} y_* &= 3.5128624172\dots, & b_* &= 0.7153318630\dots, \\ D_* &= 0.4883003258\dots, & \alpha_* &= 2.4554074823\dots \end{aligned}$$

Lemma 10.1 (uniform two-derivative Gamma-tail control). *Let $I \Subset (1, \infty)$ be compact and put*

$$A_q(y) = \frac{q!}{2} e^{qy} (qy)^{-q}.$$

There are constants $c_I, C_I > 0$ such that, for $j = 0, 1, 2$ and all sufficiently large q ,

$$\sup_{y \in I} \left| \frac{d^j}{dy^j} \left\{ \frac{L_q(qy)}{A_q(y)} - 1 \right\} \right| \leq C_I e^{-c_I q}.$$

Proof. In the positive half of the beta integral, the change of variables $u = 1 - x$ gives $A_q(y)$ times $\Pr\{G_q \leq qy\}$, where $G_q \sim \text{Gamma}(q, 1)$. For $y \in I$, Chernoff's inequality gives

$$\Pr\{G_q > qy\} \leq \exp[-q\{y - 1 - \log y\}] \leq e^{-c_I q}.$$

Differentiating the incomplete-Gamma integral once or twice produces only its boundary density and polynomial factors in q ; compactness of I lets those factors be absorbed after decreasing c_I . The negative half of the beta integral has the same exponentially small relative bound. This proves the three uniform estimates. $\square$

Theorem 10.2 (coexistence asymptotics). *As $q = n - 2 \rightarrow \infty$, the unique coexistence parameters satisfy*

$$\kappa_c(q) = qy_* - \frac{y_*}{y_* - 2} \log(\pi q/2) + O((\log q)^2/q), \quad (32)$$

$$\lambda_c(q) = \alpha_* q - \frac{y_*^2}{2(y_* - 1)^2} \log(\pi q/2) + O((\log q)^2/q), \quad (33)$$

$$b_c(q) = b_* - \frac{\log(\pi q/2)}{qy_*(y_* - 2)} + O((\log q)^2/q^2), \quad (34)$$

$$D_c(q) = D_* + \frac{2(y_* - 1)}{qy_*^2(y_* - 2)} \log(\pi q/2) + O((\log q)^2/q^2). \quad (35)$$

The information at coexistence is

$$R_c(q) = q \log y_* - \frac{y_*}{2(y_* - 2)} \log(\pi q/2) + O((\log q)^2/q). \quad (36)$$

Proof. By [theorem 10.1](#), uniformly with two y -derivatives on every compact subinterval of $(1, \infty)$,

$$L_q(qy) = \frac{q!}{2} e^{qy} (qy)^{-q} \{1 + O(e^{-cq})\}. \quad (37)$$

Stirling's expansion therefore yields

$$K_q(qy) = q(y - 1 - \log y) + \frac{1}{2} \log(\pi q/2) + \frac{1}{12q} + O(q^{-3}) + O(e^{-cq}), \quad (38)$$

$$m_q(qy) = 1 - y^{-1} + O(e^{-cq}). \quad (39)$$

To justify using this expansion at the physical contact, choose fixed $2 < y_- < y_* < y_+$. For $h(y) = y - 1 - 2 \log y$, one has $h(y_-) < 0 < h(y_+)$. The uniform expansion gives $F_q(qy_-) < 0 < F_q(qy_+)$ for all sufficiently large q ; uniqueness from [theorem 9.1](#) therefore forces $y_- < \kappa_c/q < y_+$. On this fixed compact interval the contact equation $F_q = 2K_q - \kappa m_q = 0$ becomes

$$q\{y - 1 - 2 \log y\} + \log(\pi q/2) + O(q^{-1}) = 0.$$

By [theorem 9.1](#) this equation selects the unique physical contact. Since the derivative of the bracket at y_* is $(y_* - 2)/y_* > 0$, Taylor expansion proves [equation \(32\)](#). Substitution into $b = 1 - 1/y + O(e^{-cq})$ and $\lambda = qy/(2b)$ proves [equations \(33\) to \(35\)](#). Finally, at coexistence $2K_q = \kappa m_q$, so $R_c = \kappa m_q - K_q = \kappa m_q/2 = (\kappa - q)/2 + O(e^{-cq})$, which gives [equation \(36\)](#). $\square$

The limiting root y_* , the constant α_* , and the leading logarithmic coefficient coincide with the earlier complex-projective thermodynamic result [\[21\]](#); indeed $\alpha_*/(2 \log y_*) = y_*^2/[2(y_* - 1)^2]$. The new contribution here is the finite- q uniqueness theorem and the sharper oriented-plane expansions for all five contact observables $\kappa_c, \lambda_c, b_c, D_c, R_c$.

11 High-fidelity constant

Proposition 11.1 (exact leading asymptotics). *Put $q = n - 2$. As $\kappa \rightarrow \infty$,*

$$L_n(\kappa) = \frac{q!}{2} e^\kappa \kappa^{-q} (1 + O(e^{-\kappa} \kappa^{q-1})). \quad (40)$$

Consequently, as $D \downarrow 0$,

$$\boxed{\mathcal{R}_n(D) = q \log \frac{2q}{D} - q - \log \frac{q!}{2} + o(1).} \quad (41)$$

Proof. Use [equation \(11\)](#) and retain the endpoint $x = 1$:

$$\frac{q}{2} \int_0^1 e^{\kappa x} (1-x)^{q-1} dx = \frac{q!}{2} e^\kappa \kappa^{-q} \left(1 - e^{-\kappa} \sum_{j=0}^{q-1} \frac{\kappa^j}{j!} \right).$$

The $e^{-\kappa x}$ half is exponentially smaller after the same scaling, which proves [equation \(40\)](#). Therefore

$$K_n(\kappa) = \kappa - q \log \kappa + \log(q!/2) + o(1), \quad b = K'_n(\kappa) = 1 - q/\kappa + O(e^{-\kappa} \kappa^{q-1}).$$

Equation (21) gives $D = 2q/\kappa + O(\kappa^{-2})$ and $R = q \log \kappa - q - \log(q!/2) + o(1)$. Substitution yields [equation \(41\)](#). $\square$

The pre-log $q = n - 2$ is half the manifold dimension $2(n - 2)$, in agreement with high-resolution geometry. The exact constant comes from the oriented positive-overlap cell; losing the factor $1/2$ would fail the check $L_3(\kappa) \sim e^\kappa / (2\kappa)$.

12 Worst-source capacity and finite blocks

The exact covariant channels also solve a source-universal problem. For a channel $Q(dz | w)$ and a measurable square-integrable report $y(z)$, let

$$\mathbb{C}(Q) = \sup_{P \in \mathcal{P}(\mathcal{G}_n)} I_P(W; Z)$$

and impose the uniform risk constraint

$$\sup_{w \in \mathcal{G}_n} \int \|w - y(z)\|^2 Q(dz | w) \leq D. \quad (42)$$

Let $\mathbb{C}_{1,n}^{\text{univ}}(D)$ be the infimum of $\mathbb{C}(Q)$ under [equation \(42\)](#).

For blocks of length N , allow an arbitrary joint standard-Borel channel $Q_N(dz | w^N)$ and reports $y_t(z)$, subject to

$$\sup_{w^N \in \mathcal{G}_n^N} \frac{1}{N} \sum_{t=1}^N \mathbb{E}[\|w_t - y_t(Z)\|^2 | w^N] \leq D. \quad (43)$$

Define $\mathbb{C}_{N,n}^{\text{univ}}(D)$ by minimizing $\sup_P I_P(W^N; Z)$ under [equation \(43\)](#).

Theorem 12.1 (exact universal tensorization). *For every $n \geq 3$, $N \geq 1$, and $0 \leq D \leq 1$,*

$$\boxed{\mathbb{C}_{N,n}^{\text{univ}}(D) = N \mathcal{R}_n(D).} \quad (44)$$

Arbitrary noncausal joint classical memory therefore cannot improve the single-letter oriented-plane frontier.

Proof. For $N = 1$, feed any feasible universal channel with the Haar source. Its average distortion is at most D , hence $\mathbb{C}(Q) \geq I_\mu(W; Z) \geq \mathcal{R}_n(D)$.

Conversely, at an exposed RDF point use the covariant channel from [equation \(20\)](#). Its risk and $D(Q(\cdot | w) \| \mu)$ are independent of w , and the latter equals the achieved rate. The information-radius identity then makes its capacity exactly $\mathcal{R}_n(D)$. At a nonexposed point, reveal an independent finite flag selecting active covariant components. If $Q_D = \sum_j \alpha_j \delta_j \otimes Q_j$ and $S_D = \sum_j \alpha_j \delta_j \otimes \mu$, then for every w ,

$$D(Q_D(\cdot | w) \| S_D) = \sum_j \alpha_j D(Q_j(\cdot | w) \| \mu) = \mathcal{R}_n(D).$$

Thus the one-use equality holds on linear faces and at endpoints by closure.

For the block converse, choose iid Haar inputs and put $U_t = (Z, W^{t-1})$. Conditional least squares replaces each report by $A_t = \mathbb{E}(W_t | U_t)$ without increasing its distortion D_t . Independence, the chain rule, the one-letter RDF, convexity, and monotonicity give

$$I(W^N; Z) = \sum_{t=1}^N I(W_t; U_t) \geq \sum_{t=1}^N \mathcal{R}_n(D_t) \geq N \mathcal{R}_n\left(\frac{1}{N} \sum_t D_t\right) \geq N \mathcal{R}_n(D).$$

The product of the one-use constant-divergence channels has sequence-independent risk and product information radius $N \mathcal{R}_n(D)$, proving achievability. $\square$

Fix $0 < D < 1$, choose an optimal covariant (possibly flagged) reproduction law Y^* and a supporting slope λ . Every active radius b_j has the same value $g_\lambda = K_n(2\lambda b_j) - \lambda b_j^2$. Therefore, for every source point w ,

$$\begin{aligned} j_D(w) &:= -\log \mathbb{E}_{Y^*} \exp\{\lambda(D - \|w - Y^*\|^2)\} \\ &= \lambda(1 - D) - g_\lambda = \mathcal{R}_n(D). \end{aligned}$$

The equality uses $\mathbb{E}_Q e^{2\lambda b_j \langle w, Q \rangle} = e^{K_n(2\lambda b_j)}$; source-independent mixing does not change it because all active components share g_λ . Thus the ordinary iid-Haar source dispersion vanishes.

Now consider an arbitrary randomized fixed-length encoder with at most M messages and decoder Y^N , for iid Haar W^N , and define

$$\epsilon = \Pr\left\{\frac{1}{N} \sum_{t=1}^N \|W_t - Y_t\|^2 > D\right\}.$$

The product optimal reproduction law makes the block tilted information the constant $N \mathcal{R}_n(D)$. Specializing the general nonasymptotic tilted-information converse of [\[17\]](#), and taking the limit from below if its threshold inequality is strict, gives

$$\epsilon \geq [1 - \exp\{\log M - N \mathcal{R}_n(D)\}]_+. \quad (45)$$

This is a converse statement; zero dispersion does not assert zero third-order redundancy or exact finite-block achievability.

13 Diagnostics, special cases, and scope

For $n = 3$, Hodge duality identifies unit bivectors with S^2 , and $L_3 = \sinh \kappa / \kappa$ reproduces the known spherical RDF [\[22\]](#). For $n = 4$, $\text{Gr}_2^+(\mathbb{R}^4) \simeq S^2 \times S^2$; the constrained-gradient argument in [theorem 6.1](#) is included precisely so that this reducible stabilizer geometry cannot introduce a hidden posterior-mean component. In $n = 6$, the exceptional isomorphism $\text{Spin}(6) \simeq SU(4)$ relates the oriented-plane orbit to $\text{Gr}_{\mathbb{C}}(2, 4)$ and to the exact rank-two companion model [\[23\]](#). The

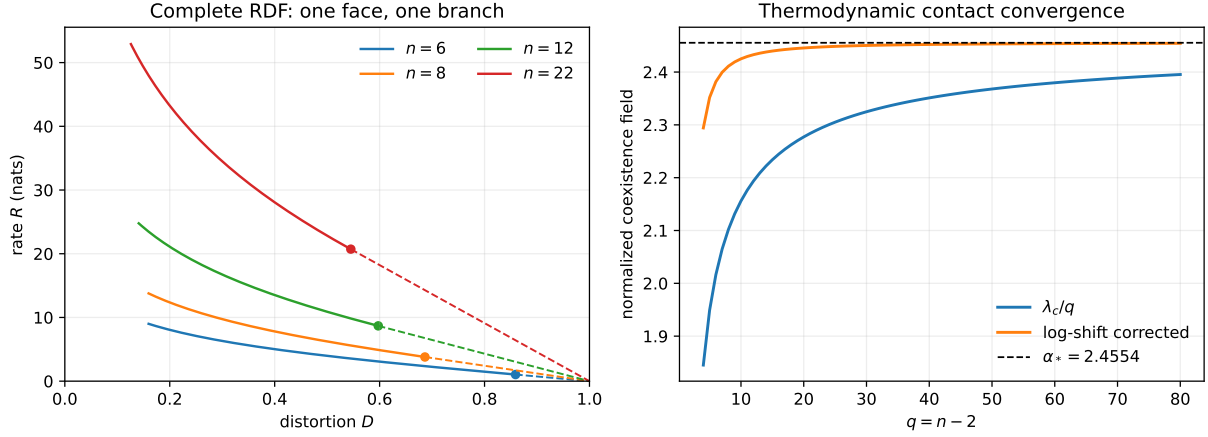


Figure 1: Left: the exact two-piece RDFs; dots are the unique coexistence contacts and dashed lines the exact time-sharing faces. Right: convergence of the normalized coexistence field, before and after the logarithmic correction in [equation \(33\)](#). The plot is diagnostic; every phase and asymptotic statement is analytic.

present theorem explains why the simple skew block appears there and extends that mechanism to every real dimension without relying on the exceptional isomorphism.

The operational object here is a signed Plücker label. It can represent an oriented plane or an antisymmetric area coordinate, and squared loss is integrated squared error over an orthonormal basis of linear Plücker queries. This is not automatically a quantum-state RDF: a physical ray identifies global sign, and Born probabilities square amplitudes, changing both the sufficient statistic and the loss. We therefore make no claim about intrinsic randomness, measurement update, click simulation, or a derivation of Born’s rule.

Reproducibility. The release includes bounded, single-process scripts. One checks the exact coefficient signs and parity-binomial reduction; one independently checks the global-phase identities; one evaluates the exact scalar contacts and their thermodynamic expansion; and one generates [figure 1](#). From the workspace root:

```
powershell -ExecutionPolicy Bypass -File work/twelfth_paper/replay_all.ps1
```

They are diagnostic replays; no finite screen or floating-point root is used to prove the all-dimensional theorems. At the frozen seal, two separate exact rational replays report 6,783 coefficient-sign checks, 297 boundary-tail checks, and zero coefficient/binomial-reduction mismatches; the scalar contact replay extends through $q = 120$. Release seal P12-ARR-UNIFIED-v1 freezes the scripts, JSON diagnostics, and figure in `Eriksson_Paper12_Replay_Artifacts_v1.zip`, whose SHA-256 is 412d36ea2abf9df035d0235064f1b20a6d25c21ac5c50c3f0980005c192b bfd9. The internal file `REPLAY_ARTIFACTS.sha256` verifies every replay input and deterministic output used by the one-command check.

14 Conclusion

The parity-binomial coefficient law closes the global phase problem for Haar oriented-plane compression. In every dimension six and above the stationary curve has one fold, the zero and informative phases meet at one positive contact, and the unique active branch cannot exchange or reenter. The exact RDF is therefore one linear coexistence face followed by one covariant branch. Combined with the continuous cases below dimension six, this is a complete all-dimensional classification rather than a local cumulant diagnosis.

The unique contact also admits a controlled thermodynamic limit with an explicit logarithmic finite-size displacement. At the operational level, the same curve is the exact minimum worst-source capacity and tensorizes under arbitrary joint block memory; transitivity supplies a zero-dispersion strong converse. The result remains deliberately specific to signed Plücker coordinates and squared ambient prediction loss. Higher exterior degree and unoriented or quantum reproduction bodies require different spectral extrema and are not claimed here.

A Exact algebra in the boundary-tail estimate

For completeness, the second rational inequality used in [theorem 8.2](#) is

$$\frac{7}{5} - \frac{P_{r+1}}{P_r} = \frac{Q(r-4)}{5(r+4)^3(3r+4)(3r+5)(4r+7)^3H(r)},$$

where

$$H(r) = 16r^6 + 176r^5 + 753r^4 + 1591r^3 + 1943r^2 + 1353r + 440$$

and

$$\begin{aligned} Q(u) = & 18432u^{14} + 1516032u^{13} + 57380608u^{12} + 1323274880u^{11} + 20744953944u^{10} \\ & + 233468375862u^9 + 1940711941655u^8 + 12066182319572u^7 + 56127825939008u^6 \\ & + 193034617284784u^5 + 477660913728601u^4 + 807579069619542u^3 \\ & + 843129988129352u^2 + 426935276758816u + 29229024927872. \end{aligned}$$

Every coefficient is positive. This makes the all- r tail step a displayed rational certificate rather than a finite numerical screen.

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