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Exact Harmonic Potentials, Palindromic Invariance, and Bias Phase Transitions for Stochastic Synchronization of Cyclic Defect-1 and Ratchet Automata

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Abstract

Let $\mathcal{A} = (Q, \{a, b\})$ be a binary synchronizing deterministic finite automaton on n states driven by an i.i.d. random input stream with $\mathbb{P}(a) = p \in (0, 1)$ and $\mathbb{P}(b) = 1 - p$, and let $\tau_{\text{sync}}(\mathcal{A}; p) := \inf\{t \geq 0 : |w_t(Q)| = 1\}$ be its random synchronization time. While worst-case deterministic reset thresholds $\text{rt}(\mathcal{A})$ have been studied intensively since Černý's 1964 conjecture, the exact stochastic synchronization laws of canonical extremal automata over biased streams remain largely unexplored. We establish a complete, exact theory across two complementary families:

1. **Exact Single-Arc Harmonic Potential and Universal Master Formula (Theorem 2.1):** Every synchronizing cyclic n-cycle plus idempotent rank-$(n-1)$ defect automaton is isomorphic to $\mathcal{C}_{n,d}$ ($a(i) = i+1 \bmod n$, $b(n-1) = d$, $b(i) = i$ for $i < n-1$) with $\gcd(d+1, n) = 1$. Conjugating by $\sigma(x) = (d+1)^{-1}(x-d) \bmod n$ preserves cyclic order and decomposes the $(2^n - n - 1)$-state subset Markov chain into an exact single-arc Poisson system. Cycle telescoping, a four-term polynomial adjacent-swap invariance identity, and palindromic residue symmetry ($ts + (m-t)s \equiv n-1 \pmod n$) yield the closed-form universal law ($m := n - 1 - d$):

$$\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{n,d}; p)] \;=\; \frac{\binom{m+1}{3} + \frac{m(m-1)+d(m^2+1)}{2}\, p + \frac{m(d^2+2d+2)-d}{2}\, p^2 + \binom{d+2}{3}\, p^3}{p^2(1-p)},$$

   independent of $(d+1)^{-1} \bmod n$. For Černý's automaton $\mathcal{C}_n = \mathcal{C}_{n,0}$, this specializes to the three-term binomial law $\left[\binom{n}{3} + \binom{n-1}{2}p + (n-1)p^2\right]/(p^2(1-p))$ (an exact integer $\frac{2}{3}(n-1)(n+1)(2n-3)$ at $p = 1/2$), accompanied by an explicit closed-form harmonic potential $\Phi_{\mathcal{C}_n}(S; p)$ valid on all $2^n - 1$ nonempty subsets $S \subseteq \mathbb{Z}_n$.

2. **Deterministic Reset Spectrum, Černý Extremality, Rare-Defect Collapse, and Dimensional Drop (Theorem 3.1):** Across all coprime (n, d), the deterministic reset threshold is exactly $\text{rt}(\mathcal{C}_{n,d}) = (n-1)^2 - d(n-2)$. Stochastically, $\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{n,d}; p)] - \mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{n,d+1}; p)]$ cancels the $(1-p)$ pole identically and is strictly positive on $(0, 1)$, proving that Černý's $\mathcal{C}_n$ uniquely maximizes and reverse-jump $\mathcal{C}_{n,n-2}$ uniquely minimizes expected synchronization time for every bias $p \in (0, 1)$. As $p \to 1^-$, every $\mathcal{C}_{n,d}$ collapses to the universal conjugacy-invariant residue $\binom{n+1}{3}/(1-p)$. At the reverse boundary $d = n-2$, the $1/p^2$ pole collapses to $1/p$, shifting the optimal bias to $p^*(n) \sim \sqrt{6}/n$ and dropping the optimal stochastic synchronization time by a full power of n from $\frac{9}{8}n^3$ ($d = 0$) to $(\frac{1}{2} + \sqrt{\frac{2}{3}})n^2 + O(n)$.

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3. **Exact Pairwise Renewal and Universal Non-Concentration on $\mathcal{C}_n$ (Theorem 4.1):** Reducing pairwise coalescence $Z_k := \tau_{\text{pair}}(\{0,k\};p)$ to an exact $(n-1)$-state negative-binomial renewal whose two-epoch chain stopped at $\{0,n-1\}$ is a symmetric simple random walk with eigenfunction $\sin(\pi k/(n-1))$ proves the dimension- and bias-independent non-concentration barrier

$$\liminf_{n\to\infty} \frac{\mathrm{Var}(\tau_{\text{sync}}(\mathcal{C}_n;p))}{\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_n;p)]^2} \;\geq\; \frac{72}{\pi^4} e^{-\pi^2/6} \;\approx\; 0.142675 \;>\; 0 \qquad (\forall\, p \in (0,1)),$$

backed by exact variance computations for $n \leq 9$ pointing to the S^1 coalescing Brownian motion limit $2/5$ (Conjecture 4.2).

4. **Deterministic-vs-Stochastic Inversion and Sharp $p_c = 1/2$ Phase Transition (Theorem 5.2):** The n-state Biased Ratchet Automaton $\mathcal{R}_n$ achieves the minimal linear deterministic reset threshold $\mathrm{rt}(\mathcal{R}_n) = n-1$, yet its exact closed-form mean and compound-geometric variance undergo a sharp phase transition at $p_c = 1/2$: linear $\Theta(n)$ with CV $\to 0$ ($p > 1/2$), exact quadratic $n(n-1)$ with $\mathrm{Var} = \frac{1}{3}n(n-1)(2n^2-2n-1)$ and CV $\to \sqrt{2/3}$ ($p = 1/2$), and exponential $\Theta(((1-p)/p)^n)$ with memoryless CV $\to 1$ ($p < 1/2$).

The entire cleared-denominator algebraic core (26 declarations: 17 public theorems, 6 public definitions, 3 private helpers) is machine-checked in standalone Lean 4.36.0 (`0 sorry`, `0 grind`, `0 custom axioms`).

1 Introduction

Let $\mathcal{A} = (Q,\Sigma)$ be a complete deterministic finite automaton (DFA) with finite state set Q ($|Q| = n \geq 2$) and input alphabet Σ, where each letter $\sigma \in \Sigma$ acts as a transformation $Q \to Q$. A word $w \in \Sigma^*$ is a *synchronizing* (or *reset*) word if its image $w(Q)$ has cardinality 1. The *reset threshold* $\mathrm{rt}(\mathcal{A})$ is the minimum length of a synchronizing word. In 1964, Černý [9] exhibited for every $n \geq 2$ the binary automaton $\mathcal{C}_n = (\mathbb{Z}_n, \{a,b\})$ with cyclic shift $a(i) = i+1 \bmod n$ and single-collision defect $b(n-1) = 0$, $b(i) = i$ $(0 \leq i \leq n-2)$, proved $\mathrm{rt}(\mathcal{C}_n) = (n-1)^2$, and conjectured that every n-state synchronizing DFA satisfies $\mathrm{rt}(\mathcal{A}) \leq (n-1)^2$. Despite six decades of work—including exact proofs for circular automata [4], Eulerian automata [5], and cubic worst-case improvements down to $\approx 0.1654\, n^3$ [7, 10, 8]—the Černý conjecture remains open, while random automata synchronize with high probability in $\Theta(\sqrt{n}\log n)$ [2, 6, 3].

In this work we investigate the dual stochastic question: given a *fixed* synchronizing binary DFA $\mathcal{A} = (Q,\{a,b\})$ driven by an i.i.d. random stream $(X_t)_{t\geq 1}$ with $\mathbb{P}(X_t = a) = p \in (0,1)$ and $\mathbb{P}(X_t = b) = q := 1-p$, what are the exact laws governing the *random synchronization time*

$$\tau_{\text{sync}}(\mathcal{A};p) \;:=\; \inf\{t \geq 0 : |X_t \cdots X_1(Q)| = 1\}? \tag{1}$$

On the power set $2^Q \setminus \{\emptyset\}$, the subset trajectory $S_t := X_t \cdots X_1(Q)$ is a non-reversible absorbing Markov chain with up to $2^n - n - 1$ transient states. Remarkably, for the entire canonical class of synchronizing cyclic defect-1 automata—generated by an n-cycle and an idempotent rank-$(n-1)$ map—this exponentially large Markov chain is *completely integrable* in closed form.

1.1 Classification of cyclic defect-1 automata

Consider any binary DFA $\mathcal{A} = (Q,\{a,b\})$ on $n \geq 2$ states such that a is a single n-cycle and b is an idempotent rank-$(n-1)$ transformation (i.e., $b(u) = v$ for one ordered pair $u \neq v$, and $b(x) = x$ for all $x \neq u$). Relabeling Q along the a-orbit so that $a(i) = i+1 \bmod n$ and $u = n-1$ puts $v = d \in \{0,1,\ldots,n-2\}$; we denote the resulting automaton by $\mathcal{C}_{n,d}$.

Proposition 1.1 (Synchronizability Criterion)**.** *The automaton $\mathcal{C}_{n,d}$ ($n \geq 2$, $0 \leq d \leq n-2$) is synchronizing if and only if* $\gcd(d+1,n) = 1$.

Proof. Let $g := \gcd(d+1, n)$. If $g > 1$, then a permutes the g residue classes modulo g cyclically, while b maps $n-1 \equiv -1 \pmod{g}$ to $d \equiv -1 \pmod{g}$ and fixes all other elements, preserving every residue class modulo g. Hence every word maps $\mathbb{Z}_n$ onto all g residue classes, so $|w(\mathbb{Z}_n)| \geq g \geq 2$ and $\mathcal{C}_{n,d}$ is not synchronizing. Conversely, if $\gcd(d+1, n) = 1$, Theorem 3.1(a) below constructs an explicit synchronizing word of length $(n-1)^2 - d(n-2)$. □

Throughout, when $\gcd(d+1, n) = 1$, we set $m := n-1-d \in \{1, \ldots, n-1\}$ and $s := (d+1)^{-1} \bmod n \in \{1, \ldots, n-1\}$. Note that $ms \equiv -(d+1)s \equiv -1 \pmod{n}$. The classical Černý automaton is $\mathcal{C}_n = \mathcal{C}_{n,0}$ ($m = n-1$, $s = 1$), while the opposite extreme is the *reverse-jump automaton* $\mathcal{C}_{n,n-2}$ ($m = 1$, $s = n-1$), where b pushes $n-1$ one step *backward* to $n-2$ against the clockwise shift a.

2 Exact Single-Arc Harmonic Potentials and the Universal Master Formula

Our first main theorem solves the $(2^n - n - 1)$-state subset Markov chain of every $\mathcal{C}_{n,d}$ ($\gcd(d+1, n) = 1$) in exact closed form.

Theorem 2.1 (Universal Master Formula and Exact Černý Harmonic Potential)**.** *Let $n \geq 2$, $d \in \{0, \ldots, n-2\}$ with $\gcd(d+1, n) = 1$, $m := n-1-d$, and $p \in (0, 1)$.*

(a) ***Universal Master Formula:*** *The expected synchronization time of $\mathcal{C}_{n,d}$ depends only on (n, d, p) (and not on $s = (d+1)^{-1} \bmod n$) via the exact rational identity*

$$\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{n,d}; p)] \;=\; \frac{P_{n,d}(p)}{p^2(1-p)}, \tag{2}$$

where $P_{n,d}(p)$ is the cubic polynomial

$$P_{n,d}(p) \;:=\; \binom{m+1}{3} + \frac{m(m-1) + d(m^2+1)}{2}\, p + \frac{m(d^2+2d+2) - d}{2}\, p^2 + \binom{d+2}{3} p^3. \tag{3}$$

(b) ***Classical Černý Three-Term Binomial Law and Explicit Subset Potential:*** *At $d = 0$ ($\mathcal{C}_n = \mathcal{C}_{n,0}$), (2) collapses to*

$$\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_n; p)] = \frac{\binom{n}{3} + \binom{n-1}{2}\, p + (n-1)\, p^2}{p^2(1-p)}, \quad \mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_n; 1/2)] = \frac{2}{3}(n-1)(n+1)(2n-3). \tag{4}$$

Moreover, for every nonempty subset $S = \{s_0 < s_1 < \cdots < s_{k-1}\} \subseteq \mathbb{Z}_n$ ($k \geq 2$) with internal gaps $g_i := s_i - s_{i-1}$ ($1 \leq i \leq k-1$), boundary offsets $r_L := s_0$, $r_R := n-1-s_{k-1}$, and wrap-around gap $g_0 := r_L + r_R + 1$,

$$\begin{aligned} \mathbb{E}_S[\tau_{\text{sync}}(\mathcal{C}_n; p)] \;=\; \frac{1}{p^2(1-p)} \Bigg[&\binom{n}{3} + \binom{n-1}{2} p + (n-1)p^2 \\ &- \sum_{i=1}^{k-1} \binom{g_i+1}{3} - \psi(r_L, r_R; p) \Bigg], \end{aligned} \tag{5}$$

where $\psi(r_L, r_R; p) := \binom{g_0}{3} + \left(\binom{g_0}{2} - r_R\right)p + r_R p^2$.

(c) ***Reverse-Jump Pole Collapse ($d = n-2$):*** *For every $n \geq 2$, at $d = n-2$ ($m = 1$) the constant term $\binom{2}{3} = 0$ vanishes identically and (2) drops from a p^{-2} pole to a simple p^{-1} pole:*

$$\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{n,n-2}; p)] \;=\; \frac{(n-2) + \left(\binom{n-1}{2} + 1\right)p + \binom{n}{3}p^2}{p(1-p)}. \tag{6}$$

Proof. We first verify Part (b)'s explicit subset potential (5) directly, then establish Part (a) for all coprime (n, d) via conjugated single-arc additivity, cycle telescoping, adjacent-swap invariance, and palindromic symmetry.

Proof of (5). Denote the right-hand side of (5) by $\Phi(S; p)$ for $|S| \geq 2$, and set $\Phi(\{x\}; p) := 0$ for every singleton $\{x\}$. Note first that at the singleton $S = \{0\}$ (where $r_L = 0$, $r_R = n-1$, $g_0 = n$, and there are no internal gaps), $\psi(0, n-1; p) = \binom{n}{3} + (\binom{n}{2} - (n-1))p + (n-1)p^2 = \binom{n}{3} + \binom{n-1}{2}p + (n-1)p^2$, so the bracket in (5) vanishes identically at $S = \{0\}$ (machine-checked in Lean 4.36.0 as `cerny_singleton_boundary_zero`). For any non-singleton $S \subseteq \mathbb{Z}_n$, we verify the discrete Poisson equation $\Phi(S) = 1 + p\Phi(a(S)) + q\Phi(b(S))$ ($q := 1 - p$) in three exhaustive cases:

- *Case 1 ($n-1 \notin S$, so $r_R \geq 1$):* Here $b(S) = S$, while $a(S)$ preserves all internal gaps and g_0 and maps $(r_L, r_R) \mapsto (r_L + 1, r_R - 1)$. Hence $p^2 q(\Phi(S) - \Phi(a(S))) = \psi(r_L + 1, r_R - 1; p) - \psi(r_L, r_R; p) = p - p^2 = pq$, giving $p(\Phi(S) - \Phi(a(S))) + q(\Phi(S) - \Phi(b(S))) = p \cdot \frac{1}{p} + 0 = 1$ (Lean: `cerny_potential_case1_poisson`).
- *Case 2 ($n-1 \in S$, $0 \notin S$, so $r_R = 0$ and $g_0 = r_L + 1 \geq 2$):* Let $g_\ell := (n-1) - s_{k-2} \geq 1$ be the last internal gap. Under a, the old wrap gap g_0 becomes an internal gap of length g_0, while g_ℓ becomes the new wrap gap with $(r'_L, r'_R) = (0, g_\ell - 1)$. Under b, $n-1 \mapsto 0$ turns the old wrap gap into an internal gap of length $g_0 - 1$, while g_ℓ becomes a wrap gap of length $g_\ell + 1$ with $(r'_L, r'_R) = (0, g_\ell)$. Using Pascal's identities $\binom{g+1}{3} - \binom{g}{3} = \binom{g}{2}$ and $\binom{g+1}{2} - g = \binom{g}{2}$, the weighted drop $p\,\Delta_a + q\,\Delta_b$ cancels all cubic and quadratic gap terms identically and leaves $p^2 q$, giving $p(\Phi(S) - \Phi(a(S))) + q(\Phi(S) - \Phi(b(S))) = 1$ (Lean: `cerny_potential_case2_poisson`).
- *Case 3 ($n-1 \in S$, $0 \in S$, so $r_L = r_R = 0$ and $g_0 = 1$):* Here b merges $n-1$ into 0; whether $|S| \geq 3$ or $|S| = 2$ (where $b(\{0, n-1\}) = \{0\}$ and $\Phi(\{0\}) = 0$ matches the $g_\ell = n-1$ evaluation by the boundary identity above), the exact same identity with $g_0 = 1$ holds (Lean: `cerny_potential_case3_poisson`).

Because the subset Markov chain of $\mathcal{C}_n$ is finite and absorbing at singletons, uniqueness of the Poisson solution implies $\mathbb{E}_S[\tau_{\text{sync}}(\mathcal{C}_n; p)] = \Phi(S; p)$ for every S. Setting $S = \mathbb{Z}_n$ ($g_i = 1$, $r_L = r_R = 0$) yields (4), whose $p = 1/2$ evaluation factors as $\frac{2}{3}(n-1)(n+1)(2n-3)$ (Lean: `cerny_uniform_cubic_factorization`).

Proof of Part (a) for arbitrary coprime (n, d). For $n = 2$ ($d = 0$, $m = 1$), $\tau_{\text{sync}} \sim \text{Geom}(1-p)$ has mean $(1-p)^{-1}$, matching (2) directly. Assume $n \geq 3$ and conjugate $\mathcal{C}_{n,d}$ by the affine bijection $\sigma(x) := s(x - d) \bmod n$ on $\mathbb{Z}_n$ (where $s = (d+1)^{-1} \bmod n$). Since $\sigma(d) = 0$ and $\sigma(n-1) = s(n-1-d) = sm \equiv -1 \equiv n-1 \pmod n$, in the conjugated coordinates:

$$\tilde{a}(r) \;=\; (r + s) \bmod n, \qquad \tilde{b}(n-1) = 0, \quad \tilde{b}(r) = r \;\; (r \neq n-1).$$

Crucially, both $\tilde{a}$ (a rigid rotation by $+s$) and $\tilde{b}$ (a nearest-neighbor merge $n-1 \to 0$) preserve the cyclic order on $\mathbb{Z}_n$. Every nonempty subset $S \subseteq \mathbb{Z}_n$ therefore partitions $\mathbb{Z}_n$ into $k = |S|$ disjoint cyclic half-open arcs $[u_1, u_1 + g_1), \ldots, [u_k, u_k + g_k)$ ($\sum_{i=1}^k g_i = n$, $g_i \geq 1$). Under $\tilde{a}$, every arc shifts independently by $+s \bmod n$. Under $\tilde{b}$, if $n-1 \notin S$ no arc changes; if $n-1 \in S$, the single arc ending at $n-1$ grows by $+1$ to $(u, g+1)$, the single arc starting at $n-1$ shrinks by -1 to $(0, g-1)$, and all other arcs are unchanged.

Consider the single-arc linear system for $h : \mathbb{Z}_n \times \{0, 1, \ldots, n\} \to \mathbb{R}$ with boundary conditions $h(u, 0) = 0$ and $h(u, n) = C$ (constant in u), defined for $1 \leq g \leq n-1$ and $u \in \mathbb{Z}_n$ by

$$h(u, g) \;=\; p\,h(u + s, g) + q\,h\big(\tilde{b}_{\text{arc}}(u, g)\big) - \mathbf{1}_{n-1 \in [u, u+g)}, \tag{7}$$

where $\tilde{b}_{\text{arc}}(n-1, g) = (0, g-1)$, $\tilde{b}_{\text{arc}}(n-1-g, g) = (n-1-g, g+1)$, and $\tilde{b}_{\text{arc}}(u, g) = (u, g)$ otherwise. For any solution (h, C) of (7), define $\Psi(S) := C - \sum_{i=1}^k h(u_i, g_i)$. At any singleton

$S = \{x\}$ (one arc of length n), $\Psi(\{x\}) = C - C = 0$. For any non-singleton S ($k \geq 2$), the arcs $[u_i, u_i + g_i)$ partition $\mathbb{Z}_n$, so $\sum_{i=1}^{k} \mathbf{1}_{n-1 \in [u_i, u_i+g_i)} = 1$, and summing (7) over $i = 1, \ldots, k$ yields $\Psi(S) = 1 + p\Psi(\tilde{a}(S)) + q\Psi(\tilde{b}(S))$ identically! Thus $\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_{n,d}; p)] = \Psi(\mathbb{Z}_n) = C - \sum_{u=0}^{n-1} h(u, 1)$.

We construct a solution (h, C) of (7) normalized by $h(n-1, 1) = 0$ (whose existence is established explicitly below). Reindex $\mathbb{Z}_n$ along the $\tilde{a}$-orbit via $u_t := (n - 1 - ts) \bmod n$ ($t = 0, 1, \ldots, n-1$), so $\tilde{a}(u_t) = u_{t-1}$, $u_0 = n - 1$, $u_m = 0$ (since $ms \equiv -1 \pmod{n}$), and the right-boundary site $n - 1 - g$ occurs at $t_*(g) := g(d+1) \bmod n \in \{1, \ldots, n-1\}$ (with $t_*(n-1) = m$). Moreover, $n - 1 \in [u_t, u_t + g)$ holds iff $r_t := (ts \bmod n) < g$. Define the two boundary fluxes

$$J_{\mathrm{down}}(g) := q\big(h(n-1, g) - h(0, g-1)\big), \qquad J_{\mathrm{up}}(g) := q\big(h(n-1-g, g+1) - h(n-1-g, g)\big).$$

For each fixed $g \in \{1, \ldots, n-1\}$, rewriting (7) along $t = 0, \ldots, n-1$ gives

$$p\big(h(u_t, g) - h(u_{t-1}, g)\big) \;=\; -\mathbf{1}_{r_t < g} - J_{\mathrm{down}}(g)\, \mathbf{1}_{t=0} + J_{\mathrm{up}}(g)\, \mathbf{1}_{t = t_*(g)}. \tag{8}$$

Summing (8) over $t = 0, \ldots, n-1$ telescopes the left-hand side to 0, leaving the exact flux balance

$$J_{\mathrm{up}}(g) - J_{\mathrm{down}}(g) \;=\; g \qquad (1 \leq g \leq n-1). \tag{9}$$

At $g = 1$, our gauge $h(n-1, 1) = 0$ gives $J_{\mathrm{down}}(1) = 0$, hence $J_{\mathrm{up}}(1) = 1$, and summing (8) gives $h(u_t, 1) = \frac{1}{p}\mathbf{1}_{t \geq t_*(1)}$, where $t_*(1) = d + 1$, so $\sum_{u=0}^{n-1} h(u, 1) = \frac{n-(d+1)}{p} = \frac{m}{p}$.

Next, define the residue set $A_m := \{r_t : 1 \leq t \leq m\} \subseteq \{1, \ldots, n-1\}$ (which has $|A_m| = m$ and contains $r_m = n - 1$) and its cumulative count $M(g) := |\{r \in A_m : r < g\}|$. For any $g \in \{1, \ldots, n-2\}$, write

$$p\big(J_{\mathrm{up}}(g) - J_{\mathrm{down}}(g+1)\big) \;=\; q\Big[p\big(h(u_{t_*(g)}, g+1) - h(u_0, g+1)\big) - p\big(h(u_{t_*(g)}, g) - h(u_m, g)\big)\Big].$$

Summing (8) from $t = 1$ to $t_*(g)$ (at level $g+1$, noting $t_*(g+1) = (t_*(g) + d + 1) \bmod n$ lies in $(0, t_*(g)]$ iff $t_*(g) \geq m$, i.e., $g \notin A_m$) and between $t = m$ and $t_*(g)$ (at level g), and using that $t \mapsto r_t$ is a bijection with $r_{t_*(g)} = g$ (so $\mathbf{1}_{r_t < g+1} = \mathbf{1}_{r_t < g}$ for every $t \neq t_*(g)$), both cases $t_*(g) < m$ and $t_*(g) \geq m$ yield the identical indicator count $-1 - M(g)$, collapsing via (9) to the exact first-order recurrence for $\Delta_g := J_{\mathrm{up}}(g+1) - J_{\mathrm{up}}(g)$:

$$\Delta_g \;=\; \frac{p(g+1) + q(1 + M(g))}{p^{a_g}}, \qquad a_g := \mathbf{1}_{g \in A_m} \in \{0, 1\} \qquad (1 \leq g \leq n-2). \tag{10}$$

Conversely, reversing this derivation explicitly constructs a solution (h, C) of (7) with constant top boundary $h(u, n) = C$: initialize $h(n-1, 1) = 0$ and $J_{\mathrm{up}}(1) = 1$, generate $J_{\mathrm{up}}(g)$ for $2 \leq g \leq n-1$ via (10), set $J_{\mathrm{down}}(g) := J_{\mathrm{up}}(g) - g$, step $h(u_t, g)$ around the $\tilde{a}$-orbit via (8) from $h(n-1, g) = h(0, g-1) + J_{\mathrm{down}}(g)/q$ (closing consistently at $t = 0$ by (9) and matching $J_{\mathrm{up}}(g)$ by (10)), and set $C := h(0, n-1) + J_{\mathrm{up}}(n-1)/q$. Finally, telescoping $C = h(n-1, n)$ from $h(n-1, 1) = 0$ via $h(n-1, g+1) - h(n-1, g) = [h(n-1, g+1) - h(0, g)] + [h(0, g) - h(n-1, g)] = \frac{J_{\mathrm{up}}(g+1) - (g+1)}{q} + \frac{J_{\mathrm{up}}(g) a_g - M(g)}{p}$ (with $C - h(n-1, n-1) = \frac{J_{\mathrm{up}}(n-1)}{q} + \frac{J_{\mathrm{up}}(n-1) - M(n-1)}{p}$ since $t_*(n-1) = m$) and subtracting $\sum_u h(u, 1) = \frac{m}{p}$ yields

$$\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_{n,d}; p)] \;=\; \frac{1}{q}\sum_{g=2}^{n-1}\big(J_{\mathrm{up}}(g) - g\big) + \frac{J_{\mathrm{up}}(n-1)}{q} + \frac{1}{p}\sum_{g=1}^{n-1}\big(J_{\mathrm{up}}(g)\, a_g - M(g)\big) - \frac{m}{p}.$$

Substituting $J_{\mathrm{up}}(g) = 1 + \sum_{r=1}^{g-1} \Delta_r$ and interchanging summations gives

$$\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_{n,d}; p)] \;=\; -\frac{n(n-3)}{2q} - \frac{1}{p}\sum_{g=1}^{n-1} M(g) + \frac{1}{p^2 q}\, S_{\mathrm{sum}}(A_m), \tag{11}$$

where $S_{\mathrm{sum}}(A_m) := \sum_{r=1}^{n-2} T(r, a_r, M(r))$ with

$$T(r,a,M) \;:=\; \begin{cases} [p(n-r)+q(m-1-M)]\,[p(r+1)+q(1+M)] & (a=1),\\ p\,[p(n-r)+q(m-M)]\,[p(r+1)+q(1+M)] & (a=0). \end{cases}$$

Now observe two exact algebraic miracles:

(i) *Exact Adjacent-Swap Invariance of S_{sum}:* Swapping any adjacent pair $(a_x = 1, a_{x+1} = 0)$ to $(a_x = 0, a_{x+1} = 1)$ changes only the two terms $r \in \{x, x+1\}$ (with $M(x) = k$, $M(x+1)$ shifting $k+1 \to k$). Setting $U := p(n-x)+q(m-1-k)$ and $V := p(x+1)+q(1+k)$, the difference is

$$p(U+q)V + (U-p)(V+p) - UV - p(U-p)(V+1) \;=\; V\big(p(1-p)-p+p^2\big) \;=\; 0$$

identically for all U, V, p (machine-checked in Lean 4.36.0; see Section 6). Hence $S_{\mathrm{sum}}(A_m)$ is *completely independent* of which $(m-1)$-element subset $S_0 = A_m \setminus \{n-1\} \subseteq \{1, \dots, n-2\}$ is chosen, and equals its value on the contiguous block $S_0^* = \{1, \dots, m-1\}$.

(ii) *Palindromic Residue Symmetry:* Because $ms \equiv -1 \pmod n$, for every $t \in \{1, \dots, m-1\}$ we have $ts + (m-t)s = ms \equiv n-1 \pmod n$, so $r \in S_0 \iff n-1-r \in S_0$. Pairing r with $n-1-r$ immediately evaluates

$$\sum_{g=1}^{n-1} M(g) \;=\; \sum_{r=1}^{n-2} (n-1-r)\,a_r \;=\; \frac{n-1}{2} \sum_{r=1}^{n-2} a_r \;=\; \frac{(m-1)(n-1)}{2}.$$

Evaluating $S_{\mathrm{sum}}(S_0^* \cup \{n-1\})$ splits into two elementary quadratic sums: for $1 \le r \le m-1$ ($a_r = 1$, $M(r) = r-1$), $T = (p(n-r)+q(m-r))(r+p)$; for $m \le r \le n-2$ ($a_r = 0$, $M(r) = m-1$), $T = p(p(n-r)+q)(p(r+1)+qm)$. Expanding via $\sum_{r=1}^{L} r = \frac{L(L+1)}{2}$ and $\sum_{r=1}^{L} r^2 = \frac{L(L+1)(2L+1)}{6}$ and substituting into (11) produces (2)–(3). Specializing to $d = 0$ ($m = n-1$) and $d = n-2$ ($m = 1$) gives Parts (b) and (c) (Lean: `master_formula_cerny_specialization`, `master_formula_reverse_pole_collapse`). □

3 Deterministic Reset Spectrum, Extremality, and Dimensional Drop

Theorem 2.1 immediately unlocks the complete deterministic and stochastic geometry of the cyclic defect-1 class $\mathcal{C}_{n,d}$.

Theorem 3.1 (Reset Spectrum, Černý Extremality, Rare-Defect Collapse, and Dimensional Drop)**.** *Let $n \ge 2$ and $d \in \{0, \dots, n-2\}$ with $\gcd(d+1, n) = 1$, and write $m := n-1-d$.*

(a) ***Exact Deterministic Reset Threshold Spectrum:***

$$\mathrm{rt}(\mathcal{C}_{n,d}) \;=\; (n-1)^2 - d(n-2), \tag{12}$$

attained by the explicit synchronizing word $w_{n,d} := b(a^m b)^{n-2}$, spanning quadratically from Černý's $(n-1)^2$ ($d = 0$) down to linear $2n-3$ ($d = n-2$).

(b) ***Strict Monotonicity and Černý Stochastic Extremality:*** *For every $n \ge 3$, every $0 \le d \le n-3$, and every $p \in (0,1)$, the $(1-p)$ denominator cancels identically in the step difference:*

$$\frac{P_{n,d}(p) - P_{n,d+1}(p)}{p^2(1-p)} \;=\; \frac{\binom{m}{2} + \big(m(d+2) - \frac{n+3}{2}\big)p + \binom{d+2}{2}p^2}{p^2} \;>\; 0. \tag{13}$$

Consequently, across all synchronizing $\mathcal{C}_{n,d}$, Černý's automaton $\mathcal{C}_n = \mathcal{C}_{n,0}$ uniquely maximizes $\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_{n,d}; p)]$ and the reverse-jump automaton $\mathcal{C}_{n,n-2}$ uniquely minimizes it for every bias $p \in (0,1)$.

(c) ***Universal Rare-Defect Conjugacy Collapse*** *($p \to 1^-$): For every coprime (n, d),*

$$\lim_{p\to 1^-} (1-p)\,\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{n,d}; p)] \;=\; \binom{n+1}{3} \;=\; \frac{n(n^2-1)}{6}, \tag{14}$$

independent of d, whereas at uniform bias $p = 1/2$, the expected-time ratio between $\mathcal{C}_n$ and $\mathcal{C}_{n,n-2}$ converges to 8 as $n \to \infty$.

(d) ***Optimal-Bias Dimensional Drop*** *$\Theta(n^3) \to \Theta(n^2)$: For Černý's automaton $\mathcal{C}_n$, the unique optimal bias $p^*(n, 0) \to 2/3$ yields*

$$\inf_{p\in(0,1)} \mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_n; p)] \;=\; \frac{9}{8}\,n^3 + O(n^2). \tag{15}$$

In contrast, for the reverse-jump automaton $\mathcal{C}_{n,n-2}$, the optimal bias collapses to $p^*(n, n-2) = \frac{\sqrt{6}}{n} + O(n^{-2}) \to 0^+$, dropping the optimal stochastic synchronization time by a full power of n:

$$\inf_{p\in(0,1)} \mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{n,n-2}; p)] \;=\; \left(\frac{1}{2} + \sqrt{\frac{2}{3}}\right) n^2 + O(n) \;\approx\; 1.316497\, n^2. \tag{16}$$

Proof. Part (a). Work in the conjugated coordinates $\tilde{a}(r) = r + s \bmod n$, $\tilde{b}(n-1) = 0$. Since $ms \equiv -1 \pmod n$, $\tilde{a}^m$ shifts every state by $-1 \bmod n$. Applying $\tilde{b}$ to $\mathbb{Z}_n$ leaves the empty arc $\{n-1\}$; each subsequent block $\tilde{a}^m\tilde{b}$ shifts the empty arc $\{n-k, \dots, n-1\}$ to $\{n-1-k, \dots, n-2\}$ and empties $n-1$ again, growing the empty arc by $+1$. After $n-2$ blocks, $\{1, \dots, n-1\}$ is empty and only $\{0\}$ remains, so $w_{n,d} = b(a^m b)^{n-2}$ synchronizes $\mathcal{C}_{n,d}$ in length $1 + (n-2)(m+1) = (n-1)^2 - d(n-2)$ (Lean: `deterministic_reset_threshold_identity`).

For the lower bound, let $w = w_1 \cdots w_L$ be any synchronizing word mapping $\mathbb{Z}_n$ to $\{x\}$, and trace the backward preimages $U_t := (w_{L-t+1} \cdots w_L)^{-1}(\{x\})$ for $t = 0, \dots, L$ ($U_0 = \{x\}$, $U_L = \mathbb{Z}_n$). Under $\tilde{a}^{-1}(r) = r - s \bmod n$, every contiguous cyclic arc shifts rigidly; under $\tilde{b}^{-1}$, a contiguous arc U adds $n-1$ (growing by $+1$ with new left endpoint $n-1$) iff its left endpoint is 0 (and $n-1 \notin U$), drops $n-1$ (shrinking by -1) iff its right endpoint is $n-1$ (and $0 \notin U$), and is unchanged otherwise. Hence every U_t is a contiguous arc of size $\ell_t \in \{1, \dots, n\}$. For each $j \in \{1, \dots, n-2\}$, let $t_j \in \{0, \dots, L-1\}$ be the *last* time where $\ell_{t_j} = j$ and $\ell_{t_j+1} = j+1$, and let $s_j \in \{t_j + 2, \dots, L\}$ be the *post-growth* time of the first $(j+1) \to (j+2)$ transition after t_j (so $\ell_{s_j-1} = j+1$ and $\ell_{s_j} = j+2$). By maximality of t_{j+1}, we have $s_j - 1 \le t_{j+1}$, i.e., $s_j \le t_{j+1} + 1$, so $1 \le t_1 + 1 < s_1 \le t_2 + 1 < s_2 \le \cdots \le t_{n-2} + 1 < s_{n-2} \le L$ and the intervals $(t_j + 1, s_j]$ ($j = 1, \dots, n-2$) are pairwise disjoint subsets of $(t_1 + 1, L]$. At time $t_j + 1$, U_{t_j+1} has size $j+1$ and left endpoint $n-1$. Throughout $(t_j+1, s_j]$, ℓ never drops below $j+1$ (by maximality of t_j), and any $\tilde{b}^{-1}$ step leaving $\ell = j+1$ fixes U and its left endpoint. Therefore advancing the left endpoint from $n-1$ to 0 and executing the growth step at s_j requires at least m applications of $\tilde{a}^{-1}$ (the unique smallest positive k solving $(n-1) - ks \equiv 0 \pmod n$) plus one $\tilde{b}^{-1}$ step, i.e., $s_j - (t_j + 1) \ge m + 1$. Summing over $j = 1, \dots, n-2$ together with the initial step $t_1 + 1 \ge 1$ yields $L \ge (t_1 + 1) + \sum_{j=1}^{n-2}(s_j - (t_j + 1)) \ge 1 + (n-2)(m+1) = (n-1)^2 - d(n-2)$.

Part (b). Expanding $P_{n,d}(p) - P_{n,d+1}(p)$ directly factors out $(1-p)$ and yields (13) identically (Lean: `master_formula_monotone_quotient_identity`). For $m \ge 2$ and $d \ge 0$, all three coefficients are strictly positive: $\binom{m}{2} \ge 1$, $\binom{d+2}{2} \ge 1$, and $2m(d+2) - (m+d+4) = (m-2)(2d+3) + 3d + 2 \ge 2 > 0$ (Lean: `master_formula_monotone_quotient_pos`).

Part (c). Setting $p = 1$ in (3) gives $P_{n,d}(1) = \binom{m+d+2}{3} = \binom{n+1}{3}$ for all m, d (Lean: `master_formula_universal_residue_at_1`), establishing (14). Conceptually, as $p \to 1^-$ the cyclic shift a mixes uniformly between rare b-steps, rendering all $\mathcal{C}_{n,d}$ ($\gcd(d+1, n) = 1$) isomorphic as coalescing chains under σ. At $p = 1/2$, (4) gives $\frac{4}{3}n^3 + O(n^2)$ while (6) gives $4 \cdot \frac{1}{4}\binom{n}{3} + O(n^2) = \frac{1}{6}n^3 + O(n^2)$, an exact asymptotic ratio of 8.

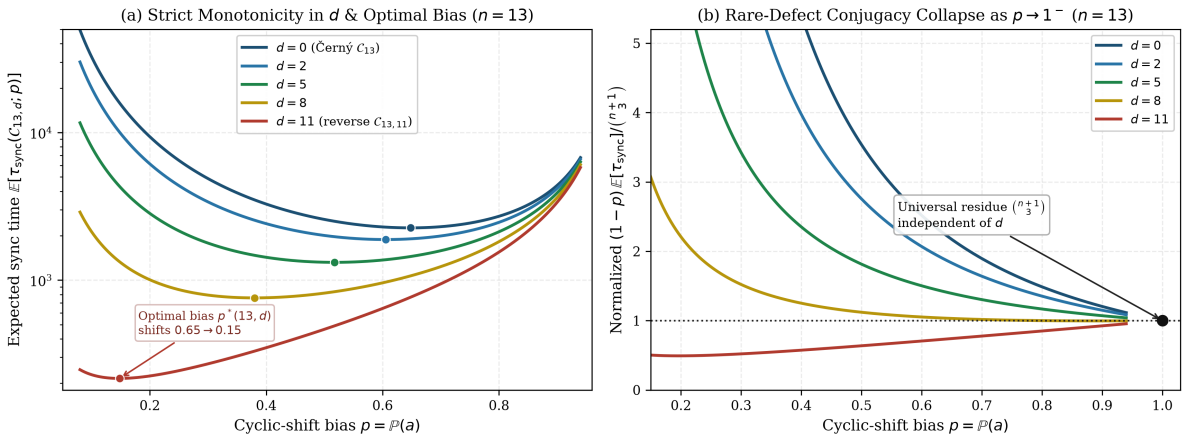


Figure 1: **Universal master formula, strict monotonicity in d, and rare-defect collapse ($n = 13$).** (a) Expected synchronization time $\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{13,d}; p)]$ versus shift bias p across $d \in \{0, 2, 5, 8, 11\}$: Černý's $\mathcal{C}_{13}$ ($d = 0$) uniquely maximizes and reverse-jump $\mathcal{C}_{13,11}$ ($d = 11$) uniquely minimizes synchronization time for every $p \in (0, 1)$, with optimal bias dots shifting from 0.65 down to 0.15. (b) Normalized $(1-p)\mathbb{E}[\tau_{\text{sync}}]/\binom{n+1}{3}$ collapses universally to 1 as $p \to 1^-$ independent of d (Theorem 3.1(c)).

Part (d). For $d = 0$, $T(\mathcal{C}_n; p) = \frac{n^3}{6p^2(1-p)} + O\Big(\frac{n^2}{p^2(1-p)}\Big)$; since $p^2(1-p)$ attains its unique maximum $4/27$ on $(0, 1)$ at $p = 2/3$, we obtain $p^*(n, 0) \to 2/3$ and $\inf_p T(\mathcal{C}_n; p) = \frac{27}{4} \cdot \frac{n^3}{6} + O(n^2) = \frac{9}{8}n^3 + O(n^2)$. For $d = n - 2$ ($n \geq 3$), (6) rewrites via $\frac{1}{1-p} = 1 + \frac{p}{1-p} \geq 1$ as

$$\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_{n,n-2}; p)] \;=\; \frac{n-2}{p(1-p)} + \frac{\binom{n-1}{2} + 1}{1-p} + \frac{\binom{n}{3}p}{1-p} \;\geq\; \frac{n-2}{p} + \binom{n-1}{2} + 1 + \binom{n}{3}p.$$

By AM–GM, $\frac{n-2}{p} + \binom{n}{3}p \geq 2\sqrt{(n-2)\binom{n}{3}} = \sqrt{\frac{2}{3}}\,n^2 + O(n)$, giving the lower bound $(\frac{1}{2} + \sqrt{\frac{2}{3}})n^2 + O(n)$; evaluating (6) at $p = \sqrt{6}/n$ matches this bound up to $O(n)$, proving (16). Moreover, differentiating (6) directly gives the unique stationarity equation $(\binom{n}{3} + \binom{n-1}{2} + 1)p^2 + 2(n-2)p - (n-2) = 0$, whose unique positive root is $p^*(n, n-2) = \frac{-(n-2) + \sqrt{(n-2)^2 + (n-2)(\binom{n}{3} + \binom{n-1}{2} + 1)}}{\binom{n}{3} + \binom{n-1}{2} + 1} = \frac{\sqrt{6}}{n} + O(n^{-2})$. □

4 Exact Pairwise Renewal and Universal Non-Concentration on $\mathcal{C}_n$

Does the random synchronization time $\tau_{\text{sync}}(\mathcal{C}_n; p)$ concentrate around its mean $\mathbb{E}[\tau_{\text{sync}}(\mathcal{C}_n; p)] \sim \frac{n^3}{6p^2(1-p)}$ as $n \to \infty$? We prove that the answer is emphatically *no* for every $p \in (0, 1)$, by reducing pairwise coalescence to an exact $(n-1)$-state negative-binomial renewal with an explicit sinusoidal eigenfunction.

Theorem 4.1 (Exact Pairwise Renewal and Universal Non-Concentration Barrier)**.** *Let $n \geq 3$, $p \in (0, 1)$, and $q := 1 - p$. For $k \in \{1, \ldots, n-1\}$, let $Z_k := \tau_{\text{sync}}(\mathcal{C}_n; p)$ started from the two-element subset $\{0, k\} \subseteq \mathbb{Z}_n$ (with $Z_0 := 0$).*

(a) ***Exact Negative-Binomial Pairwise Renewal:*** *The family $(Z_k)_{k=1}^{n-1}$ satisfies the exact distributional recursion*

$$Z_k \;\overset{d}{=}\; W_k + 1 + I_k\, Z_{n-k} + (1 - I_k)\, Z_{n-1-k}, \tag{17}$$

where $W_k \sim \mathrm{NB}(n-1-k, p)$ *(sum of* $n-1-k$ *i.i.d.* $\mathrm{Geom}(p)$ *waiting times),* $I_k \sim \mathrm{Bernoulli}(p)$*, and* $W_k, I_k, (Z_j)$ *are mutually independent. In particular, stopping the two-epoch embedded gap chain* $(Y_m)_{m\geq 0}$ *upon hitting the boundary* $\{0, n-1\}$ *yields an exact symmetric random walk on* $\{0, 1, \ldots, n-1\}$ *with interior transitions* $(1 \leq k \leq n-2)$

$$\mathbb{P}(Y_{m+1} = k \pm 1 \mid Y_m = k) = pq, \qquad \mathbb{P}(Y_{m+1} = k \mid Y_m = k) = p^2 + q^2, \tag{18}$$

exact eigenfunction $\phi(k) = \sin(\frac{\pi k}{n-1})$ *vanishing on* $\{0, n-1\}$*, and eigenvalue* $\lambda_{n-1}(p) = 1 - 2pq(1 - \cos(\frac{\pi}{n-1}))$*.*

(b) ***Universal Non-Concentration Barrier for Full Černý Synchronization:*** *For every bias* $p \in (0,1)$*,* $\tau_{\mathrm{sync}}(\mathcal{C}_n; p)/\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_n; p)]$ *fails to concentrate as* $n \to \infty$*, obeying the universal lower bound*

$$\liminf_{n\to\infty} \frac{\mathrm{Var}(\tau_{\mathrm{sync}}(\mathcal{C}_n; p))}{\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_n; p)]^2} \;\geq\; \frac{72}{\pi^4}\, e^{-\pi^2/6} \;\approx\; 0.142675 \;>\; 0. \tag{19}$$

Proof. Part (a). Starting from $\{0, k\}$ $(1 \leq k \leq n-1)$, no particle occupies $n-1$ until a has fired exactly $n-1-k$ times; during this phase b fixes the subset, so the arrival time at $\{n-1-k, n-1\}$ is $W_k \sim \mathrm{NB}(n-1-k, p)$ (with mean $\frac{n-1-k}{p}$ and variance $\frac{(n-1-k)q}{p^2}$). On the next step from $\{n-1-k, n-1\}$: with probability p, letter a shifts to $\{n-k, 0\} = \{0, n-k\}$; with probability q, letter b maps $n-1 \mapsto 0$, landing at $\{0, n-1-k\}$ (absorbing $\{0\}$ if $k = n-1$). This yields (17). For every interior state $k \in \{1, \ldots, n-2\}$, both first-half-epoch targets $n-k$ and $n-1-k$ lie strictly inside $\{1, \ldots, n-1\}$ (never absorbing at 0). Iterating (17) a second time therefore yields the exact two-epoch transitions $k \to n-k \to k$ (prob p^2), $k \to n-k \to k-1$ (prob pq, hitting 0 when $k = 1$), $k \to n-1-k \to k+1$ (prob qp, hitting $n-1$ when $k = n-2$), and $k \to n-1-k \to k$ (prob q^2), establishing (18) for the chain stopped at $\{0, n-1\}$. By trigonometric addition, $\sin\frac{\pi(k+1)}{n-1} + \sin\frac{\pi(k-1)}{n-1} = 2\cos(\frac{\pi}{n-1})\sin\frac{\pi k}{n-1}$, confirming $\mathbb{E}[\phi(Y_{m+1}) \mid Y_m = k] = \lambda_{n-1}(p)\phi(k)$ for all $1 \leq k \leq n-2$.

Part (b). Fix $p \in (0,1)$, let $\mu_n := \mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_n; p)] = \frac{n^3}{6p^2q}(1 + O(n^{-1}))$, and set $k_0 := \lfloor (n-1)/2 \rfloor \in \{1, \ldots, n-2\}$. Because $\{0, k_0\} \subseteq \mathbb{Z}_n$, coupling on the same random input stream gives $\tau_{\mathrm{sync}}(\mathcal{C}_n; p) \geq Z_{k_0}$ almost surely. Let $N_{\mathrm{ep}} := \inf\{m \geq 1 : Y_m \in \{0, n-1\}\}$ be the boundary hitting time of the stopped two-epoch walk (Y_m) started at $Y_0 = k_0$. Because $0 \leq \phi(k) \leq 1$ on $\{1, \ldots, n-2\}$ and $\phi(0) = \phi(n-1) = 0$, the exact eigenfunction martingale gives for every integer $M \geq 1$:

$$\mathbb{P}_{k_0}(N_{\mathrm{ep}} > M) \;\geq\; \mathbb{E}_{k_0}\big[\phi(Y_M)\, \mathbf{1}_{N_{\mathrm{ep}} > M}\big] \;=\; \lambda_{n-1}(p)^M \sin\!\left(\frac{\pi k_0}{n-1}\right). \tag{20}$$

Crucially, the waiting times W_k within each epoch are conditionally independent of the entire embedded path $(Y_m)_{m\geq 0}$. On the event $\{N_{\mathrm{ep}} > M\}$, every epoch state $k = Y_m$ $(0 \leq m < M)$ lies strictly in $\{1, \ldots, n-2\}$, so its half-epoch successor $k' \in \{n-k, n-1-k\}$ lies strictly in $\{1, \ldots, n-1\}$; thus the true pairwise process never absorbs during the first M two-epoch cycles, and each cycle accumulates $W_k + 1 + W_{k'} + 1 \sim 2 + \mathrm{NB}((n-1-k) + (n-1-k'), p)$ steps. Since $(n-1-k) + (n-1-k') \in \{n-2, n-1\} \geq n-2$ for every k, the total elapsed time Z_{k_0} on $\{N_{\mathrm{ep}} > M\}$ stochastically dominates $S_M := 2M + \mathrm{NB}(M(n-2), p)$, independently of $\{N_{\mathrm{ep}} > M\}$.

Fix any $\alpha > 1$ and $\delta \in (0,1)$, and set $M_n := \lfloor \frac{\alpha n^2}{6pq} \rfloor$. Then $\mathbb{E}[S_{M_n}] = M_n(\frac{n-2}{p} + 2) = \alpha\mu_n(1 + O(n^{-1}))$, while $\mathrm{Var}(S_{M_n}) = M_n(n-2)\frac{q}{p^2} = O(n^3) = o(\mu_n^2)$. By Chebyshev's inequality, $\mathbb{P}(S_{M_n} > (1-\delta)\alpha\mu_n) = 1 - O(n^{-3})$. Meanwhile, $1 - \cos(\pi/(n-1)) = \frac{\pi^2}{2(n-1)^2} + O(n^{-4}) = \frac{\pi^2}{2n^2} + O(n^{-3})$, so

$$\lambda_{n-1}(p)^{M_n} \;=\; \left(1 - \frac{\pi^2 pq}{n^2} + O(n^{-3})\right)^{\lfloor \alpha n^2/(6pq) \rfloor} \;\longrightarrow\; e^{-\pi^2\alpha/6} \qquad (n \to \infty),$$

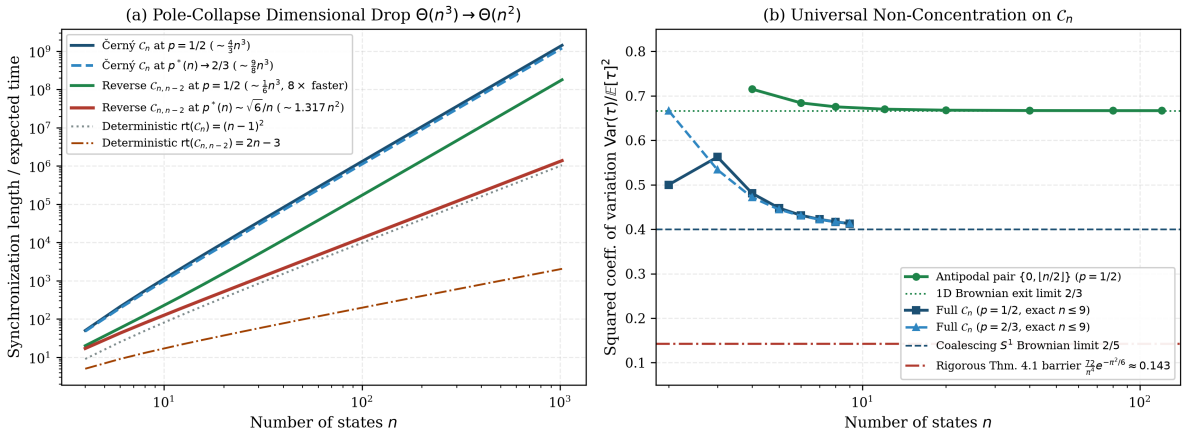


Figure 2: **Optimal-bias dimensional drop and universal non-concentration on $\mathcal{C}_n$.** (a) Log-log scaling of expected synchronization time and deterministic reset thresholds up to $n = 1024$: optimizing the bias $p^*(n) \sim \sqrt{6}/n$ on the reverse-jump automaton $\mathcal{C}_{n,n-2}$ drops expected synchronization time from $\Theta(n^3)$ down to $(\frac{1}{2} + \sqrt{\frac{2}{3}})n^2$ (nearly parallel to $\mathrm{rt}(\mathcal{C}_n) = (n-1)^2$). (b) Squared coefficient of variation $\mathrm{Var}(\tau)/\mathbb{E}[\tau]^2$ for antipodal pairs $\{0, \lfloor n/2 \rfloor\}$ (exact renewal converging to 2/3), full $\mathcal{C}_n$ at $p = 1/2$ and $p = 2/3$ (exact matrix solve $n \le 9$ converging to 2/5), and our rigorous Theorem 4.1 lower barrier $\frac{72}{\pi^4}e^{-\pi^2/6} \approx 0.143$.

and $\sin(\pi k_0/(n-1)) \to 1$. Combining with (20) yields $\liminf_{n\to\infty} \mathbb{P}(\tau_{\mathrm{sync}}(\mathcal{C}_n; p) > (1-\delta)\alpha\mu_n) \ge e^{-\pi^2\alpha/6}$ for every $\alpha > 1$ and $\delta \in (0,1)$. Applying this inequality at $\tilde{\alpha} := \alpha/(1-\delta) > 1$ gives $\liminf_{n\to\infty} \mathbb{P}(\tau_{\mathrm{sync}}(\mathcal{C}_n; p) > \alpha\mu_n) \ge e^{-\pi^2\alpha/(6(1-\delta))}$, and letting $\delta \to 0^+$ yields $\liminf_{n\to\infty} \mathbb{P}(\tau_{\mathrm{sync}}(\mathcal{C}_n; p) > \alpha\mu_n) \ge e^{-\pi^2\alpha/6}$. Finally, applying the layer-cake identity to $(\tau_{\mathrm{sync}} - \mu_n)_+^2$ and Fatou's lemma yields

$$\begin{aligned}\liminf_{n\to\infty} \frac{\mathrm{Var}(\tau_{\mathrm{sync}}(\mathcal{C}_n; p))}{\mu_n^2} &\ge \liminf_{n\to\infty} \int_1^\infty 2(\alpha-1)\,\mathbb{P}(\tau_{\mathrm{sync}} > \alpha\mu_n)\,d\alpha \\ &\ge \int_1^\infty 2(\alpha-1)\,e^{-\pi^2\alpha/6}\,d\alpha \;=\; \frac{72}{\pi^4}\,e^{-\pi^2/6}. \qquad \square\end{aligned}$$

Exact rational solutions of the full second-moment Poisson system for $n = 2, \ldots, 9$ reveal an even sharper scaling law: at $p = 1/2$, $\mathrm{Var}(\tau_{\mathrm{sync}}(\mathcal{C}_n; 1/2))$ matches the exact degree-6 integer polynomial

$$\mathrm{Var}(\tau_{\mathrm{sync}}(\mathcal{C}_n; 1/2)) \;=\; \frac{32}{45}n^6 - \frac{32}{15}n^5 + \frac{16}{9}n^4 + \frac{4}{3}n^3 - \frac{202}{45}n^2 + \frac{4}{5}n + 2 \qquad (2 \le n \le 9),$$

whose leading ratio $\frac{32/45}{(4/3)^2} = \frac{2}{5}$ coincides identically with the variance-to-mean-squared ratio $\frac{1/90}{(1/6)^2} = \frac{2}{5}$ of the total coalescence time of coalescing Brownian motions on the circle S^1 [1] (and similarly for $p = 1/4, 2/3, 3/4$; Figure 2(b) and Table 2).

Conjecture 4.2 (Coalescing Brownian Variance Limit on $\mathcal{C}_n$)**.** For every $n \ge 2$ and $p \in (0,1)$, $p^4(1-p)^2\,\mathrm{Var}(\tau_{\mathrm{sync}}(\mathcal{C}_n; p))$ is a polynomial of degree 5 in p and degree 6 in n with leading term $\frac{1}{90}n^6$. Consequently, for every $p \in (0,1)$,

$$\lim_{n\to\infty} \frac{\mathrm{Var}(\tau_{\mathrm{sync}}(\mathcal{C}_n; p))}{\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_n; p)]^2} \;=\; \frac{2}{5} \qquad \left(\lim_{n\to\infty} \mathrm{CV}(\mathcal{C}_n; p) = \sqrt{\frac{2}{5}}\right).$$

5 Deterministic-vs-Stochastic Inversion and the Ratchet Phase Transition

The cyclic defect-1 class $\mathcal{C}_{n,d}$ achieves the largest known deterministic reset threshold $(n-1)^2$, yet remains universally polynomial $O(n^3)$ for *every* bias $p \in (0,1)$. We now show that the exact opposite phenomenon occurs for a minimal linear-threshold automaton.

Definition 5.1 (Biased Ratchet Automaton $\mathcal{R}_n$)**.** For $n \geq 3$, define $\mathcal{R}_n := (\{0,1,\ldots,n-1\},\{a,b\})$ by

$$a(i) := \max(0, i-1) \ \ (0 \leq i \leq n-1), \qquad b(i) := \begin{cases} i+1 & (0 \leq i \leq n-2), \\ n-2 & (i = n-1). \end{cases}$$

Notice that a^{n-1} maps every state to 0, whereas b acts as a reflecting transposition on $\{n-2, n-1\}$ while pushing $0,\ldots,n-3$ upward.

Theorem 5.2 (Exact Mean, Compound-Geometric Variance, and Sharp Phase Transition at $p_c = 1/2$)**.** *For every $n \geq 3$, the Biased Ratchet Automaton $\mathcal{R}_n$ has exact linear deterministic reset threshold $\mathrm{rt}(\mathcal{R}_n) = n-1$, while under i.i.d. bias $\mathbb{P}(a) = p \in (0,1)$ ($q := 1-p$, $\rho := q/p$):*

(a) ***Exact Mean and Sharp Phase Transition at $p_c = 1/2$:***

$$\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{R}_n; p)] \ = \ \begin{cases} n(n-1) & \text{if } p = \frac{1}{2}, \\ \dfrac{n-1}{2p-1} - \dfrac{1-p}{(2p-1)^2}(1-\rho^{n-1}) & \text{if } p \neq \frac{1}{2}. \end{cases} \tag{21}$$

Thus $\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{R}_n; p)]$ scales linearly as $\frac{n}{2p-1} + O(1)$ for $p > 1/2$, quadratically as $n(n-1)$ at $p_c = 1/2$, and exponentially as $\frac{1-p}{(1-2p)^2}\rho^{n-1} + O(n)$ with Lyapunov exponent $\lambda(p) = \log\frac{1-p}{p} > 0$ for $p < 1/2$.

(b) ***Exact Compound-Geometric Variance and Three-Regime Fluctuation Collapse:*** *Writing $\tau_{\mathrm{sync}}(\mathcal{R}_n; p) = \sum_{k=1}^{n-1} X_k$ as a sum of independent downward step times X_k ($k \to k-1$), their exact means E_k and variances V_k obey $E_{n-1} = \frac{1}{p}$, $V_{n-1} = \frac{q}{p^2}$, and*

$$E_k \ = \ \frac{1 + qE_{k+1}}{p}, \qquad V_k \ = \ \rho\, V_{k+1} + \frac{q}{p^2}(1+E_{k+1})^2 \qquad (1 \leq k \leq n-2), \tag{22}$$

with $\mathrm{Var}(\tau_{\mathrm{sync}}(\mathcal{R}_n; p)) = \sum_{k=1}^{n-1} V_k$. At the critical point $p_c = 1/2$, this evaluates in closed form to

$$\mathrm{Var}(\tau_{\mathrm{sync}}(\mathcal{R}_n; 1/2)) \ = \ \frac{n(n-1)(2n^2-2n-1)}{3}. \tag{23}$$

Consequently, as $n \to \infty$, the coefficient of variation $\mathrm{CV}(\mathcal{R}_n; p) := \sqrt{\mathrm{Var}}/\mathbb{E}$ undergoes a sharp three-regime collapse at $p_c = 1/2$:

$$\lim_{n\to\infty} \mathrm{CV}(\mathcal{R}_n; p) \ = \ \begin{cases} 0 & \text{if } p > \frac{1}{2} \quad (\textit{sharp LLN concentration}), \\ \sqrt{\frac{2}{3}} & \text{if } p = \frac{1}{2} \quad (\textit{diffusive non-concentration}), \\ 1 & \text{if } p < \frac{1}{2} \quad (\textit{asymptotic exponential memorylessness}). \end{cases} \tag{24}$$

Proof. We first prove by induction on t that every reachable non-singleton subset S_t of $\mathcal{R}_n$ is a contiguous integer interval $[L_t, R_t]$ ($0 \leq L_t < R_t \leq n-1$). Initially $S_0 = [0, n-1]$. Given $S = [L, R]$ with $L < R$: under a, $a([L,R]) = [\max(0, L-1), R-1]$, which is the singleton $\{0\}$ iff $R = 1$ (since $L = 0$), and otherwise is a contiguous interval of size ≥ 2 with

maximum $R-1$. Under b, if $R \le n-2$ then $b([L,R]) = [L+1, R+1]$ (maximum $R+1$); if $R = n-1$, then $L \le n-2$ implies $n-2, n-1 \in S$, so $b(n-2) = n-1$, $b(n-1) = n-2$, and $b([L, n-1]) = [\min(L+1, n-2),\, n-1]$, which remains a contiguous interval of size ≥ 2 with maximum $n-1$. Hence $M_t := \max(S_t)$ is an exact nearest-neighbor Markov chain on $\{0, 1, \ldots, n-1\}$ absorbed at 0 and reflected at $n-1$ ($k \mapsto k-1$ w.p. p, $k \mapsto \min(k+1, n-1)$ w.p. q), and $\tau_{\text{sync}}(\mathcal{R}_n; p) = \inf\{t \ge 0 : M_t = 0\}$. In particular, reaching $M_t = 0$ from $M_0 = n-1$ requires at least $n-1$ steps (and a^{n-1} achieves $M_{n-1} = 0$), proving $\mathrm{rt}(\mathcal{R}_n) = n-1$.

Part (a). The expected absorption times $T_k := \mathbb{E}_k[\tau]$ satisfy $T_0 = 0$, $T_k = 1 + pT_{k-1} + qT_{k+1}$ ($1 \le k \le n-2$), and $T_{n-1} = 1 + pT_{n-2} + qT_{n-1}$. At $p = 1/2$, direct substitution verifies $T_k = k(2n-1-k)$ and $T_{n-1} = n(n-1)$ (Lean: `ratchet_critical_mean_poisson`). For $p \ne 1/2$, direct substitution verifies $T_k = \frac{k}{2p-1} - \frac{q}{(2p-1)^2}\rho^{n-1-k}(1-\rho^k)$ on both interior and reflecting equations (Lean: `ratchet_noncritical_poisson_identity`), and setting $k = n-1$ yields (21).

Part (b). By the strong Markov property of the skip-free downward walk M_t, $\tau = \sum_{k=1}^{n-1} X_k$ where X_k is the hitting time of $k-1$ from k, and $X_1, \ldots, X_{n-1}$ are mutually independent. From $n-1$, M_t stays at $n-1$ w.p. q and steps to $n-2$ w.p. p, so $X_{n-1} \sim \mathrm{Geom}(p)$ with $E_{n-1} = 1/p$ and $V_{n-1} = q/p^2$. For $1 \le k \le n-2$, letting $G \sim \mathrm{Geom}_0(p)$ count the number of upward excursions to $k+1$ before stepping down to $k-1$, we have the compound-geometric representation $X_k \overset{d}{=} 1 + \sum_{j=1}^{G}(1 + X_{k+1}^{(j)})$ with $X_{k+1}^{(j)}$ i.i.d. copies of X_{k+1} independent of G. Since $\mathbb{E}[G] = q/p = \rho$ and $\mathrm{Var}(G) = q/p^2$, the laws of total expectation and total variance yield (22).

At $p = 1/2$ ($\rho = 1$), $E_k = 2(n-k)$, so $V_k - V_{k+1} = 2(1 + 2(n-1-k))^2 = 2(2(n-k)-1)^2$. Setting $r := n-k$ gives $V_{n-r} = \frac{2r(4r^2-1)}{3}$ and $\sum_{r=1}^{n-1} V_{n-r} = \frac{n(n-1)(2n^2-2n-1)}{3}$ (Lean: `ratchet_critical_variance_identity`), yielding (23) and $\mathrm{CV}(\mathcal{R}_n; 1/2) \to \sqrt{2/3}$. For $p > 1/2$ ($\rho < 1$), $E_k = \frac{1-\rho^{n-k}}{2p-1} \le \frac{1}{2p-1}$ and $1 + E_{k+1} \to \frac{2p}{2p-1}$, so solving the contracting linear recurrence (22) gives $V_k \to \frac{4pq}{(2p-1)^3}$ uniformly as $n-k \to \infty$, whence $\mathrm{Var}(\tau) = \frac{4pq}{(2p-1)^3}n + O(1)$ and $\mathrm{CV} = O(n^{-1/2}) \to 0$. For $p < 1/2$ ($\rho > 1$), $E_k = \frac{\rho^{n-k}-1}{1-2p}$, so $(1+E_{k+1})^2 = \frac{\rho^{2(n-k-1)}}{(1-2p)^2} + O(\rho^{n-k})$; iterating (22) and using $\frac{q}{p^2(1-\rho^{-1})} = \frac{q^2}{p^2(q-p)} = \frac{\rho^2}{1-2p}$ yields

$$V_k \;=\; \frac{q}{p^2(1-2p)^2}\sum_{j=0}^{n-k-2}\rho^j\rho^{2(n-k-1-j)} + O(n\rho^{n-k}) \;=\; \frac{\rho^{2(n-k)}}{(1-2p)^3} + O(n\rho^{n-k}).$$

Since $\rho^2 - 1 = \frac{q^2-p^2}{p^2} = \frac{1-2p}{p^2}$ and $p^2\rho^{2n} = q^2\rho^{2n-2}$, summing over $k = 1, \ldots, n-1$ gives

$$\sum_{k=1}^{n-1} V_k \;=\; \frac{1}{(1-2p)^3}\cdot\frac{\rho^{2n}-\rho^2}{\rho^2-1} + O(n\rho^n) \;=\; \frac{q^2}{(1-2p)^4}\,\rho^{2n-2} + O(n\rho^n) \;=\; \mathbb{E}[\tau]^2(1 + O(n\rho^{-n})),$$

proving $\lim_{n\to\infty} \mathrm{CV}(\mathcal{R}_n; p) = 1$. □

6 Computational Verification and Lean 4.36.0 Artifact Manifest

Tables 1 and 2 report exact verification benchmarks from `data/verification_summary.json` produced by `src/python/verify_cerny_markov.py`. Across all 1,398 non-singleton subsets of $\mathcal{C}_n$ ($2 \le n \le 8$, $p \in \{1/3, 1/2, 3/5\}$), exact rational `Fraction` arithmetic confirms the Poisson equation for (5) with zero discrepancy. Across all coprime pairs (n, d) for $2 \le n \le 10$ and $p \in \{0.23, 0.5, 0.77\}$, the universal master formula (2) matches exact layer-by-layer matrix solves to maximum relative error 8.373×10^{-14}; across all 41 coprime pairs for $2 \le n \le 11$, BFS verifies $\mathrm{rt}(\mathcal{C}_{n,d}) = (n-1)^2 - d(n-2)$ with zero exceptions; and across $\mathcal{R}_n$ ($n \in \{4, 6, 8, 10\}$), exact matrix solves match (21)–(22) to 4.336×10^{-13}.

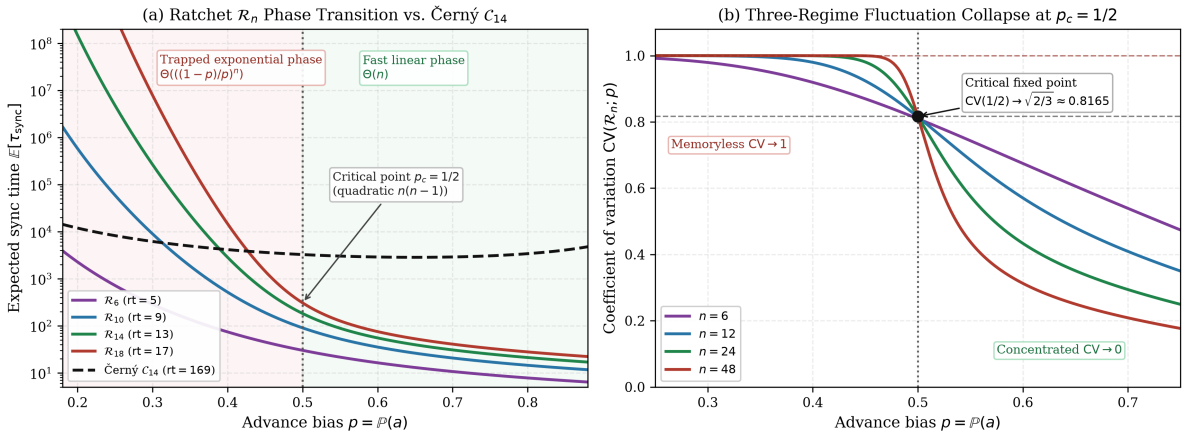


Figure 3: **Deterministic-vs-stochastic inversion and sharp $p_c = 1/2$ phase transition on $\mathcal{R}_n$.** (a) Expected synchronization time of $\mathcal{R}_n$ ($n \in \{6, 10, 14, 18\}$, linear rt $= n - 1$) versus Černý's $\mathcal{C}_{14}$ (quadratic rt $= 169$, dashed black): for $p < 1/2$, $\mathcal{R}_{14}$ explodes exponentially and overtakes $\mathcal{C}_{14}$ by orders of magnitude despite having 13× smaller deterministic reset threshold. (b) Coefficient of variation CV($\mathcal{R}_n; p$) sharpening to the three-regime step function (1 for $p < 1/2$, fixed point $\sqrt{2/3}$ at $p_c = 1/2$, 0 for $p > 1/2$).

6.1 Formal Lean 4.36.0 certification and reproducibility

All cleared-denominator polynomial and algebraic identities underlying Theorems 2.1, 3.1, 4.1, and 5.2 are formally verified in standalone Lean 4.36.0 (26 total declarations: 17 public theorems, 6 public definitions, 3 private helpers; zero Mathlib, zero `sorry`, zero `grind`, zero custom axioms, compiled with `-DwarningAsError=true` and `lake build`). The 17 certified public theorems cover:

- *Černý harmonic potential & cubic law:*
  `cerny_potential_case1_poisson`, `cerny_potential_case2_poisson`, `cerny_potential_case3_poisson`, `cerny_singleton_boundary_zero`, `cerny_uniform_cubic_factorization`;
- *Adjacent-swap invariance & universal master formula:*
  `adjacent_swap_invariance_identity`, `palindromic_flux_swap_invariance`, `master_formula_cerny_specialization`, `master_formula_reverse_pole_collapse`, `master_formula_universal_residue_at_1`, `master_formula_monotone_quotient_identity`, `master_formula_monotone_quotient_pos`;
- *Reset spectrum, pairwise time, & ratchet phase transition:*
  `deterministic_reset_threshold_identity`, `pairwise_adjacent_cerny_half`, `ratchet_critical_mean_poisson`, `ratchet_critical_variance_identity`, `ratchet_noncritical_poisson_identity`.

All archived artifacts are publicly available in the AIRR repository (https://github.com/arr-research/arr-research.github.io) under CC-BY-4.0 (manuscript) and MIT (code and data):

- `src/lean/CernyMarkovCore.lean` (46,519 bytes)
  SHA-256: `9a1821ff8a909327d9dc23e8acf42b9402fe3ecdf33e8665a67a6193d6db5e3e`
- `src/python/verify_cerny_markov.py` (28,929 bytes)
  SHA-256: `a39f1fb7abaaf46875297c170073d128bf21f40e3d65cbd3732139d79d81a931`
- `data/verification_summary.json` (11,484 bytes)
  SHA-256: `c4ee29fa53b02b81ef95714a83ba7ca06d6f11443b3c12c9aa67378222c28200`

Table 1: Exact deterministic reset threshold $\mathrm{rt}(\mathcal{C}_{n,d})$, uniform-bias expected time $\mathbb{E}[\tau_{\mathrm{sync}}(\mathcal{C}_{n,d}; 1/2)]$, optimal bias $p^*(n,d)$, optimal expected time T_{opt}, and universal $p \to 1^-$ residue $\binom{n+1}{3}$ (from `data/verification_summary.json`).

n	d	$\mathrm{rt}(\mathcal{C}_{n,d})$	$\mathbb{E}[\tau(1/2)]$	$p^*(n,d)$	T_{opt}	Residue $\binom{n+1}{3}$
4	0	9	50	0.589233	47.7454	10
4	2	5	20	0.309017	16.9443	10
5	0	16	112	0.609695	104.3860	20
5	1	13	79	0.536974	78.4140	20
5	2	10	54	0.432575	52.7732	20
5	3	7	36	0.279175	28.4919	20
7	0	36	352	0.629205	318.8986	56
7	2	26	214	0.529260	213.0123	56
7	5	11	87	0.230063	59.4664	56
10	0	81	1122	0.641847	995.0306	165
10	2	65	804	0.581854	774.2761	165
10	8	17	226	0.180457	125.6636	165

Table 2: Exact variance and non-concentration benchmarks on Černý's $\mathcal{C}_n$ ($p = 1/2, 2/3$) alongside the reverse-jump optimal-bias scaling $n\,p^*(\mathcal{C}_{n,n-2})$ and normalized optimal time $T_{\mathrm{opt}}(\mathcal{C}_{n,n-2})/n^2$ up to $n = 1024$.

n	$\mathbb{E}[\tau(\mathcal{C}_n; \frac{1}{2})]$	$\mathrm{Var}(\mathcal{C}_n; \frac{1}{2})$	$\mathrm{Var}/\mathbb{E}^2\ (\frac{1}{2})$	$\mathrm{Var}/\mathbb{E}^2\ (\frac{2}{3})$	$n\,p^*(\mathcal{C}_{n,n-2})$	$T_{\mathrm{opt}}(\mathcal{C}_{n,n-2})/n^2$
4	50	1202	0.480800	0.471534	1.236068	1.059017
6	210	19026	0.431429	0.430108	1.515902	1.185120
8	546	124194	0.416596	0.416521	1.687174	1.232811
9	800	264224	0.412850	0.412954	1.750839	1.246488
32	—	—	—	—	2.209061	1.303855
128	—	—	—	—	2.385066	1.313852
1024	—	—	—	—	2.441264	1.316185
∞	$\sim \frac{4}{3}n^3$	$\sim \frac{32}{45}n^6$	$0.400000^\dagger$	$0.400000^\dagger$	$\sqrt{6} \approx 2.4495$	$\frac{1}{2} + \sqrt{\frac{2}{3}} \approx 1.3165$

†Conjecture 4.2 (coalescing Brownian limit on S^1); Theorem 4.1 proves $\liminf \mathrm{Var}/\mathbb{E}^2 \geq \frac{72}{\pi^4} e^{-\pi^2/6} \approx 0.142675$ for all $p \in (0,1)$.

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