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THE EXACT KREIN–DIRICHLET-TO-NEUMANN LAW FOR CRITICAL BRAESS TRANSITIONS IN QUANTUM GRAPHS, CUBIC TANGENT SCALING, AND UNIVERSAL MULTIPLICITY-2 COLLAPSE

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Abstract. Attaching a new edge e_ℓ of length $\ell > 0$ between two points u, v of a compact connected metric graph Γ equipped with the Neumann–Kirchhoff Laplacian $-\Delta_\Gamma$ creates a fundamental competition between additional connectivity (which tends to raise the spectral gap λ_1) and additional mass/length (which tends to lower λ_1). When does adding e_ℓ strictly *lower* the first non-zero eigenvalue—the spectral Braess paradox $\lambda_1(\Gamma \cup_{u,v} e_\ell) < \lambda_1(\Gamma)$—and what is the *exact* critical length $\ell_*(u, v)$ separating spectral-gap improvement from Braess degradation?

We establish a complete, closed-form analytical theory resolving both questions across arbitrary compact metric graphs. First, via a mean-zero k-harmonic extension and a complementary high-length cosine trial function, we prove an exact universal Rayleigh deficit formula $\mathcal{Q}_k(\ell) = \frac{k}{2}\left[D^2 \cot(\frac{k\ell}{2}) - S^2 \frac{kL \tan(k\ell/2)}{kL + 2\tan(k\ell/2)}\right]$ ($k = \sqrt{\lambda_1(\Gamma)}$, $D = \psi_1(u) - \psi_1(v)$, $S = \psi_1(u) + \psi_1(v)$), yielding a universal upper bound $\ell_*(u, v) \le \ell_{\mathrm{var}}(u, v) \le \pi/k$, proving universal strict collapse for all $\ell > \pi/k$ whenever $(\psi_1(u), \psi_1(v)) \neq (0, 0)$, and establishing the *zero-length equipotential/loop Braess paradox*: whenever $\psi_1(u) = \psi_1(v) \neq 0$ (including any self-loop off the nodal set), $\lambda_1(\Gamma \cup e_\ell) < \lambda_1(\Gamma)$ for *every* $\ell > 0$. Second, whenever $\lambda_1(\Gamma) = k^2$ is simple and $u \neq v$ is a non-nodal pair (i.e., $(\psi_1(u), \psi_1(v)) \neq (0, 0)$), we combine Krein–Birman–Schwinger resolvent reduction, edge-lengthening monotonicity, and the Hadamard derivative to prove that the exact critical length $\ell_*(u, v) \in [0, \pi/k]$ is determined uniquely by the universal quadratic law in $t = \tan(k\ell_*/2)$:

$$\frac{S^2}{2}\, t^2 + kQ_M\, t - \frac{D^2}{2} = 0, \qquad Q_M := (\mathbf{p}^\perp)^\top M_\Gamma(k^2)\mathbf{p}^\perp = -\frac{D^2}{k^2 L} + \sum_{j \ge 2} \frac{(\psi_j(u)\psi_1(v) - \psi_j(v)\psi_1(u))^2}{\lambda_j - k^2},$$

where $M_\Gamma(k^2) = (G'_\Gamma(x, y; k^2))_{x,y \in \{u,v\}}$ is the reduced Green's matrix at λ_1. This proves that $\ell_{\mathrm{var}}(u, v)$ is literally the zero-mode truncation of the exact Krein law, that higher-mode relaxation universally lowers $\ell_*(u, v) \le \ell_{\mathrm{var}}(u, v)$, and that anti-equipotential pairs ($S = 0, D \neq 0$) undergo a sharp *sign transition* in Q_M: $\ell_* = \pi/k$ when $Q_M \le 0$, whereas $Q_M > 0$ (realized on asymmetric 3-stars) triggers a finite Braess transition strictly below π/k. Third, we prove the *Universal Multiplicity-*≥ 2 *Collapse Theorem*: whenever $\dim E_{\lambda_1}(\Gamma) \ge 2$ (circles, equilateral stars, pumpkins, equilateral complete graphs), $\lambda_1(\Gamma \cup_{u,v} e_\ell) \le \lambda_1(\Gamma)$ for *every* $u, v \in \Gamma$ and *every* $\ell > 0$. Fourth, on the interval $[0, L]$ we evaluate $M_\Gamma(k^2)$ in closed form, obtaining $kQ_M = \frac{2}{L}\sin(\frac{\pi(v-u)}{L})\cos(\frac{\pi u}{L})\cos(\frac{\pi v}{L})$, the exact cubic tangent law $\tan(\frac{\pi \ell_*(0,a)}{2L}) = \tan^3(\frac{\pi a}{2L})$ (with duality $\ell_*(0, a) + \ell_*(0, L - a) = L$), the midpoint law $\ell_*(u, L - u) = L$, and the short-chord limit $\ell_*(u, u + h)/h \to \frac{1}{2}(\sqrt{1 + 4\tan^2(\pi u/L)} - 1)$, revealing a *magic tetrahedral detour angle* $u_{\mathrm{crit}}/L = \frac{1}{\pi}\arctan(\sqrt{2}) = \frac{1}{2\pi}\arccos(-1/3) \approx 0.304087$ beyond which even detour chords strictly longer than the geodesic distance improve the spectral gap.

1. Introduction

On a finite combinatorial graph, adding an edge can never decrease the algebraic connectivity (Fiedler value) of the combinatorial Laplacian $L = D - A$: the quadratic form $\sum_{\{i,j\} \in E}(f_i - f_j)^2$

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increases monotonically while the vertex mass $\|f\|_2^2$ remains unchanged. On a *metric (quantum) graph* Γ equipped with the standard Neumann–Kirchhoff Laplacian $-\Delta_\Gamma$ [3, 7], the physics is fundamentally richer. Every edge e carries one-dimensional Lebesgue measure of length $\ell_e > 0$; attaching a new edge e_ℓ of length ℓ between two points $u, v \in \Gamma$ simultaneously adds kinetic energy $\int_{e_\ell} |f'|^2$ and adds L^2-mass $\int_{e_\ell} |f|^2$. As a result, adding an edge can counterintuitively *lower* the first non-zero eigenvalue (spectral gap) $\lambda_1(\Gamma)$—a spectral manifestation of the *Braess paradox* in quantum networks, diffusive transport, and elastic frames [6, 1, 8, 2].

Despite intensive study of spectral surgery and isoperimetric bounds on quantum graphs [4, 6, 2, 7], three fundamental quantitative questions concerning edge-attachment Braess transitions have remained open:

(1) *Exact Critical Length Formula:* For an arbitrary compact connected metric graph Γ and arbitrary attachment points $u, v \in \Gamma$, is there a closed-form algebraic law determining the exact critical edge length $\ell_*(u, v)$ separating spectral-gap improvement ($\lambda_1(\Gamma \cup_{u,v} e_\ell) > \lambda_1(\Gamma)$) from Braess degradation ($\lambda_1(\Gamma \cup_{u,v} e_\ell) < \lambda_1(\Gamma)$)?
(2) *Role of Higher Eigenmodes and Anti-Equipotential Pairs:* If $\psi_1(u) = -\psi_1(v) \neq 0$ (maximally opposite first eigenfunction values), classical intuition suggests the added edge should help up to the half-wavelength $\ell = \pi/k_1$. Can higher-mode relaxation break this threshold and induce a Braess paradox strictly below π/k_1?
(3) *Multiplicity Obstruction and Exact Interval Geometry:* What happens universally when $\lambda_1(\Gamma)$ is degenerate ($\dim E_{\lambda_1}(\Gamma) \geq 2$), and what is the exact closed-form phase boundary $\ell_*(u, v)$ on the fundamental benchmark $\Gamma = [0, L]$?

Setup and Notation. Let $\Gamma = (V, E, \{\ell_e\}_{e\in E})$ be a compact connected finite metric graph of total length $L := |\Gamma| = \sum_{e\in E} \ell_e > 0$. Throughout the paper, all function spaces $L^2(\Gamma) = L^2(\Gamma; \mathbb{R})$ and $H^1(\Gamma) = H^1(\Gamma; \mathbb{R})$ are real-valued: $H^1(\Gamma)$ consists of continuous real-valued functions $f \in C(\Gamma; \mathbb{R})$ whose restriction to each edge $e \cong [0, \ell_e]$ lies in $H^1(0, \ell_e; \mathbb{R})$. The Neumann–Kirchhoff Laplacian $-\Delta_\Gamma$ is the self-adjoint operator on $L^2(\Gamma)$ associated with the closed symmetric quadratic form

$$\mathcal{E}_\Gamma(f) := \int_\Gamma |f'(x)|^2 \, dx, \qquad \mathrm{dom}(\mathcal{E}_\Gamma) = H^1(\Gamma).$$

Its domain consists of $f \in H^1(\Gamma)$ that are edgewise H^2 and satisfy the Kirchhoff current-conservation condition $\sum_{e\sim w} \partial_e f(w) = 0$ at every vertex $w \in V$ (where $\partial_e f(w)$ denotes the derivative directed *into* e from w). Its discrete spectrum is

$$0 = \lambda_0(\Gamma) < \lambda_1(\Gamma) \leq \lambda_2(\Gamma) \leq \cdots \to \infty,$$

with a real-valued $L^2(\Gamma)$-orthonormal eigenbasis $\{\psi_j\}_{j=0}^\infty$ ($\psi_0 \equiv L^{-1/2}$). We write $k := \sqrt{\lambda_1(\Gamma)} > 0$ and $E_{\lambda_1}(\Gamma) := \ker(-\Delta_\Gamma - \lambda_1(\Gamma))$. Given any two points $u, v \in \Gamma$ (which may be interior points of edges; inserting dummy degree-2 vertices at u and v leaves $H^1(\Gamma)$ and $-\Delta_\Gamma$ unchanged) and any $\ell > 0$, we denote by $\Gamma_\ell := \Gamma \cup_{u,v} e_\ell$ the compact connected metric graph obtained by attaching a new edge $e_\ell \cong [0, \ell]$ with endpoint 0 identified to u and endpoint ℓ identified to v (forming a self-loop if $u = v$).

Overview of Main Results.

- **Universal Variational Formula, Equipotential Paradox, and High-Length Collapse (Theorem 2.1).** For any eigenfunction $\psi_1 \in E_{\lambda_1}(\Gamma)$ ($\|\psi_1\|_{L^2(\Gamma)} = 1$), setting $D := \psi_1(u) - \psi_1(v)$ and $S := \psi_1(u) + \psi_1(v)$, the mean-zero k-harmonic extension on Γ_ℓ ($\ell \in (0, \pi/k)$) yields the exact Rayleigh deficit

$$\mathcal{Q}_k(\ell) = \frac{k}{2}\left[D^2 \cot\left(\frac{k\ell}{2}\right) - S^2 \frac{kL\tan(k\ell/2)}{kL + 2\tan(k\ell/2)}\right],$$

implying whenever $(\psi_1(u),\psi_1(v)) \neq (0,0)$ that $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ for every $\ell \in (\ell_{\mathrm{var}}(u,v), \pi/k)$, where $\ell_{\mathrm{var}}(u,v) = \frac{2}{k}\arctan\!\Big(\frac{|D|(|D|+\sqrt{D^2+(kL)^2S^2})}{kLS^2}\Big)$ for $D \neq 0, S \neq 0$ (with $\ell_{\mathrm{var}} = 0$ if $D = 0, S \neq 0$ and $\ell_{\mathrm{var}} = \pi/k$ if $S = 0, D \neq 0$). In particular, whenever $\psi_1(u) = \psi_1(v) \neq 0$ (equipotential pair or non-nodal self-loop), $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ for *every* $\ell > 0$. A complementary cosine trial function proves $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ universally for every $\ell > \pi/k$ whenever $(\psi_1(u),\psi_1(v)) \neq (0,0)$.

- **Exact Krein–Dirichlet-to-Neumann Law (Theorem 3.2).** Whenever $\lambda_1(\Gamma) = k^2$ is simple and $u \neq v$ with $(\psi_1(u),\psi_1(v)) \neq (0,0)$, combining edge-lengthening monotonicity, Cauchy interlacing, and Krein resolvent reduction shows that $\ell \mapsto \lambda_1(\Gamma_\ell)$ crosses $\lambda_1(\Gamma)$ at a unique sharp threshold $\ell_*(u,v) \in [0,\pi/k]$ governed by the universal quadratic in $t = \tan(k\ell_*/2)$:

$$\frac{S^2}{2}\,t^2 + kQ_M\,t - \frac{D^2}{2} \;=\; 0, \qquad Q_M := (\mathbf{p}^\perp)^\top M_\Gamma(k^2)\mathbf{p}^\perp,$$

  where $\mathbf{p}^\perp = (\psi_1(v), -\psi_1(u))^\top$ and $M_\Gamma(k^2)$ is the 2×2 reduced Green's matrix at k^2. The spectral expansion $Q_M = -\frac{D^2}{k^2L} + \sum_{j\geq 2}\frac{(\psi_j(u)\psi_1(v)-\psi_j(v)\psi_1(u))^2}{\lambda_j - k^2}$ proves $\ell_*(u,v) \leq \ell_{\mathrm{var}}(u,v)$ universally and uncovers a sharp sign transition for anti-equipotential pairs ($S = 0, D \neq 0$): $\ell_* = \pi/k$ when $Q_M \leq 0$, whereas $Q_M > 0$ yields $\ell_* = \frac{2}{k}\arctan(\frac{D^2}{2kQ_M}) < \frac{\pi}{k}$ (realized explicitly on the asymmetric 3-star $(1.0, 0.7, 0.35)$).
- **Universal Multiplicity-≥ 2 Collapse (Theorem 4.1).** Whenever $\dim E_{\lambda_1}(\Gamma) \geq 2$, $\lambda_1(\Gamma_\ell) \leq \lambda_1(\Gamma)$ holds for *every* $u, v \in \Gamma$ and *every* $\ell > 0$, with strict inequality for all $\ell > 0$ whenever $\exists\,\psi \in E_{\lambda_1}(\Gamma)$ with $\psi(u) = \psi(v) \neq 0$.
- **Exact Interval Laws and Magic Tetrahedral Detour Angle (Theorem 5.1).** On $\Gamma = [0,L]$, we prove $kQ_M(u,v) = \frac{2}{L}\sin(\frac{\pi(v-u)}{L})\cos(\frac{\pi u}{L})\cos(\frac{\pi v}{L})$, giving the exact cubic tangent law $\tan(\frac{\pi\ell_*(0,a)}{2L}) = \tan^3(\frac{\pi a}{2L})$ (with duality $\ell_*(0,a) + \ell_*(0,L-a) = L$ and cubic small-a scaling $\frac{\pi^2}{4L^2}a^3$), the midpoint law $\ell_*(u,L-u) = L$, and the short-chord limit $\ell_*(u,u+h)/h \to \frac{1}{2}(\sqrt{1+4\tan^2(\pi u/L)} - 1)$ $(u \in (0,L/2))$, which crosses 1 at the magic tetrahedral angle $u_{\mathrm{crit}}/L = \frac{1}{\pi}\arctan(\sqrt{2}) = \frac{1}{2\pi}\arccos(-1/3) \approx 0.304087$.

2. Universal Variational Formula, Equipotential Paradox, and High-Length Collapse

We first construct two explicit mean-zero trial functions in $H^1(\Gamma_\ell)$ requiring no knowledge of higher eigenmodes ψ_j $(j \geq 2)$: a k-harmonic extension valid on $(0,\pi/k)$ and a cosine interpolation valid on $(0,\infty)$.

Theorem 2.1 (Universal variational Braess bound, equipotential paradox, and high-length collapse)**.** *Let Γ be a compact connected metric graph of total length $L > 0$ with first non-zero eigenvalue $\lambda_1(\Gamma) = k^2 > 0$, let $\psi_1 \in E_{\lambda_1}(\Gamma)$ satisfy $\|\psi_1\|_{L^2(\Gamma)} = 1$, and let $u, v \in \Gamma$ (possibly $u = v$). Set $D := \psi_1(u) - \psi_1(v)$ and $S := \psi_1(u) + \psi_1(v)$.*

(a) Mean-zero k-harmonic Rayleigh deficit. *For every $\ell \in (0,\pi/k)$, there exists a non-zero mean-zero function $f_\ell \in H^1(\Gamma_\ell)$ ($\int_{\Gamma_\ell} f_\ell = 0$) satisfying*

$$\mathcal{Q}_k(\ell) \;:=\; \int_{\Gamma_\ell} |f_\ell'|^2\,dx - k^2\int_{\Gamma_\ell}|f_\ell|^2\,dx \;=\; \frac{k}{2}\left[D^2\cot\left(\frac{k\ell}{2}\right) - S^2\,\frac{kL\tan(k\ell/2)}{kL + 2\tan(k\ell/2)}\right]. \tag{1}$$

Consequently, whenever $(\psi_1(u),\psi_1(v)) \neq (0,0)$*, we have* $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ *for every* $\ell \in (\ell_{\mathrm{var}}(u,v), \pi/k)$*, where*

$$\ell_{\mathrm{var}}(u,v) \;:=\; \begin{cases} 0, & D=0,\ S\neq 0,\\ \dfrac{2}{k}\arctan\!\left(\dfrac{|D|\big(|D|+\sqrt{D^2+(kL)^2S^2}\big)}{kL\,S^2}\right), & D\neq 0,\ S\neq 0,\\ \dfrac{\pi}{k}, & S=0,\ D\neq 0. \end{cases} \tag{2}$$

(b) High-length cosine deficit and universal collapse for $\ell > \pi/k$. *For every* $\ell > 0$*, there exists a non-zero mean-zero function* $f_\ell^{\mathrm{hi}} \in H^1(\Gamma_\ell)$ *satisfying*

$$\mathcal{Q}_k^{\mathrm{hi}}(\ell) \;:=\; \int_{\Gamma_\ell} |(f_\ell^{\mathrm{hi}})'|^2\,dx - k^2\int_{\Gamma_\ell}|f_\ell^{\mathrm{hi}}|^2\,dx \;=\; \frac{D^2}{8\ell}\big(\pi^2-k^2\ell^2\big) - \frac{k^2L\ell}{4(L+\ell)}\,S^2. \tag{3}$$

In particular, whenever $(\psi_1(u),\psi_1(v)) \neq (0,0)$*:*

- $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ *strictly for* every $\ell > \pi/k$*;*
- *if* $S \neq 0$*, then* $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ *strictly for* every $\ell \geq \pi/k$*;*
- (Zero-length equipotential and loop Braess paradox) *if* $\psi_1(u) = \psi_1(v) \neq 0$ *(including any self-loop* $u = v$ *with* $\psi_1(u) \neq 0$*), then* $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ *strictly for* every $\ell > 0$.

Proof. Proof of (a). Fix $\ell \in (0,\pi/k)$, so $\cos(k\ell/2) > 0$ and $\sin(k\ell/2) > 0$, and let $c \in \mathbb{R}$ be a constant to be chosen shortly. Define $f_\ell \in H^1(\Gamma_\ell)$ by $f_\ell(x) := \psi_1(x) - c$ for $x \in \Gamma$, and on $e_\ell \cong [0,\ell]$ by the unique solution of $-g'' = k^2 g$ matching the boundary values $g(0) = \psi_1(u) - c$ and $g(\ell) = \psi_1(v) - c$:

$$g(s) \;:=\; \frac{S-2c}{2}\,\frac{\cos\big(k(s-\ell/2)\big)}{\cos(k\ell/2)} \;-\; \frac{D}{2}\,\frac{\sin\big(k(s-\ell/2)\big)}{\sin(k\ell/2)}, \qquad s\in[0,\ell]. \tag{4}$$

Indeed, at $s = 0$ we obtain $g(0) = \frac{S-2c}{2} + \frac{D}{2} = \psi_1(u) - c$, and at $s = \ell$ we obtain $g(\ell) = \frac{S-2c}{2} - \frac{D}{2} = \psi_1(v) - c$, so $f_\ell \in H^1(\Gamma_\ell)$ (including when $u = v$, where $D = 0$ and $g(0) = g(\ell) = \psi_1(u) - c$). Because $\int_\Gamma \psi_1 = 0$ and the sin term in (4) is odd about $s = \ell/2$, the total integral of f_ℓ is

$$\int_{\Gamma_\ell} f_\ell\,dx \;=\; -cL + \int_0^\ell g(s)\,ds \;=\; -cL + \frac{S-2c}{k}\tan\!\left(\frac{k\ell}{2}\right).$$

Choosing

$$c \;:=\; \frac{S\tan(k\ell/2)}{kL + 2\tan(k\ell/2)}$$

(whose denominator is strictly positive since $\tan(k\ell/2) > 0$) enforces $\int_{\Gamma_\ell} f_\ell = 0$ and yields $S - 2c = \frac{kLS}{kL+2\tan(k\ell/2)}$. Moreover, $\|f_\ell\|^2_{L^2(\Gamma_\ell)} \geq \|\psi_1 - c\|^2_{L^2(\Gamma)} = 1 + c^2L \geq 1 > 0$, so $f_\ell \not\equiv 0$.

On Γ, since $\int_\Gamma |\psi_1'|^2 = k^2\int_\Gamma \psi_1^2 = k^2$ and $\int_\Gamma \psi_1 = 0$, we have $\int_\Gamma(|f_\ell'|^2 - k^2 f_\ell^2) = -k^2c^2L$. On e_ℓ, since $-g'' = k^2 g$, integration by parts gives

$$\int_0^\ell \big(|g'|^2 - k^2g^2\big)\,ds \;=\; g(\ell)g'(\ell) - g(0)g'(0).$$

Differentiating (4) gives $g'(\ell) = -\frac{k(S-2c)}{2}\tan(\frac{k\ell}{2}) - \frac{kD}{2}\cot(\frac{k\ell}{2})$ and $g'(0) = +\frac{k(S-2c)}{2}\tan(\frac{k\ell}{2}) - \frac{kD}{2}\cot(\frac{k\ell}{2})$, whence

$$g(\ell)g'(\ell) - g(0)g'(0) \;=\; \frac{k}{2}\left[D^2\cot\!\left(\frac{k\ell}{2}\right) - (S-2c)^2\tan\!\left(\frac{k\ell}{2}\right)\right].$$

Summing the contributions from Γ and e_ℓ with $t := \tan(k\ell/2) > 0$:

$$\begin{aligned}\mathcal{Q}_k(\ell) &= \frac{k}{2}D^2\cot\left(\frac{k\ell}{2}\right) - \frac{k}{2}(S-2c)^2 t - k^2 L c^2 \\ &= \frac{k}{2}D^2\cot\left(\frac{k\ell}{2}\right) - \frac{kS^2 t}{2(kL+2t)^2}\Big[(kL)^2 + 2kLt\Big] \;=\; \frac{k}{2}\left[D^2\cot\left(\frac{k\ell}{2}\right) - S^2\,\frac{kLt}{kL+2t}\right],\end{aligned}$$

proving (1). By the Rayleigh–Ritz principle on $\mathbf{1}^\perp \cap H^1(\Gamma_\ell)$, $\lambda_1(\Gamma_\ell) - k^2 \le \mathcal{Q}_k(\ell)/\|f_\ell\|^2_{L^2(\Gamma_\ell)}$, so $\mathcal{Q}_k(\ell) < 0$ implies $\lambda_1(\Gamma_\ell) < k^2 = \lambda_1(\Gamma)$. Writing $\mathcal{Q}_k(\ell) = -\frac{k}{t(kL+2t)}\big(\frac{kLS^2}{2}t^2 - D^2 t - \frac{kLD^2}{2}\big)$, we see that $\mathcal{Q}_k(\ell) < 0$ for $t > 0$ iff $\frac{kLS^2}{2}t^2 - D^2 t - \frac{kLD^2}{2} > 0$: when $D = 0, S \neq 0$ this holds for all $t > 0$; when $D \neq 0, S \neq 0$ it holds for all $t > t_{\mathrm{var}} := \frac{|D|(|D|+\sqrt{D^2+(kL)^2S^2})}{kLS^2}$, i.e., $\ell \in (\ell_{\mathrm{var}}(u,v), \pi/k)$.

Proof of (b). For any $\ell > 0$, set $c_{\mathrm{hi}} := \frac{S\ell}{2(L+\ell)}$ and define $f^{\mathrm{hi}}_\ell \in H^1(\Gamma_\ell)$ by $f^{\mathrm{hi}}_\ell(x) := \psi_1(x) - c_{\mathrm{hi}}$ on Γ and

$$h(s) \;:=\; \frac{D}{2}\cos\Big(\frac{\pi s}{\ell}\Big) + \frac{S}{2} - c_{\mathrm{hi}}, \qquad s \in [0,\ell],$$

on e_ℓ. Then $h(0) = \psi_1(u) - c_{\mathrm{hi}}$, $h(\ell) = \psi_1(v) - c_{\mathrm{hi}}$, $\int_{\Gamma_\ell} f^{\mathrm{hi}}_\ell = -c_{\mathrm{hi}}L + (\frac{S}{2} - c_{\mathrm{hi}})\ell = 0$, and $\|f^{\mathrm{hi}}_\ell\|^2_{L^2(\Gamma_\ell)} \ge 1 + c^2_{\mathrm{hi}}L > 0$. Since $\int_0^\ell \cos(\pi s/\ell)\,ds = 0$ and $\int_0^\ell \cos^2(\pi s/\ell)\,ds = \int_0^\ell \sin^2(\pi s/\ell)\,ds = \ell/2$, we compute directly:

$$\begin{aligned}\mathcal{Q}^{\mathrm{hi}}_k(\ell) &= -k^2 c^2_{\mathrm{hi}} L + \frac{D^2}{4}\left(\frac{\pi^2}{\ell^2} - k^2\right)\frac{\ell}{2} - k^2\ell\left(\frac{S}{2} - c_{\mathrm{hi}}\right)^2 \\ &= \frac{D^2}{8\ell}(\pi^2 - k^2\ell^2) - \frac{k^2 L\ell}{4(L+\ell)}\,S^2,\end{aligned}$$

proving (3). If $(\psi_1(u), \psi_1(v)) \neq (0,0)$, then $D^2 + S^2 = 2(\psi_1(u)^2 + \psi_1(v)^2) > 0$: for $\ell > \pi/k$, both terms in (3) are non-positive and at least one is strictly negative, so $\mathcal{Q}^{\mathrm{hi}}_k(\ell) < 0$ and $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$. For $S \neq 0$, $\mathcal{Q}^{\mathrm{hi}}_k(\pi/k) < 0$ as well. Finally, if $D = 0$ and $S \neq 0$, Part (a) gives $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ on $(0, \pi/k)$ while (3) gives $\mathcal{Q}^{\mathrm{hi}}_k(\ell) < 0$ on $[\pi/k, \infty)$, covering all $\ell > 0$. □

3. Exact Krein–Dirichlet-to-Neumann Law for Simple λ_1

To upgrade the variational upper bound $\ell_{\mathrm{var}}(u,v)$ into an *exact* formula for the critical length $\ell_*(u,v)$, we first collect three structural properties of the eigenvalue curve $\ell \mapsto \lambda_1(\Gamma_\ell)$ in a self-contained lemma.

Lemma 3.1 (Edge-lengthening monotonicity, Cauchy interlacing, and Hadamard derivative)**.** *Let Γ be a compact connected metric graph with $\lambda_1(\Gamma) = k^2 > 0$, and let $\Gamma_\ell = \Gamma \cup_{u,v} e_\ell$ for $\ell > 0$.*

(a) Monotonicity in ℓ. *For every $j \ge 1$, the map $\ell \mapsto \lambda_j(\Gamma_\ell)$ is continuous and monotonically non-increasing on $(0,\infty)$.*

(b) Cauchy interlacing on $(0, \pi/k)$. *If $\lambda_1(\Gamma) = k^2$ is simple, then $\lambda_2(\Gamma_\ell) \ge k^2$ for every $\ell \in (0, \pi/k)$.*

(c) Hadamard derivative. *If $\lambda(\ell_0)$ is a simple positive eigenvalue of $-\Delta_{\Gamma_{\ell_0}}$ with normalized eigenfunction $F = (f,g)$ ($f = F|_\Gamma$, $g = F|_{e_{\ell_0}}$), then $\lambda(\ell)$ is real-analytic in a neighborhood of ℓ_0 with derivative*

$$\lambda'(\ell_0) \;=\; -\frac{1}{\ell_0}\int_0^{\ell_0}\big(|g'(s)|^2 + \lambda(\ell_0)|g(s)|^2\big)\,ds \;=\; -\big(|g'(s)|^2 + \lambda(\ell_0)|g(s)|^2\big)\Big|_{e_{\ell_0}} \;\le\; 0, \tag{5}$$

where the energy density $|g'(s)|^2 + \lambda(\ell_0)|g(s)|^2$ is constant along e_{ℓ_0}, and $\lambda'(\ell_0) < 0$ strictly whenever $g \not\equiv 0$.

Proof. Proof of (a). Fix $0 < \ell < \ell'$. Rescaling $e_\ell \cong [0,\ell]$ linearly onto $e_{\ell'} \cong [0,\ell']$ via $s \mapsto (\ell'/\ell)s$ defines a linear pullback isomorphism $T : H^1(\Gamma_\ell) \to H^1(\Gamma_{\ell'})$ mapping $F = (f,g)$ to $TF = (f,\tilde{g})$ with $\tilde{g}(y) := g((\ell/\ell')y)$. For every non-zero $F \in H^1(\Gamma_\ell)$,

$$\frac{\ell}{\ell'}\int_{\Gamma_\ell}|F'|^2 \le \int_{\Gamma_{\ell'}}|(TF)'|^2 = \int_\Gamma |f'|^2 + \frac{\ell}{\ell'}\int_0^\ell |g'|^2 \le \int_{\Gamma_\ell}|F'|^2,$$

and

$$\int_{\Gamma_\ell}|F|^2 \le \int_{\Gamma_{\ell'}}|TF|^2 = \int_\Gamma |f|^2 + \frac{\ell'}{\ell}\int_0^\ell |g|^2 \le \frac{\ell'}{\ell}\int_{\Gamma_\ell}|F|^2.$$

Hence the Rayleigh quotient satisfies

$$\left(\frac{\ell}{\ell'}\right)^2 \frac{\mathcal{E}_{\Gamma_\ell}(F)}{\|F\|^2_{L^2(\Gamma_\ell)}} \;\le\; \frac{\mathcal{E}_{\Gamma_{\ell'}}(TF)}{\|TF\|^2_{L^2(\Gamma_{\ell'})}} \;\le\; \frac{\mathcal{E}_{\Gamma_\ell}(F)}{\|F\|^2_{L^2(\Gamma_\ell)}}$$

for every non-zero $F \in H^1(\Gamma_\ell)$. Applying the Courant–Fischer min-max characterization over $(j+1)$-dimensional subspaces of $H^1(\Gamma_\ell)$ immediately yields

$$\left(\frac{\ell}{\ell'}\right)^2 \lambda_j(\Gamma_\ell) \;\le\; \lambda_j(\Gamma_{\ell'}) \;\le\; \lambda_j(\Gamma_\ell) \qquad (0 < \ell < \ell'),$$

which simultaneously proves that $\ell \mapsto \lambda_j(\Gamma_\ell)$ is monotonically non-increasing and continuous on $(0,\infty)$ for every $j \ge 1$.

Proof of (b). Detaching e_ℓ at u and v yields the disconnected graph $\Gamma \sqcup e_\ell$ with form domain $H^1(\Gamma)\oplus H^1(e_\ell)$, whose Neumann spectrum is the multiset union $\sigma(-\Delta_\Gamma)\cup\{0,(\pi/\ell)^2,(2\pi/\ell)^2,\dots\}$. Because $\ell < \pi/k$ and $\lambda_1(\Gamma) = k^2$ is simple, the three lowest eigenvalues of $-\Delta_{\Gamma\sqcup e_\ell}$ are $\lambda_0 = 0$, $\lambda_1 = 0$, and $\lambda_2 = k^2$. Re-attaching e_ℓ restricts the form domain to the closed subspace $\{F \in H^1(\Gamma)\oplus H^1(e_\ell) : g(0) = f(u),\, g(\ell) = f(v)\}$, which increases every eigenvalue by Courant–Fischer: $\lambda_j(\Gamma_\ell) \ge \lambda_j(\Gamma \sqcup e_\ell)$ for all $j \ge 0$. Taking $j = 2$ gives $\lambda_2(\Gamma_\ell) \ge k^2$.

Proof of (c). Pull back e_ℓ to $[0,1]$ via $y = s/\ell$ and work on the *fixed* Hilbert space $\mathcal{H} := L^2(\Gamma)\oplus L^2(0,1)$ equipped with its standard ℓ-independent inner product $\langle (f_1,G_1),(f_2,G_2)\rangle_{\mathcal{H}} := \langle f_1,f_2\rangle_{L^2(\Gamma)} + \langle G_1,G_2\rangle_{L^2(0,1)}$. Define the bounded positive self-adjoint multiplication operator $J_\ell(f,G) := (f,\ell G)$ on $\mathcal{H}$, and let $B_\ell \ge 0$ be the self-adjoint operator on $\mathcal{H}$ associated with the closed symmetric quadratic form $q_\ell(f,G) := \int_\Gamma |f'|^2 + \ell^{-1}\int_0^1 |G'|^2$ on the ℓ-independent domain $\mathcal{D}_0 := \{(f,G) \in H^1(\Gamma)\oplus H^1(0,1) : G(0) = f(u),\, G(1) = f(v)\}$. Under the unitary isomorphism $U_\ell : L^2(\Gamma_\ell) \to \mathcal{H}$ given by $U_\ell(f,g) := (f, \sqrt{\ell}\, g(\ell\,\cdot))$, the resolvent $(-\Delta_{\Gamma_\ell} + I)^{-1}$ is unitarily equivalent to the compact self-adjoint operator

$$R(\ell) \;:=\; J_\ell^{1/2}(B_\ell + J_\ell)^{-1}J_\ell^{1/2}$$

on $\mathcal{H}$. Because $\mathcal{D}_0$ is independent of ℓ and $\ell \mapsto q_\ell(h) + \langle h, J_\ell h\rangle_{\mathcal{H}}$ is analytic and uniformly sectorial for ℓ in a complex neighborhood of $\ell_0 > 0$, $B_\ell + J_\ell$ is a holomorphic family of type (B) on $\mathcal{H}$ with bounded inverse [5, Chapter VII]; since $\ell \mapsto J_\ell^{1/2}$ is also norm-analytic near ℓ_0, $R(\ell)$ is a norm-analytic family of compact self-adjoint operators on $\mathcal{H}$. By Kato's perturbation theory for simple eigenvalues [5], the simple eigenvalue $(\lambda(\ell)+1)^{-1}$ of $R(\ell)$ (hence $\lambda(\ell)$) and a normalized eigenvector $\widetilde{F}(\ell)$ of $R(\ell)$ depend real-analytically on ℓ near ℓ_0, and hence $F(\ell) := J_\ell^{-1/2}\widetilde{F}(\ell) = (f(\ell),G(\ell)) \in \mathcal{D}_0$ satisfies $\langle F(\ell), J_\ell F(\ell)\rangle_{\mathcal{H}} = 1$ and varies analytically in $\mathcal{D}_0$ (since $F(\ell) = (B_\ell + J_\ell)^{-1}J_\ell^{1/2}(\lambda(\ell)+1)\widetilde{F}(\ell)$). Differentiating $q_\ell(F(\ell)) = \lambda(\ell)\langle F(\ell), J_\ell F(\ell)\rangle_{\mathcal{H}}$ at ℓ_0 and using the stationarity identity $q_{\ell_0}(F'(\ell_0),F(\ell_0)) = \lambda(\ell_0)\langle F'(\ell_0), J_{\ell_0}F(\ell_0)\rangle_{\mathcal{H}}$ yields

$$\lambda'(\ell_0) \;=\; -\frac{1}{\ell_0^2}\int_0^1 |G'(y)|^2\,dy - \lambda(\ell_0)\int_0^1 |G(y)|^2\,dy \;=\; -\frac{1}{\ell_0}\int_0^{\ell_0}\big(|g'(s)|^2 + \lambda(\ell_0)|g(s)|^2\big)\,ds.$$

Since $-g'' = \lambda(\ell_0)g$ on $[0,\ell_0]$, $\frac{d}{ds}(|g'|^2 + \lambda(\ell_0)|g|^2) = 2g'(g'' + \lambda(\ell_0)g) = 0$, so the integrand is constant along e_{ℓ_0} and strictly positive whenever $g \not\equiv 0$. □

We now introduce the reduced Green's matrix of Γ at the simple eigenvalue $\lambda_1(\Gamma) = k^2$. For any $x, y \in \Gamma$, the *reduced Green's function* at $\lambda_1 = k^2$ is the absolutely and uniformly convergent spectral series

$$G'_\Gamma(x,y;k^2) \;:=\; \sum_{j\neq 1} \frac{\psi_j(x)\psi_j(y)}{\lambda_j - k^2} \;=\; -\frac{1}{k^2 L} + \sum_{j\geq 2} \frac{\psi_j(x)\psi_j(y)}{\lambda_j - k^2}, \tag{6}$$

which is continuous on $\Gamma\times\Gamma$ (by Weyl's law $\lambda_j \asymp j^2$ and uniform boundedness of $\|\psi_j\|_\infty$ on compact metric graphs [3]) and equals the integral kernel of the reduced resolvent $(-\Delta_\Gamma - k^2)^{-1}P_{\psi_1^\perp}$ on $L^2(\Gamma)$. For any two points $u, v \in \Gamma$, define the symmetric 2×2 *reduced Green's matrix*, the eigenfunction evaluation vector $\mathbf{p}$, its orthogonal complement $\mathbf{p}^\perp$, and the *Krein spectral-relaxation invariant* Q_M:

$$\begin{aligned} M_\Gamma(k^2) &:= \begin{pmatrix} G'_\Gamma(u,u;k^2) & G'_\Gamma(u,v;k^2) \\ G'_\Gamma(v,u;k^2) & G'_\Gamma(v,v;k^2) \end{pmatrix}, \qquad \mathbf{p} := \begin{pmatrix} \psi_1(u) \\ \psi_1(v) \end{pmatrix}, \\ \mathbf{p}^\perp &:= \begin{pmatrix} \psi_1(v) \\ -\psi_1(u) \end{pmatrix}, \qquad Q_M := (\mathbf{p}^\perp)^\top M_\Gamma(k^2)\mathbf{p}^\perp. \end{aligned} \tag{7}$$

From (6) and $\psi_0 \equiv L^{-1/2}$, Q_M admits the exact spectral decomposition

$$Q_M \;=\; -\frac{D^2}{k^2 L} + \sum_{j\geq 2} \frac{\big(\psi_j(u)\psi_1(v) - \psi_j(v)\psi_1(u)\big)^2}{\lambda_j - k^2} \;\geq\; -\frac{D^2}{k^2 L}, \tag{8}$$

since $\lambda_j > k^2$ for all $j \geq 2$. Let $\Lambda_k(\ell)$ denote the symmetric 2×2 Dirichlet-to-Neumann (DtN) matrix of the single edge e_ℓ at wavenumber $k \in (0, \pi/\ell)$:

$$\Lambda_k(\ell) \;:=\; k \begin{pmatrix} \cot(k\ell) & -\csc(k\ell) \\ -\csc(k\ell) & \cot(k\ell) \end{pmatrix}. \tag{9}$$

Theorem 3.2 (Exact Krein–Dirichlet-to-Neumann law for critical Braess transitions)**.** *Let Γ be a compact connected metric graph of total length $L > 0$ whose first non-zero eigenvalue $\lambda_1(\Gamma) = k^2 > 0$ is simple with normalized eigenfunction ψ_1. Let $u, v \in \Gamma$ with $u \neq v$, and set $D := \psi_1(u) - \psi_1(v)$, $S := \psi_1(u) + \psi_1(v)$, and $Q_M := (\mathbf{p}^\perp)^\top M_\Gamma(k^2)\mathbf{p}^\perp$.*

(a) Non-nodal exact critical length law. *Suppose $\mathbf{p} = (\psi_1(u), \psi_1(v))^\top \neq (0,0)$. Then there exists a unique critical length $\ell_*(u,v) \in [0, \pi/k]$ separating strict spectral-gap improvement from strict Braess degradation:*

$$\lambda_1(\Gamma_\ell) > \lambda_1(\Gamma) \;\; \forall\, \ell \in (0,\ell_*), \qquad \lambda_1(\Gamma_{\ell_*}) = \lambda_1(\Gamma) \text{ (if } \ell_* > 0), \qquad \lambda_1(\Gamma_\ell) < \lambda_1(\Gamma) \;\; \forall\, \ell > \ell_*.$$

Specifically, setting $t_* := \tan(k\ell_*/2) \in [0,\infty]$:

(i) Generic regime ($D \neq 0$, $S \neq 0$): *$t_* \in (0,\infty)$ is the unique positive root of the universal Krein quadratic*

$$\frac{S^2}{2}\, t^2 + kQ_M\, t - \frac{D^2}{2} \;=\; 0, \tag{10}$$

giving the exact closed form

$$\ell_*(u,v) \;=\; \frac{2}{k} \arctan\left(\frac{-kQ_M + \sqrt{(kQ_M)^2 + S^2D^2}}{S^2} \right) \;\in\; \left(0, \frac{\pi}{k}\right). \tag{11}$$

(ii) Anti-equipotential sign transition ($S = 0$, $D \neq 0$, i.e., $\psi_1(u) = -\psi_1(v) \neq 0$):

$$\ell_*(u,v) \;=\; \begin{cases} \dfrac{\pi}{k}, & \text{if } Q_M \le 0, \\ \dfrac{2}{k}\arctan\!\left(\dfrac{D^2}{2kQ_M}\right) \;<\; \dfrac{\pi}{k}, & \text{if } Q_M > 0. \end{cases} \tag{12}$$

(iii) Equipotential regime ($D = 0$, $S \neq 0$, i.e., $\psi_1(u) = \psi_1(v) \neq 0$): $\ell_*(u,v) = 0$ *(and the same holds for a non-nodal self-loop $u = v$ by Theorem 2.1(b)).*

(b) Higher-mode relaxation inequality. *In every non-nodal case, $\ell_*(u,v) \le \ell_{\mathrm{var}}(u,v)$, with* strict *inequality $\ell_*(u,v) < \ell_{\mathrm{var}}(u,v)$ whenever $D \neq 0, S \neq 0$ (or $S = 0, D \neq 0, Q_M > 0$), and equality if and only if either $D = 0$ or ($S = 0$ and $Q_M \le 0$). In particular, the variational bound (2) is the exact zero-mode ($j = 0$) truncation of (10).*

(c) Nodal plateau ($\psi_1(u) = \psi_1(v) = 0$). *If both attachment points lie on the nodal set of ψ_1, then $\lambda_1(\Gamma_\ell) \le \lambda_1(\Gamma)$ for all $\ell > 0$, with $\lambda_1(\Gamma_\ell) = \lambda_1(\Gamma)$ on a non-empty nodal plateau $(0, \ell_{\mathrm{nod}}]$ ($\ell_{\mathrm{nod}} > 0$) and $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ for all $\ell > 2\pi/k$; on $(0, \pi/k)$, $\lambda_1(\Gamma_\ell)$ drops strictly below k^2 at the smallest root (if any) of* $\det(I_2 + M_\Gamma(k^2)\Lambda_k(\ell)) = 0$.

Proof. Step 1: Exact Krein–Birman–Schwinger reduction at $\lambda = k^2$ for $\ell \in (0, \pi/k)$. Fix $\ell \in (0, \pi/k)$ and suppose $F = (f, g) \in \mathrm{dom}(-\Delta_{\Gamma_\ell}) \setminus \{0\}$ satisfies $-\Delta_{\Gamma_\ell} F = k^2 F$. On $e_\ell \cong [0, \ell]$, $-g'' = k^2 g$ with $g(0) = f(u)$ and $g(\ell) = f(v)$. Because $k\ell \in (0, \pi)$, $\sin(k\ell) \neq 0$, so g is uniquely determined by $\mathbf{f}_{uv} := (f(u), f(v))^\top$ via

$$g(s) \;=\; f(u)\frac{\sin(k(\ell - s))}{\sin(k\ell)} + f(v)\frac{\sin(ks)}{\sin(k\ell)}.$$

The derivatives of g directed *into* e_ℓ from 0 and ℓ are $g'(0) = -k\cot(k\ell)f(u) + k\csc(k\ell)f(v)$ and $-g'(\ell) = k\csc(k\ell)f(u) - k\cot(k\ell)f(v)$, so the outward normal derivative vector $\mathbf{j} := (-g'(0), g'(\ell))^\top$ on e_ℓ (which by Kirchhoff conservation equals the sum of inward derivatives entering Γ at u and v) satisfies $\mathbf{j} = \Lambda_k(\ell)\mathbf{f}_{uv}$. By integration by parts on Γ, $f \in H^1(\Gamma)$ satisfies the weak equation

$$\int_\Gamma f'\varphi'\,dx - k^2\int_\Gamma f\varphi\,dx \;=\; -j_u\,\varphi(u) - j_v\,\varphi(v) \qquad \forall\,\varphi \in H^1(\Gamma). \tag{13}$$

Decompose $f = \alpha\psi_1 + w$ with $\alpha = \langle f, \psi_1\rangle_{L^2(\Gamma)}$ and $w \perp_{L^2(\Gamma)} \psi_1$. Testing (13) with $\varphi = \psi_1$ gives $0 = -\mathbf{p}^\top\mathbf{j}$. Testing (13) with $\varphi = \psi_j$ ($j \neq 1$) gives $(\lambda_j - k^2)\langle w, \psi_j\rangle_{L^2(\Gamma)} = -(\psi_j(u)j_u + \psi_j(v)j_v)$, whence by uniform convergence of (6),

$$w(x) \;=\; -G'_\Gamma(x, u; k^2)j_u - G'_\Gamma(x, v; k^2)j_v \qquad \forall\, x \in \Gamma.$$

Evaluating $f = \alpha\psi_1 + w$ at u and v yields the exact finite-dimensional system

$$\mathbf{p}^\top\mathbf{j} \;=\; 0, \qquad \mathbf{f}_{uv} \;=\; \alpha\mathbf{p} - M_\Gamma(k^2)\mathbf{j}, \qquad \mathbf{j} \;=\; \Lambda_k(\ell)\mathbf{f}_{uv}. \tag{14}$$

Conversely, any triple $(\alpha, \mathbf{f}_{uv}, \mathbf{j})$ solving (14) reconstructs $f := \alpha\psi_1 - G'_\Gamma(\cdot, u; k^2)j_u - G'_\Gamma(\cdot, v; k^2)j_v \in H^1(\Gamma)$ and $g \in H^2(0, \ell)$ satisfying (13) and continuity at u, v, hence an eigenfunction $F = (f, g) \in \mathrm{dom}(-\Delta_{\Gamma_\ell})$ at eigenvalue k^2.

Step 2: Exact equivalence with (10) when $\mathbf{p} \neq 0$. Observe that $\Lambda_k(\ell)$ in (9) is orthogonally diagonalized by $\mathbf{e}_+ := \frac{1}{\sqrt{2}}(1, 1)^\top$ and $\mathbf{e}_- := \frac{1}{\sqrt{2}}(1, -1)^\top$: using $\cot(2\theta) - \csc(2\theta) = -\tan\theta$ and $\cot(2\theta) + \csc(2\theta) = \cot\theta$ with $t := \tan(k\ell/2) \in (0, \infty)$,

$$\Lambda_k(\ell)\mathbf{e}_+ = (-kt)\,\mathbf{e}_+, \qquad \Lambda_k(\ell)\mathbf{e}_- = \left(\frac{k}{t}\right)\mathbf{e}_-. \tag{15}$$

In particular, both eigenvalues of $\Lambda_k(\ell)$ are non-zero for every $\ell \in (0, \pi/k)$, so $\Lambda_k(\ell)$ is invertible.

Assume now $\mathbf{p} \neq 0$. The condition $\mathbf{p}^\top\mathbf{j} = 0$ in $\mathbb{R}^2$ holds iff $\mathbf{j} = c\,\mathbf{p}^\perp$ for some $c \in \mathbb{R}$. If $c = 0$, then $\mathbf{j} = 0 \implies \mathbf{f}_{uv} = \Lambda_k(\ell)^{-1}\mathbf{j} = 0 \implies \alpha\mathbf{p} = 0 \implies \alpha = 0 \implies F \equiv 0$. Therefore every

non-trivial eigenfunction at k^2 has $c \neq 0$ (and hence $\mathbf{j} \neq 0$, so $g \not\equiv 0$). Substituting $\mathbf{f}_{uv} = \Lambda_k(\ell)^{-1}\mathbf{j}$ into the second equation of (14) gives

$$\big(\Lambda_k(\ell)^{-1} + M_\Gamma(k^2)\big)(c\,\mathbf{p}^\perp) \;=\; \alpha\mathbf{p}. \tag{16}$$

Taking the inner product of (16) with $\mathbf{p}^\perp$ eliminates α (since $(\mathbf{p}^\perp)^\top\mathbf{p} = 0$) and yields the necessary condition $(\mathbf{p}^\perp)^\top(\Lambda_k(\ell)^{-1} + M_\Gamma(k^2))\mathbf{p}^\perp = 0$. Conversely, whenever this scalar condition holds, taking $c = 1$ and $\alpha := \|\mathbf{p}\|^{-2}\mathbf{p}^\top(\Lambda_k(\ell)^{-1} + M_\Gamma(k^2))\mathbf{p}^\perp$ uniquely solves (16) (since $\{\mathbf{p}, \mathbf{p}^\perp\}$ is an orthogonal basis of $\mathbb{R}^2$). Thus, for $\ell \in (0, \pi/k)$ and $\mathbf{p} \neq 0$:

$$k^2 \in \sigma(-\Delta_{\Gamma_\ell}) \iff (\mathbf{p}^\perp)^\top\big(\Lambda_k(\ell)^{-1} + M_\Gamma(k^2)\big)\mathbf{p}^\perp = 0,$$

and whenever this holds, k^2 is a *simple* eigenvalue of $-\Delta_{\Gamma_\ell}$ whose eigenfunction satisfies $g \not\equiv 0$.

Now evaluate $(\mathbf{p}^\perp)^\top\Lambda_k(\ell)^{-1}\mathbf{p}^\perp$ via (15). Since $\mathbf{e}_+^\top\mathbf{p}^\perp = \frac{\psi_1(v)-\psi_1(u)}{\sqrt{2}} = -\frac{D}{\sqrt{2}}$ and $\mathbf{e}_-^\top\mathbf{p}^\perp = \frac{\psi_1(v)+\psi_1(u)}{\sqrt{2}} = \frac{S}{\sqrt{2}}$, we obtain

$$(\mathbf{p}^\perp)^\top\Lambda_k(\ell)^{-1}\mathbf{p}^\perp \;=\; \frac{D^2}{2}\left(-\frac{1}{kt}\right) + \frac{S^2}{2}\left(\frac{t}{k}\right) \;=\; \frac{1}{kt}\left(\frac{S^2}{2}t^2 - \frac{D^2}{2}\right).$$

Adding $Q_M = (\mathbf{p}^\perp)^\top M_\Gamma(k^2)\mathbf{p}^\perp$ and multiplying by $kt > 0$ proves that $k^2 \in \sigma(-\Delta_{\Gamma_\ell})$ for $\ell \in (0, \pi/k)$ if and only if $t = \tan(k\ell/2) \in (0, \infty)$ satisfies (10):

$$\phi(t) \;:=\; \frac{S^2}{2}\,t^2 + kQ_M\,t - \frac{D^2}{2} \;=\; 0.$$

Step 3: Identifying $\ell_*(u, v)$ across all three non-nodal regimes. Case (iii) ($D = 0, S \neq 0$) is already established in Theorem 2.1(b) ($\lambda_1(\Gamma_\ell) < k^2$ for all $\ell > 0$, so $\ell_* = 0$). Assume henceforth $D \neq 0$.

First, we show $\lambda_1(\Gamma_\ell) > k^2$ for all sufficiently small $\ell > 0$. Let $\Gamma/\{u \sim v\}$ be the metric graph obtained by identifying u and v, whose form domain is $H^1_0 := \{f \in H^1(\Gamma) : f(u) = f(v)\}$. Because $\lambda_1(\Gamma) = k^2$ is simple and $\psi_1(u) - \psi_1(v) = D \neq 0$, we have $\psi_1 \notin H^1_0$, so by compactness of the embedding $H^1(\Gamma) \hookrightarrow L^2(\Gamma)$, $\lambda_1(\Gamma/\{u \sim v\}) = \min_{f \in H^1_0 \cap \mathbf{1}^\perp,\, \|f\|_2 = 1} \mathcal{E}_\Gamma(f) > k^2$. Suppose for contradiction there existed a sequence $\ell_m \to 0^+$ with $\lambda_1(\Gamma_{\ell_m}) \leq k^2$, and let $F_m = (f_m, g_m)$ be normalized eigenfunctions ($\|F_m\|_{L^2(\Gamma_{\ell_m})} = 1$, $\int_{\Gamma_{\ell_m}} F_m = 0$). Then $\int_{\Gamma_{\ell_m}} |F_m'|^2 = \lambda_1(\Gamma_{\ell_m}) \leq k^2$, so by Cauchy–Schwarz

$$|f_m(u) - f_m(v)| \;=\; |g_m(0) - g_m(\ell_m)| \;\leq\; \sqrt{\ell_m}\left(\int_0^{\ell_m} |g_m'|^2\right)^{1/2} \;\leq\; k\sqrt{\ell_m} \;\to\; 0,$$

and $\|g_m\|^2_{L^2(e_{\ell_m})} \leq \ell_m\|g_m\|^2_\infty = O(\ell_m) \to 0$ (since $\|f_m\|_{H^1(\Gamma)}$ is uniformly bounded, $H^1(\Gamma) \hookrightarrow C(\Gamma)$ bounds $|f_m(u)|$, and $|g_m(s)| \leq |f_m(u)| + \sqrt{\ell_m}\|g_m'\|_{L^2(0,\ell_m)} = O(1)$). Passing to a weakly convergent subsequence $f_m \rightharpoonup f_\infty$ in $H^1(\Gamma)$ (strongly in $C(\Gamma)$ and $L^2(\Gamma)$) yields $f_\infty \in H^1_0$, $\|f_\infty\|_{L^2(\Gamma)} = 1$, $\int_\Gamma f_\infty = 0$, and $\mathcal{E}_\Gamma(f_\infty) \leq \liminf \mathcal{E}_\Gamma(f_m) \leq k^2 < \lambda_1(\Gamma/\{u \sim v\})$, a contradiction. Hence $\lambda_1(\Gamma_\ell) > k^2$ for all sufficiently small $\ell > 0$.

Now apply Lemma 3.1(a)–(c):

- *Subcase (i) ($D \neq 0$, $S \neq 0$):* Since $\phi(0) = -D^2/2 < 0$ and $\phi(t) \to +\infty$ as $t \to \infty$, ϕ has a unique positive root $t_* = \frac{-kQ_M + \sqrt{(kQ_M)^2 + S^2D^2}}{S^2} \in (0, \infty)$, giving a unique $\ell_* = \frac{2}{k}\arctan(t_*) \in (0, \pi/k)$ where $k^2 \in \sigma(-\Delta_{\Gamma_\ell})$. Because $\lambda_1(\Gamma_\ell) > k^2$ for small $\ell > 0$, $\lambda_1(\Gamma_\ell)$ is continuous and non-increasing (Lemma 3.1(a)), and $\lambda_1(\Gamma_\ell) < k^2$ for $\ell \geq \pi/k$ (Theorem 2.1(b)), $\lambda_1(\Gamma_\ell)$ must cross k^2 at ℓ_*. At ℓ_*, Step 2 showed k^2 is simple with $g \not\equiv 0$, so Lemma 3.1(c) gives $\lambda_1'(\ell_*) < 0$ strictly. Combining with monotonicity gives $\lambda_1(\Gamma_\ell) > k^2$ for $\ell \in (0, \ell_*)$ and $\lambda_1(\Gamma_\ell) < k^2$ for all $\ell > \ell_*$.

- *Subcase (ii) ($S = 0$, $D \neq 0$):* Here $\phi(t) = kQ_M t - D^2/2$.
  - If $Q_M \leq 0$, then $\phi(t) \leq -D^2/2 < 0$ for all $t > 0$, so $k^2 \notin \sigma(-\Delta_{\Gamma_\ell})$ for any $\ell \in (0, \pi/k)$. Since $\lambda_1(\Gamma_\ell) > k^2$ near 0^+, continuity forces $\lambda_1(\Gamma_\ell) > k^2$ for all $\ell \in (0, \pi/k)$. At $\ell = \pi/k$, defining $f = \psi_1$ on Γ and $g(s) = \frac{D}{2}\cos(ks)$ on $e_{\pi/k}$ gives $g(0) = \psi_1(u)$, $g(\pi/k) = \psi_1(v)$ (since $S = 0$), $g'(0) = g'(\pi/k) = 0$, and $-g'' = k^2 g$, so $F = (f, g)$ is an exact eigenfunction of $-\Delta_{\Gamma_{\pi/k}}$ at k^2; by continuity $\lambda_1(\Gamma_{\pi/k}) = k^2$. For all $\ell > \pi/k$, Theorem 2.1(b) gives $\lambda_1(\Gamma_\ell) < k^2$. Hence $\ell_* = \pi/k$.
  - If $Q_M > 0$, then $\phi(t) = 0$ has the unique positive root $t_* = D^2/(2kQ_M)$, giving a unique $\ell_* = \frac{2}{k}\arctan(D^2/(2kQ_M)) \in (0, \pi/k)$ where $k^2 \in \sigma(-\Delta_{\Gamma_\ell})$. Could k^2 be $\lambda_2(\Gamma_{\ell_*})$ rather than $\lambda_1(\Gamma_{\ell_*})$? No: if $\lambda_1(\Gamma_{\ell_*}) < k^2$, then by continuity $\lambda_1(\Gamma_\ell)$ would have crossed k^2 at some $\ell_1 \in (0, \ell_*)$, contradicting uniqueness of the root ℓ_* of ϕ on $(0, \pi/k)$. Hence $\lambda_1(\Gamma_{\ell_*}) = k^2$ (simple, with $g \not\equiv 0$ by Step 2), so Lemma 3.1(c) gives $\lambda_1'(\ell_*) < 0$ strictly, and monotonicity yields $\lambda_1(\Gamma_\ell) > k^2$ on $(0, \ell_*)$ and $\lambda_1(\Gamma_\ell) < k^2$ for all $\ell > \ell_*$.

Proof of (b). By (8), $kQ_M \geq -\frac{D^2}{kL}$, with equality iff $\psi_j(u)\psi_1(v) = \psi_j(v)\psi_1(u)$ for every $j \geq 2$. Because $u \neq v$ and $\mathbf{p} = (\psi_1(u), \psi_1(v))^\top \neq 0$, this equality can never hold: if it did, then every ψ_j ($j \geq 1$) would satisfy $\psi_1(v)\psi_j(u) - \psi_1(u)\psi_j(v) = 0$, so by density of $\operatorname{span}\{\psi_j : j \geq 1\}$ in $H^1(\Gamma) \cap \mathbf{1}^\perp$ (which embeds continuously into $C(\Gamma)$), every mean-zero $f \in H^1(\Gamma)$ would satisfy $\psi_1(v)f(u) = \psi_1(u)f(v)$, contradicting the existence of smooth bump functions in $H^1(\Gamma) \cap \mathbf{1}^\perp$ separating u and v. Hence $kQ_M > -\frac{D^2}{kL}$ strictly. In (11), $q \mapsto \frac{-q+\sqrt{q^2+S^2D^2}}{S^2} = \frac{D^2}{q+\sqrt{q^2+S^2D^2}}$ is strictly decreasing in $q \in \mathbb{R}$ and evaluates at $q = -\frac{D^2}{kL}$ to $t_{\mathrm{var}} = \frac{|D|(|D|+\sqrt{D^2+(kL)^2S^2})}{kLS^2}$, proving $\ell_*(u, v) < \ell_{\mathrm{var}}(u, v)$ strictly whenever $D \neq 0, S \neq 0$ (while for $S = 0, D \neq 0$, $\ell_{\mathrm{var}} = \pi/k$, so $\ell_* < \ell_{\mathrm{var}}$ iff $Q_M > 0$, and for $D = 0$, $\ell_* = \ell_{\mathrm{var}} = 0$).

Proof of (c). When $\mathbf{p} = (0, 0)^\top$, extending ψ_1 by $g \equiv 0$ on e_ℓ defines an exact mean-zero eigenfunction $\tilde{\psi}_1 = (\psi_1, 0)$ of $-\Delta_{\Gamma_\ell}$ at eigenvalue k^2 for every $\ell > 0$, so $k^2 \in \sigma(-\Delta_{\Gamma_\ell})$ and $\lambda_1(\Gamma_\ell) \leq k^2$ for all $\ell > 0$. On $\Gamma/\{u \sim v\}$, $\psi_1 \in H_0^1$ remains the simple first non-zero eigenfunction since $\lambda_2(\Gamma/\{u \sim v\}) \geq \lambda_2(\Gamma) > k^2$ (by form-domain restriction). Suppose for contradiction there existed a sequence $\ell_m \to 0^+$ along which either $\lambda_1(\Gamma_{\ell_m}) < k^2$ or $\dim E_{k^2}(\Gamma_{\ell_m}) \geq 2$; in either case there exists a normalized eigenfunction $F_m = (f_m, g_m)$ of $-\Delta_{\Gamma_{\ell_m}}$ with eigenvalue $\mu_m \in (0, k^2]$ that is $L^2(\Gamma_{\ell_m})$-orthogonal to both $\mathbf{1}$ and $\tilde{\psi}_1 = (\psi_1, 0)$ (taking $F_m \in E_{\lambda_1}(\Gamma_{\ell_m})$ when $\lambda_1(\Gamma_{\ell_m}) < k^2$, or $F_m \in E_{k^2}(\Gamma_{\ell_m}) \cap \tilde{\psi}_1^\perp$ when $\lambda_1(\Gamma_{\ell_m}) = \lambda_2(\Gamma_{\ell_m}) = k^2$), which implies $\langle f_m, \psi_1\rangle_{L^2(\Gamma)} = 0$. Applying the exact same compactness argument as in Step 3 yields a limit $f_\infty \in H_0^1$ with $\|f_\infty\|_{L^2(\Gamma)} = 1$, $\int_\Gamma f_\infty = 0$, $\langle f_\infty, \psi_1\rangle_{L^2(\Gamma)} = 0$, and $\mathcal{E}_\Gamma(f_\infty) \leq k^2 < \lambda_2(\Gamma/\{u \sim v\})$, a contradiction. Hence $\lambda_1(\Gamma_\ell) = k^2 < \lambda_2(\Gamma_\ell)$ for all sufficiently small $\ell > 0$, proving $\ell_{\mathrm{nod}} > 0$.

For $\ell > 2\pi/k$, the function equal to $\sin(2\pi s/\ell)$ on e_ℓ and 0 on Γ is continuous, non-zero, and mean-zero with Rayleigh quotient $(2\pi/\ell)^2 < k^2$, proving $\lambda_1(\Gamma_\ell) < k^2$. Finally, for $\ell \in (0, \pi/k)$, $\mathbf{p} = 0$ turns (14) into $(I_2 + M_\Gamma(k^2)\Lambda_k(\ell))\mathbf{f}_{uv} = 0$ (with α free, corresponding to $\tilde{\psi}_1$); hence $\dim E_{k^2}(\Gamma_\ell) \geq 2$ iff $\det(I_2 + M_\Gamma(k^2)\Lambda_k(\ell)) = 0$. Let $\ell_0 \in (0, \pi/k)$ be the smallest such root (if one exists). On $(0, \ell_0)$, k^2 is a *simple* eigenvalue of $-\Delta_{\Gamma_\ell}$ for every ℓ. We claim $\lambda_1(\Gamma_\ell) = k^2$ for all $\ell \in (0, \ell_0]$: otherwise, setting $\ell_1 := \inf\{\ell \in (0, \ell_0) : \lambda_1(\Gamma_\ell) < k^2\} \in (0, \ell_0)$, for every $\ell \in (\ell_1, \ell_0)$ we have $\lambda_1(\Gamma_\ell) < k^2 \in \sigma(-\Delta_{\Gamma_\ell})$ while $\lambda_2(\Gamma_\ell) \geq k^2$ (Lemma 3.1(b)), forcing $\lambda_2(\Gamma_\ell) = k^2$; letting $\ell \downarrow \ell_1$ by continuity (Lemma 3.1(a)) gives $\lambda_1(\Gamma_{\ell_1}) = \lambda_2(\Gamma_{\ell_1}) = k^2$, contradicting simplicity of k^2 at $\ell_1 \in (0, \ell_0)$. Conversely, at ℓ_0 itself, choosing $\alpha = 0$ and a non-zero $\mathbf{f}_{uv} \in \ker(I_2 + M_\Gamma(k^2)\Lambda_k(\ell_0))$ produces an eigenfunction $F_0 = (f_0, g_0)$ of $-\Delta_{\Gamma_{\ell_0}}$ at eigenvalue k^2 with $\int_{\Gamma_{\ell_0}} F_0 = 0$ and $\mathbf{f}_{uv} = (g_0(0), g_0(\ell_0))^\top \neq 0$; since $-g_0'' = k^2 g_0$ ($k > 0$) and $g_0 \not\equiv 0$, g_0 is non-constant, so $\int_0^{\ell_0} |g_0'|^2 > 0$. For any $\ell > \ell_0$, let $T : H^1(\Gamma_{\ell_0}) \to H^1(\Gamma_\ell)$ be the rescaling pullback of Lemma 3.1(a) and set $H_\ell :=$

$T(F_0) - c_\ell \mathbf{1} \in H^1(\Gamma_\ell)$ with $c_\ell := (L+\ell)^{-1} \int_{\Gamma_\ell} T(F_0)$, so $\int_{\Gamma_\ell} H_\ell = 0$. Because $H_\ell = T(F_0 - c_\ell \mathbf{1})$, $\ell_0/\ell < 1 < \ell/\ell_0$, and $\int_{\Gamma_{\ell_0}} F_0 = 0$, we obtain

$$\int_{\Gamma_\ell} |H_\ell'|^2 = \int_\Gamma |f_0'|^2 + \frac{\ell_0}{\ell} \int_0^{\ell_0} |g_0'|^2 < \int_{\Gamma_{\ell_0}} |F_0'|^2 = k^2 \|F_0\|^2_{L^2(\Gamma_{\ell_0})},$$

while

$$\int_{\Gamma_\ell} |H_\ell|^2 = \int_\Gamma |f_0 - c_\ell|^2 + \frac{\ell}{\ell_0} \int_0^{\ell_0} |g_0 - c_\ell|^2 \geq \|F_0 - c_\ell \mathbf{1}\|^2_{L^2(\Gamma_{\ell_0})} = \|F_0\|^2_{L^2(\Gamma_{\ell_0})} + c_\ell^2 (L + \ell_0) \geq \|F_0\|^2_{L^2(\Gamma_{\ell_0})}.$$

Hence $\lambda_1(\Gamma_\ell) \leq \mathcal{E}_{\Gamma_\ell}(H_\ell)/\|H_\ell\|^2_{L^2(\Gamma_\ell)} < k^2$ strictly for every $\ell > \ell_0$, so $\ell_{\mathrm{nod}} = \ell_0$. □

Remark 3.3 (Asymmetric 3-star realization of the $S = 0, Q_M > 0$ sign transition). Consider the asymmetric 3-star Γ with edge lengths $(1.0, 0.7, 0.35)$ joined at a central vertex 0 with leaves $1, 2, 3$. Its first non-zero eigenvalue is simple: $k_1 \approx 1.816827$ ($\pi/k_1 \approx 1.729164$), with $\psi_1(1) \approx -1.150650$, $\psi_1(2) \approx +0.951343$, and $\psi_1(3) \approx +0.348331$. Choosing u on edge $(0,1)$ at distance $x_{\mathrm{anti}} = 1 - \frac{1}{k_1} \arccos(-\psi_1(2)/\psi_1(1)) \approx 0.671091$ from vertex 0 and $v = 2$ yields an exact anti-equipotential pair $\psi_1(u) = -\psi_1(v) \approx -0.951343$ ($S = 0$, $D \approx -1.902686$). Here the variational bound predicts $\ell_{\mathrm{var}} = \pi/k_1 \approx 1.729164$, yet higher-mode relaxation produces $Q_M \approx +0.028295 > 0$, so Theorem 3.2(a)(ii) predicts a strict early Braess transition at

$$\ell_*(u,v) \;=\; \frac{2}{k_1} \arctan\!\left(\frac{D^2}{2k_1 Q_M}\right) \;\approx\; 1.697909 \;<\; 1.729164 \;=\; \frac{\pi}{k_1},$$

matching direct secular-determinant root finding to 7.64×10^{-13} (Table 1 and Figure 2(a)).

4. Universal Multiplicity-≥ 2 Quantum Braess Collapse

When $\lambda_1(\Gamma)$ is degenerate, the linear geometry of the eigenspace $E_{\lambda_1}(\Gamma)$ eliminates the spectral-gap improvement regime $(0, \ell_*)$ *everywhere* on $\Gamma \times \Gamma$.

Theorem 4.1 (Universal multiplicity-≥ 2 collapse)**.** *Let Γ be a compact connected metric graph with* $\dim E_{\lambda_1}(\Gamma) \geq 2$. *Then for* every *pair of points $u, v \in \Gamma$ and* every *$\ell > 0$:*

(a) $\lambda_1(\Gamma_\ell) \leq \lambda_1(\Gamma)$ *(no edge attachment of any positive length can ever strictly increase the spectral gap);*
(b) *if there exists $\psi \in E_{\lambda_1}(\Gamma)$ such that $\psi(u) = \psi(v) \neq 0$, then* $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ strictly *for every $\ell > 0$.*

In particular, $\lambda_1(\Gamma_\ell) \leq \lambda_1(\Gamma)$ *holds universally on every circle C_L, equilateral star S_d ($d \geq 3$), d-edge pumpkin P_d ($d \geq 3$), and equilateral complete graph K_n ($n \geq 3$), with strict inequality for all $\ell > 0$ on all non-antipodal pairs of C_L, all pairs of distinct leaves of S_d, and all non-nodal equipotential pairs of P_d and K_n.*

Proof. Fix arbitrary $u, v \in \Gamma$. Consider the linear evaluation-difference functional $\Phi_{u,v} : E_{\lambda_1}(\Gamma) \to \mathbb{R}$ defined by $\Phi_{u,v}(\psi) := \psi(u) - \psi(v)$. Because $\dim E_{\lambda_1}(\Gamma) \geq 2$ and $\dim \mathbb{R} = 1$, rank–nullity yields $\dim \ker(\Phi_{u,v}) \geq 1$, so there exists a normalized eigenfunction $\psi \in E_{\lambda_1}(\Gamma)$ ($\|\psi\|_{L^2(\Gamma)} = 1$) satisfying $\psi(u) = \psi(v)$.

- If $\psi(u) = \psi(v) \neq 0$, then $D = 0$ and $S = 2\psi(u) \neq 0$, so Theorem 2.1(b) immediately yields $\lambda_1(\Gamma_\ell) < \lambda_1(\Gamma)$ strictly for every $\ell > 0$.
- If $\psi(u) = \psi(v) = 0$, then extending ψ by 0 on e_ℓ produces a non-zero mean-zero function $\tilde{\psi} \in H^1(\Gamma_\ell)$ with Rayleigh quotient $\mathcal{E}_{\Gamma_\ell}(\tilde{\psi})/\|\tilde{\psi}\|^2_{L^2(\Gamma_\ell)} = \lambda_1(\Gamma)$, proving $\lambda_1(\Gamma_\ell) \leq \lambda_1(\Gamma)$ for every $\ell > 0$.

For the canonical families: on a circle C_L ($k_1 = 2\pi/L$, $E_{\lambda_1} = \operatorname{span}\{\cos(\frac{2\pi x}{L}), \sin(\frac{2\pi x}{L})\}$), for any non-antipodal u, v the symmetric cosine centered at the midpoint of u, v satisfies $\psi(u) = \psi(v) \neq 0$ (whereas for antipodal u, v every $\psi \in E_{\lambda_1}$ has $\psi(u) = -\psi(v)$, so $\psi(u) = \psi(v)$ forces $\psi(u) = \psi(v) = 0$, yielding a nodal plateau on $(0, L/2]$). On an equilateral d-star S_d ($d \geq 3$, arm length a, $k_1 = \pi/(2a)$, $\dim E_{\lambda_1} = d - 1 \geq 2$), every $\psi \in E_{\lambda_1}$ vanishes at the center and takes the form $c_r \sin(\frac{\pi x_r}{2a})$ on arm r with $\sum_{r=1}^{d} c_r = 0$: for any two distinct leaves r_1, r_2, choosing $c_{r_1} = c_{r_2} = 1$ and $c_{r_3} = -2$ (0 elsewhere) gives $\psi(r_1) = \psi(r_2) = 1 \neq 0$ (strict drop for all $\ell > 0$), whereas between the center 0 and leaf r_1, $\psi(0) = 0$ forces $\psi(r_1) = 0$ (nodal plateau). On a d-edge pumpkin P_d ($d \geq 3$, parallel edges of length a joining two vertices, $k_1 = \pi/a$), $\dim E_{\lambda_1} = d \geq 3$ (spanned by the $d - 1$ zero-sum Dirichlet sine modes $\sin(\pi x_r/a)$ together with the symmetric Neumann mode $\cos(\pi x_r/a)$ on every edge), and on equilateral K_n ($n \geq 3$), $\dim E_{\lambda_1} = n - 1 \geq 2$. □

5. Exact Interval Chord Law, Cubic Tangent Scaling, and Magic Tetrahedral Angle

On the fundamental benchmark $\Gamma = [0, L]$, Theorem 3.2 evaluates in complete elementary closed form, revealing striking algebraic and geometric laws.

Theorem 5.1 (Exact interval chord law, cubic tangent scaling, and magic tetrahedral angle)**.** *Let $\Gamma = [0, L]$ ($L > 0$), so $\lambda_1(\Gamma) = (\pi/L)^2$, $k = \pi/L$, and $\psi_1(x) = \sqrt{2/L}\cos(\pi x/L)$. For any $0 \leq u < v \leq L$:*

(a) Closed-form reduced Green's invariant.

$$k\, Q_M(u, v) \;=\; \frac{2}{L} \sin\!\left(\frac{\pi(v-u)}{L}\right) \cos\!\left(\frac{\pi u}{L}\right) \cos\!\left(\frac{\pi v}{L}\right). \tag{17}$$

Consequently, the exact critical chord length $\ell_*(u, v) \in (0, L]$ is given explicitly by (11)–(12) with $D = \sqrt{2/L}(\cos(\pi u/L) - \cos(\pi v/L))$, $S = \sqrt{2/L}(\cos(\pi u/L) + \cos(\pi v/L))$, and (17).

(b) Endpoint cubic tangent law and duality. *When $u = 0$ and $v = a \in (0, L)$,*

$$\tan\!\left(\frac{\pi\, \ell_*(0, a)}{2L}\right) \;=\; \tan^3\!\left(\frac{\pi a}{2L}\right). \tag{18}$$

In particular, $\ell_*(0, a)$ obeys the exact reflection duality $\ell_*(0, a) + \ell_*(0, L - a) = L$ (with fixed point $\ell_*(0, L/2) = L/2$) and the cubic small-a scaling

$$\ell_*(0, a) \;=\; \frac{\pi^2}{4L^2}\, a^3 + O(a^5) \qquad (a \to 0^+),$$

in sharp contrast to the quadratic variational bound $\ell_{\mathrm{var}}(0, a) = \frac{\pi}{2L} a^2 + O(a^4)$.

(c) Midpoint reflection law. *For every symmetric pair $v = L - u$ ($u \in [0, L/2)$), $S = 0$ and $kQ_M(u, L - u) = -\frac{2}{L}\sin(\frac{2\pi u}{L})\cos^2(\frac{\pi u}{L}) \leq 0$, whence $\ell_*(u, L - u) = L = \pi/k$ identically.*

(d) Short-chord limit and magic tetrahedral detour angle. *For every fixed $u \in (0, L/2)$, as $h \to 0^+$,*

$$\lim_{h \to 0^+} \frac{\ell_*(u, u + h)}{h} \;=\; \frac{\sqrt{1 + 4\tan^2(\pi u/L)} - 1}{2}. \tag{19}$$

This limiting slope equals 1 at the unique magic tetrahedral angle

$$\frac{u_{\mathrm{crit}}}{L} \;=\; \frac{1}{\pi}\arctan(\sqrt{2}) \;=\; \frac{1}{2\pi}\arccos\!\left(-\frac{1}{3}\right) \;\approx\; 0.304086724. \tag{20}$$

For $u \in (u_{\mathrm{crit}}, L/2)$, the slope strictly exceeds 1: there exists $h_0(u) > 0$ such that for all $h \in (0, h_0(u))$, attaching a detour chord *of length $\ell \in (h, \ell_*(u, u + h))$ strictly longer than the geodesic distance h still strictly* increases *the spectral gap $\lambda_1(\Gamma_\ell) > \lambda_1(\Gamma)$.*

Proof. Proof of (a). On $[0,L]$, the Neumann eigenbasis is $\psi_0 \equiv L^{-1/2}$ and $\psi_n(x) = \sqrt{2/L}\cos(n\pi x/L)$ with $\lambda_n = (n\pi/L)^2$ $(n \geq 1)$. We first evaluate the cosine series $H(\theta) := \sum_{n=2}^{\infty} \frac{\cos(n\theta)}{n^2-1}$ for $\theta \in [0,2\pi]$. Define the continuous function

$$h(\theta) \;:=\; \frac{1}{2} + \frac{1}{4}\cos\theta - \frac{\pi-\theta}{2}\sin\theta, \qquad \theta \in [0,2\pi].$$

Observe that $h(2\pi-\theta) = h(\theta)$, so h is even about π and determined on $[0,2\pi]$ by its cosine series on $[0,\pi]$. For every integer $m \geq 1$, integration by parts gives $\int_0^\pi (\pi-\theta)\sin(m\theta)\,d\theta = \big[-(\pi-\theta)\frac{\cos(m\theta)}{m}\big]_0^\pi - \frac{1}{m}\int_0^\pi \cos(m\theta)\,d\theta = \frac{\pi}{m}$. Using $\sin\theta\cos(n\theta) = \frac{1}{2}[\sin((n+1)\theta) - \sin((n-1)\theta)]$, the cosine coefficients $a_n = \frac{2}{\pi}\int_0^\pi h(\theta)\cos(n\theta)\,d\theta$ evaluate to:

- $a_0/2 = \frac{1}{2} - \frac{1}{2\pi}\int_0^\pi (\pi-\theta)\sin\theta\,d\theta = \frac{1}{2} - \frac{1}{2} = 0$;
- $a_1 = \frac{1}{4} - \frac{1}{\pi}\cdot\frac{1}{2}\int_0^\pi (\pi-\theta)\sin(2\theta)\,d\theta = \frac{1}{4} - \frac{1}{4} = 0$;
- for $n \geq 2$: $a_n = -\frac{1}{\pi}\cdot\frac{1}{2}\Big(\frac{\pi}{n+1} - \frac{\pi}{n-1}\Big) = \frac{1}{n^2-1}$.

Hence $H(\theta) = h(\theta)$ uniformly on $[0,2\pi]$.

Now write $\alpha := \frac{\pi u}{L}$ and $\beta := \frac{\pi v}{L}$ $(0 \leq \alpha < \beta \leq \pi)$. By the product-to-sum identity $\cos(n\alpha)\cos(n\beta) = \frac{1}{2}[\cos(n(\beta-\alpha)) + \cos(n(\alpha+\beta))]$, the reduced Green's function (6) on $[0,L]$ equals

$$G'_{[0,L]}(x,y;k^2) \;=\; \frac{L}{\pi^2}\Big[-1 + h\big(|\theta_x - \theta_y|\big) + h\big(\theta_x + \theta_y\big)\Big], \qquad \theta_x := \frac{\pi x}{L},\ \theta_y := \frac{\pi y}{L}.$$

Substituting $h(0) = \frac{3}{4}$ and $h(2\theta) = \frac{1}{2} + \frac{1}{4}\cos(2\theta) - \frac{\pi-2\theta}{2}\sin(2\theta)$ gives:

$$\begin{aligned}
\frac{\pi^2}{L}G'(u,u) &= \frac{1}{4} + \frac{1}{4}\cos(2\alpha) - \frac{\pi-2\alpha}{2}\sin(2\alpha) = \frac{1}{2}\cos^2\alpha - (\pi-2\alpha)\sin\alpha\cos\alpha,\\
\frac{\pi^2}{L}G'(v,v) &= \frac{1}{2}\cos^2\beta - (\pi-2\beta)\sin\beta\cos\beta,\\
\frac{\pi^2}{L}G'(u,v) &= \frac{1}{4}\big(\cos(\beta-\alpha) + \cos(\alpha+\beta)\big) - \frac{\pi-(\beta-\alpha)}{2}\sin(\beta-\alpha) - \frac{\pi-(\alpha+\beta)}{2}\sin(\alpha+\beta)\\
&= \frac{1}{2}\cos\alpha\cos\beta - (\pi-\beta)\sin\beta\cos\alpha + \alpha\sin\alpha\cos\beta,
\end{aligned}$$

where the last line uses $\sin(\beta-\alpha) + \sin(\alpha+\beta) = 2\sin\beta\cos\alpha$ and $\sin(\beta-\alpha) - \sin(\alpha+\beta) = -2\sin\alpha\cos\beta$. Now compute $kQ_M = \frac{\pi}{L}\cdot\frac{2}{L}\big[\cos^2\beta\,G'(u,u) - 2\cos\alpha\cos\beta\,G'(u,v) + \cos^2\alpha\,G'(v,v)\big]$:

- The $\frac{1}{2}\cos^2\alpha\cos^2\beta$ terms cancel identically: $\frac{1}{2} - 2\cdot\frac{1}{2} + \frac{1}{2} = 0$.
- Factoring out $\cos\alpha\cos\beta$ from the remaining terms gives

$$\begin{aligned}
&-\cos\beta(\pi-2\alpha)\sin\alpha + 2\big((\pi-\beta)\sin\beta\cos\alpha - \alpha\sin\alpha\cos\beta\big) - \cos\alpha(\pi-2\beta)\sin\beta\\
&\quad = -\pi\sin\alpha\cos\beta + \pi\sin\beta\cos\alpha \;=\; \pi\sin(\beta-\alpha).
\end{aligned}$$

Multiplying by $\frac{2}{\pi L}\cos\alpha\cos\beta$ yields

$$kQ_M(u,v) \;=\; \frac{2}{L}\sin(\beta-\alpha)\cos\alpha\cos\beta \;=\; \frac{2}{L}\sin\left(\frac{\pi(v-u)}{L}\right)\cos\left(\frac{\pi u}{L}\right)\cos\left(\frac{\pi v}{L}\right),$$

proving (17).

Proof of (b). Set $u = 0$ ($\alpha = 0$) and $v = a \in (0, L)$ ($\beta = \pi a/L$), and let $\tau := \tan(\beta/2) \in (0, \infty)$. Then $\cos\beta = \frac{1-\tau^2}{1+\tau^2}$ and $\sin\beta = \frac{2\tau}{1+\tau^2}$, so

$$\frac{L}{2}D^2 = (1-\cos\beta)^2 = \frac{4\tau^4}{(1+\tau^2)^2}, \qquad \frac{L}{2}S^2 = (1+\cos\beta)^2 = \frac{4}{(1+\tau^2)^2},$$
$$\frac{L}{2}kQ_M = \sin\beta\cos\beta = \frac{2\tau(1-\tau^2)}{(1+\tau^2)^2}.$$

Multiplying the Krein quadratic (10) by $\frac{L}{2}(1+\tau^2)^2$ produces

$$2t^2 + 2\tau(1-\tau^2)\,t - 2\tau^4 \;=\; 2(t-\tau^3)(t+\tau) \;=\; 0.$$

Since $t > 0$ and $\tau > 0$, the unique positive root is $t_* = \tau^3$, proving (18). Replacing a by $L - a$ maps $\tau = \tan(\frac{\pi a}{2L})$ to $\cot(\frac{\pi a}{2L}) = \tau^{-1}$, hence $t_* = \tau^3 \mapsto \tau^{-3} = t_*^{-1}$, so $\frac{\pi\ell_*(0,L-a)}{2L} = \frac{\pi}{2} - \frac{\pi\ell_*(0,a)}{2L}$, i.e., $\ell_*(0,a) + \ell_*(0,L-a) = L$. Expanding $\ell_*(0,a) = \frac{2L}{\pi}\arctan(\tan^3(\frac{\pi a}{2L}))$ as $a \to 0^+$ gives $\frac{2L}{\pi}(\frac{\pi a}{2L})^3 + O(a^5) = \frac{\pi^2}{4L^2}a^3 + O(a^5)$, whereas in (2), $D \sim \frac{\sqrt{2/L}}{2}(\frac{\pi a}{L})^2$ and $S \sim 2\sqrt{2/L}$ give $t_{\mathrm{var}} \sim \frac{|D|}{S} \sim \frac{\pi^2 a^2}{4L^2}$, so $\ell_{\mathrm{var}}(0,a) = \frac{\pi}{2L}a^2 + O(a^4)$.

Proof of (c). When $v = L - u$ ($u \in [0, L/2)$), $\beta = \pi - \alpha$, so $\cos\beta = -\cos\alpha$ ($S = 0$, $D = 2\sqrt{2/L}\cos\alpha \neq 0$), and (17) gives $kQ_M(u, L-u) = -\frac{2}{L}\sin(2\alpha)\cos^2\alpha \leq 0$. Theorem 3.2(a)(ii) immediately yields $\ell_*(u, L-u) = \pi/k = L$.

Proof of (d). Fix $u \in (0, L/2)$ ($\alpha = \pi u/L \in (0, \pi/2)$) and let $\delta := \pi h/L \to 0^+$ ($\beta = \alpha + \delta$). Then:

$$D = \sqrt{\frac{2}{L}}\big(\cos\alpha - \cos(\alpha+\delta)\big) = \sqrt{\frac{2}{L}}\,(\sin\alpha)\,\delta + O(\delta^2),$$
$$S = \sqrt{\frac{2}{L}}\big(2\cos\alpha + O(\delta)\big),$$
$$kQ_M = \frac{2}{L}\big(\cos^2\alpha\big)\,\delta + O(\delta^2).$$

Substituting into (11) gives

$$t_* \;=\; \frac{-2\cos^2\alpha + \sqrt{4\cos^4\alpha + 16\cos^2\alpha\sin^2\alpha}}{8\cos^2\alpha}\,\delta + O(\delta^2) \;=\; \frac{\sqrt{1+4\tan^2\alpha}-1}{4}\,\delta + O(\delta^2).$$

Since $\ell_* = \frac{2L}{\pi}\arctan(t_*) = \frac{2L}{\pi}t_* + O(t_*^3)$ and $\frac{2L}{\pi}\delta = 2h$, dividing by h yields (19). Setting $\frac{1}{2}(\sqrt{1+4\tan^2\alpha}-1) = 1$ gives $1 + 4\tan^2\alpha = 9 \iff \tan\alpha = \sqrt{2}$ (for $\alpha \in (0, \pi/2)$), i.e., $\cos(2\alpha) = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = -\frac{1}{3}$, proving (20). Whenever $u \in (u_{\mathrm{crit}}, L/2)$, the limiting slope strictly exceeds 1, so $\ell_*(u, u+h) > h$ for all sufficiently small $h > 0$, and by Theorem 3.2(a) every chord of length $\ell \in (h, \ell_*(u, u+h))$ satisfies $\lambda_1(\Gamma_\ell) > \lambda_1(\Gamma)$. □

6. High-Precision Numerical Verification and Open Problems

All four theorems have been independently verified via our reproducible Python suite (the verification script `verify_quantum_braess.py` and pytest suite `test_quantum_braess.py` in the archived repository):

- **50-digit `mpmath` verification of Theorem 5.1:** Across multiple interval lengths $L \in \{1.0, 2.5, 3.7, \pi\}$ and attachment pairs (u, v), the closed form (17) matches the exact Fourier-series reduced Green's form to 3.11×10^{-61}, the cubic tangent law (18) matches the Krein quadratic root to 1.49×10^{-59}, the magic tetrahedral angle identities (20) match to 0.0 (60-digit working precision), and the short-chord ratio $\ell_*(u, u+h)/h$ at $h = 10^{-12}$ matches (19) to 6.40×10^{-11}.

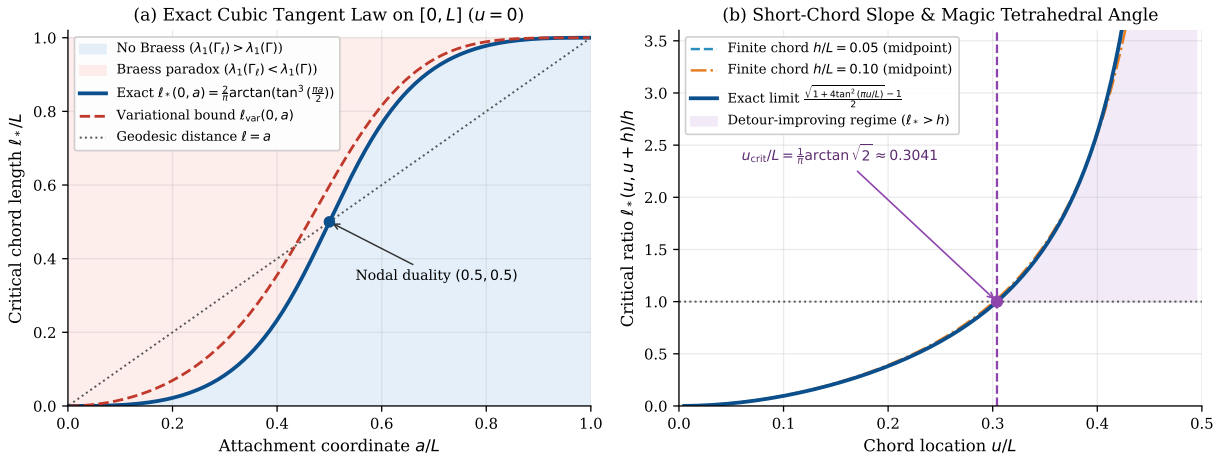


FIGURE 1. (a) Exact cubic tangent phase boundary $\ell_*(0,a) = \frac{2L}{\pi}\arctan(\tan^3(\frac{\pi a}{2L}))$ on $[0,L]$ (Theorem 5.1(b)) compared against the universal variational upper bound $\ell_{\mathrm{var}}(0,a)$ (Theorem 2.1(a)) and the geodesic line $\ell = a$, separating the spectral-gap improvement phase (blue) from the Braess paradox phase (red). (b) Exact short-chord limit $\lim_{h\to 0^+} \ell_*(u,u+h)/h = \frac{1}{2}(\sqrt{1+4\tan^2(\pi u/L)}-1)$ and finite-h curves on $[0,L]$, crossing 1 at the magic tetrahedral angle $u_{\mathrm{crit}}/L = \frac{1}{\pi}\arctan\sqrt{2} \approx 0.3041$ (Theorem 5.1(d)).

- **Multi-graph DtN secular verification of Theorems 2.1 and 3.2:** Across 12 benchmark configurations spanning intervals $[0,1]$ (including nodal $v = 0.5$, midpoint-symmetric $(0.2, 0.8)$, and detour-improving $(0.35, 0.45)$ where $h = 0.10$ yet $\ell_* = 0.259660 > 2.59h$), asymmetric 3-stars $(1.0, 0.7, 0.35)$ (including the anti-equipotential $S = 0, Q_M = +0.028295 > 0$ early transition), tadpole graphs (loop 1.4 + tail 1.0), and asymmetric dumbbells (loops of lengths 1.2 and 0.8 joined by a bridge of length 1.0, with attachment pair $(1,3)$ denoting the two loop antipodes opposite the bridge vertices), the Krein formula (11)–(12) matches direct secular-determinant root finding for $\lambda_1(\Gamma_\ell) = \lambda_1(\Gamma)$ to maximum absolute error 1.13×10^{-12} (Table 1).
- **Verification of Theorem 4.1 and high-length/loop collapse:** Direct secular computations on the circle C_1 (both non-antipodal and antipodal chords), the equilateral 3-star S_3 (leaf-to-leaf and center-to-leaf), and the 3-edge pumpkin P_3 confirm $\lambda_1(\Gamma_\ell) \le \lambda_1(\Gamma)$ for all tested lengths $\ell > 0$, with strict immediate drops for every non-nodal equipotential pair and flat nodal plateaus for antipodal C_1 and center-to-leaf S_3 (representative curves shown in Figure 2(b)).

Open Problems. Two natural directions extend the Krein–DtN framework developed here:

(1) *Simultaneous multi-edge attachment and Braess hypersurfaces:* When attaching $m \ge 2$ edges $e_{\ell_1}, \dots, e_{\ell_m}$ simultaneously to Γ, the Krein reduction yields an m-parameter determinantal hypersurface $\det_{\mathbf{p}^\perp}(\Lambda_k(\boldsymbol{\ell})^{-1} + M_\Gamma(k^2)) = 0$ in $\mathbb{R}^m_{>0}$. Can multiple edges cooperate to *raise* λ_1 on a multiplicity-2 graph (overcoming Theorem 4.1's single-edge obstruction), and what is the minimal number of edges required in terms of $\dim E_{\lambda_1}(\Gamma)$?

(2) *Higher-eigenvalue and Schrödinger-operator transitions:* Extending Theorem 3.2 to higher simple eigenvalues $\lambda_r(\Gamma)$ $(r \ge 2)$ and to quantum graphs with non-zero potential $q \in L^\infty(\Gamma)$ or δ-type vertex couplings (where $\Lambda_k(\ell)$ is replaced by the Sturm–Liouville Weyl–Titchmarsh M-function of the added edge).

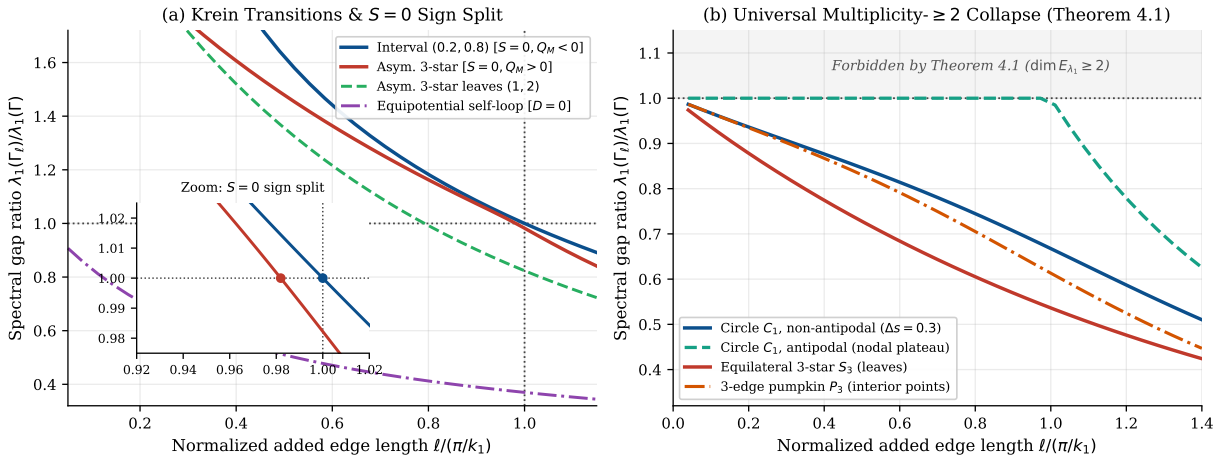


FIGURE 2. (a) Spectral-gap ratio $\lambda_1(\Gamma_\ell)/\lambda_1(\Gamma)$ vs. normalized edge length $\ell/(\pi/k_1)$ across simple-λ_1 topologies, with inset zooming near $\ell = \pi/k_1$ to highlight the anti-equipotential ($S = 0$) sign split: interval $(0.2, 0.8)$ ($Q_M < 0$) crosses at exactly $\ell = \pi/k_1$, whereas the asymmetric 3-star ($Q_M = +0.028295 > 0$) crosses early at $\ell_* = 0.98192\,(\pi/k_1)$. (b) Universal multiplicity-≥ 2 collapse (Theorem 4.1): no chord of any positive length can exceed $\lambda_1(\Gamma)$ on C_1, S_3, or P_3; non-nodal equipotential pairs drop strictly for all $\ell > 0$, while antipodal C_1 displays a nodal plateau on $(0, \pi/k_1]$.

TABLE 1. Verification of the exact Krein–DtN law (Theorem 3.2) against direct metric-graph secular root finding across 12 multi-graph benchmarks, together with the universal variational upper bound ℓ_{var} (Theorem 2.1) and half-wavelength π/k_1.

Graph Γ and pair (u, v)	k_1	π/k_1	Q_M	ℓ_{var}	ℓ_* (Krein)	ℓ_* (num.)	$\lvert\Delta\ell_*\rvert$
$[0,1]$, $(0, 0.25)$	3.14159	1.00000	+0.318310	0.114117	0.045167	0.045167	3.24×10^{-14}
$[0,1]$, $(0, 0.50)$ [nodal v]	3.14159	1.00000	+0.000000	0.598093	0.500000	0.500000	2.50×10^{-14}
$[0,1]$, $(0, 0.75)$	3.14159	1.00000	−0.318310	0.972454	0.954833	0.954833	1.78×10^{-14}
$[0,1]$, $(0.2, 0.8)$ $[S = 0, Q_M < 0]$	3.14159	1.00000	−0.396280	1.000000	1.000000	1.000000	0.00×10^{0}
$[0,1]$, $(0.35, 0.45)$ [detour $> h$]	3.14159	1.00000	+0.013971	0.329296	0.259660	0.259660	7.08×10^{-14}
$[0,1]$, $(0.15, 0.65)$	3.14159	1.00000	−0.257518	0.913606	0.892292	0.892292	5.02×10^{-14}
3-star $(1, 0.7, 0.35)$ $[S = 0, Q_M > 0]$	1.81683	1.72916	+0.028295	1.729164	1.697909	1.697909	7.64×10^{-13}
3-star $(1, 0.7, 0.35)$, leaves $(1, 2)$	1.81683	1.72916	+0.380315	1.711277	1.369104	1.369104	1.13×10^{-12}
3-star $(1, 0.7, 0.35)$, leaves $(1, 3)$	1.81683	1.72916	+0.271991	1.377966	0.987343	0.987343	3.22×10^{-14}
3-star $(1, 0.7, 0.35)$, leaves $(2, 3)$	1.81683	1.72916	+0.584370	0.532495	0.166804	0.166804	9.44×10^{-14}
Tadpole $(1.4; 1.0)$, (antipode, tip)	1.74929	1.79592	−0.108328	1.619866	1.503311	1.503311	1.07×10^{-12}
Dumbbell $(1.2, 1.0, 0.8)$, $(1, 3)$	1.24179	2.52990	−0.279786	2.507966	2.496561	2.496561	6.38×10^{-13}

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