Non-Asymptotic Loewner Sensitivity of Discrete Lyapunov Equations, Concavity-Certified Riccati Contraction, Global Quadratic Kleinman–Hewer Policy Iteration, and Deadbeat Gramian DARE Ceilings

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Abstract

We establish four explicit, non-asymptotic matrix-inequality theorems governing stability sensitivity, Riccati value iteration, policy iteration, and controllability bounds for discrete-time linear-quadratic systems $x_{k+1} = Ax_k + Bu_k$ with $Q \succ 0$ and $R \succ 0$. First, for any Schur-stable A with Lyapunov solution $A^\top PA - P = -Q$ and contraction factor $\gamma := 1-\lambda_{\min}(P^{-1/2}QP^{-1/2}) \in [0,1)$, we prove a sharp P-metric perturbation stability radius $\|P^{1/2}\Delta A\, P^{-1/2}\|_2 < 1-\sqrt{\gamma}$ together with an explicit two-sided Loewner sensitivity sandwich $\frac{1}{1+\eta_E}P \preceq \widetilde{P} \preceq \frac{1}{1-\eta_E}P$ for the perturbed solution $\widetilde{P}$, where $\eta_E := (2\sqrt{\gamma}\|E\|_2+\|E\|_2^2)/(1-\gamma) < 1$. Second, exploiting an exact completing-the-square suboptimality deficit and Loewner concavity of the discrete Riccati operator $\mathcal{R}$, we prove an a priori computable geometric contraction bound $0 \preceq P^* - P_t \preceq (1-\alpha)^t P^* \preceq (1-\underline{\alpha})^t P_{K_0}$ for zero-initialized Riccati iteration ($P_0 = 0$) alongside a computable two-sided Loewner bracket $P_t \preceq P^* \preceq \overline{P}_t$. Third, from an exact algebraic gain-error identity $K_{j+1}-K^* = (R+B^\top X_j B)^{-1}B^\top(X_j-P^*)A_{\mathrm{cl}}^*$ and a uniform Lyapunov-operator domination $\mathcal{T}_{j+1}(Q) \preceq X_j$, we prove an explicit, global, zero-local-neighborhood quadratic contraction inequality $0 \preceq X_{j+1}-P^* \preceq \frac{\|A_{\mathrm{cl}}^*\|_2^2\|BR^{-1}B^\top\|_2}{\lambda_{\min}(Q)}\|X_j - P^*\|_2^2\, X_j$ valid from iteration $j = 0$ of discrete Kleinman–Hewer policy iteration. Fourth, pairing minimum-energy controllability-Gramian steering with a deadbeat finite-horizon certificate, we derive an explicit closed-form Loewner upper ceiling on P^* requiring no prior stabilizing feedback gain and attaining sharpness at $N = 1$. All identities and Loewner bounds are verified across 99 deterministic numerical checks.

1 Introduction and Relation to Prior Work

Discrete-time Lyapunov equations and discrete algebraic Riccati equations (DAREs) form the analytical core of linear-quadratic optimal control, robust stability certification, and modern reinforcement learning for linear-quadratic regulators (LQR) [4, 1]. Classical control theory establishes existence, uniqueness, and asymptotic convergence of Riccati value iteration and Kleinman–Hewer policy iteration [3, 4], while recent learning-based LQR analyses demand explicit, non-asymptotic perturbation bounds and quantitative convergence rates in the Loewner semidefinite order [2, 5].

Here we develop four completely self-contained, elementary, non-asymptotic matrix-inequality theorems requiring no complex-analytic Riccati subspace theory or local implicit-function neighborhoods: (i) exact two-sided Lyapunov energy envelopes, the P-metric Schur stability radius $\|E\|_2 < 1-\sqrt{\gamma}$, and a two-sided Loewner sensitivity sandwich for perturbed Lyapunov

solutions (Theorem 2.1); (ii) exact completing-the-square LQR suboptimality, Loewner concavity of $\mathcal{R}$, and an explicit three-line a priori geometric convergence rate for zero-initialized Riccati iteration paired with a computable two-sided bracket $P_t \preceq P^* \preceq \overline{P}_t$ (Theorem 3.1); (iii) an exact algebraic gain-error identity for Kleinman–Hewer policy iteration yielding an explicit, global quadratic Loewner contraction bound valid from $j = 0$ (Theorem 4.1); and (iv) exact minimum-energy controllability-Gramian steering yielding a gain-free, closed-form Loewner upper ceiling on P^* (Theorem 5.1).

Throughout, $\mathbb{S}^d_+$ and $\mathbb{S}^d_{++}$ denote symmetric positive semidefinite and positive definite $d \times d$ real matrices, $\preceq$ denotes the Loewner order ($X \preceq Y \iff Y - X \in \mathbb{S}^d_+$), $\|M\|_2$ is the spectral norm, and $\rho(A)$ is the spectral radius.

2 Discrete Lyapunov Energy Envelopes and Loewner Perturbation Sensitivity

Theorem 2.1 (Lyapunov Contraction Sandwich, P-Metric Stability Radius, and Loewner Sensitivity)**.** *Let $A \in \mathbb{R}^{d\times d}$ ($d \geq 1$) and $Q \in \mathbb{S}^d_{++}$.*

(a) (Existence, Uniqueness, and Two-Sided Energy Envelope.) *The discrete Lyapunov equation $A^\top PA - P = -Q$ admits a solution $P \in \mathbb{S}^d_{++}$ if and only if $\rho(A) < 1$, in which case the solution is unique and given by the absolutely convergent series $P = \sum_{k=0}^{\infty}(A^\top)^k Q A^k \succeq Q$. Setting*

$$\gamma := 1 - \lambda_{\min}(P^{-1/2}QP^{-1/2}) \in [0, 1), \qquad \gamma_+ := 1 - \lambda_{\max}(P^{-1/2}QP^{-1/2}) \in [0, \gamma], \tag{1}$$

every trajectory $x_{k+1} = Ax_k$ ($k \geq 0$) obeys $\rho(A)^2 \leq \gamma$, $\sum_{k=N}^{\infty} x_k^\top Q x_k = x_N^\top P x_N$, and

$$\gamma_+^k(x_0^\top P x_0) \leq x_k^\top P x_k \leq \gamma^k(x_0^\top P x_0) \qquad (\forall k \geq 0). \tag{2}$$

(b) (Explicit P-Metric Stability Radius and Two-Sided Loewner Sensitivity Sandwich.) *Suppose $\rho(A) < 1$, let $P \in \mathbb{S}^d_{++}$ solve $A^\top PA - P = -Q$, and for any $\Delta A \in \mathbb{R}^{d\times d}$ define $E := P^{1/2}\Delta A\, P^{-1/2}$. Whenever $\|E\|_2 < 1 - \sqrt{\gamma}$, the perturbed matrix $A + \Delta A$ is strictly Schur-stable with $\rho(A + \Delta A) \leq \sqrt{\gamma} + \|E\|_2 < 1$. Moreover, setting*

$$\eta_E := \frac{2\sqrt{\gamma}\,\|E\|_2 + \|E\|_2^2}{1 - \gamma} \in [0, 1), \tag{3}$$

the unique perturbed Lyapunov solution $\widetilde{P} \in \mathbb{S}^d_{++}$ of $(A + \Delta A)^\top \widetilde{P}(A + \Delta A) - \widetilde{P} = -Q$ satisfies the explicit two-sided Loewner sandwich

$$\frac{1}{1 + \eta_E}\, P \preceq \widetilde{P} \preceq \frac{1}{1 - \eta_E}\, P, \qquad \left\|P^{-1/2}(\widetilde{P} - P)P^{-1/2}\right\|_2 \leq \frac{\eta_E}{1 - \eta_E}. \tag{4}$$

Proof. Part (a). If $A^\top PA - P = -Q$ with $P, Q \succ 0$ and $Av = \lambda v$ ($v \in \mathbb{C}^d \setminus \{0\}$), then $(|\lambda|^2 - 1)v^*Pv = -v^*Qv < 0$, forcing $|\lambda| < 1$ and $\rho(A) < 1$. Conversely, if $\rho(A) < 1$, Gelfand's formula gives $\|A^k\|_2 \leq Cr^k$ for some $r \in (\rho(A), 1)$, so $P := \sum_{k=0}^{\infty}(A^\top)^k Q A^k$ converges absolutely, satisfies $P \succeq Q \succ 0$, and telescopes to $A^\top PA - P = -Q$. If P_1, P_2 both solve the equation, $\Delta := P_1 - P_2$ obeys $\Delta = (A^\top)^k \Delta A^k \to 0$ as $k \to \infty$, proving uniqueness. Setting $M := P^{1/2}AP^{-1/2}$, conjugating $A^\top PA = P - Q$ yields $M^\top M = I - P^{-1/2}QP^{-1/2}$. Because $0 \prec Q \preceq P$, the eigenvalues of $P^{-1/2}QP^{-1/2}$ lie in $(0, 1]$, so $\gamma_+ I \preceq M^\top M \preceq \gamma I$ with $0 \leq \gamma_+ \leq \gamma < 1$, giving $\|M\|_2 = \sqrt{\gamma}$ and $\rho(A) = \rho(M) \leq \sqrt{\gamma}$. For $x_{k+1} = Ax_k$, writing $y_k := P^{1/2}x_k$ gives $\|y_{k+1}\|_2^2 = y_k^\top M^\top M y_k \in [\gamma_+\|y_k\|_2^2, \gamma\|y_k\|_2^2]$, yielding (2) by induction, while telescoping $x_k^\top Q x_k = x_k^\top P x_k - x_{k+1}^\top P x_{k+1}$ from $k = N$ to ∞ gives $\sum_{k=N}^{\infty} x_k^\top Q x_k = x_N^\top P x_N$.

Part (b). Using $M = P^{1/2}AP^{-1/2}$ and $E = P^{1/2}\Delta A\, P^{-1/2}$, similarity gives $\rho(A+\Delta A) = \rho(M+E) \le \|M+E\|_2 \le \|M\|_2 + \|E\|_2 = \sqrt{\gamma} + \|E\|_2 < 1$, so $\widetilde{P} \succ 0$ exists uniquely by Part (a). Note that $\|E\|_2 < 1-\sqrt{\gamma} \iff 2\sqrt{\gamma}\|E\|_2 + \|E\|_2^2 < 1-\gamma \iff \eta_E \in [0,1)$. Define $\Delta Q := (A+\Delta A)^\top P(A+\Delta A) - A^\top PA = P^{1/2}(M^\top E + E^\top M + E^\top E)P^{1/2}$. Then $\|P^{-1/2}\Delta Q\, P^{-1/2}\|_2 \le 2\|M\|_2\|E\|_2 + \|E\|_2^2 = \delta_E := \eta_E(1-\gamma)$, so $-\delta_E P \preceq \Delta Q \preceq \delta_E P$. Because $P^{-1/2}QP^{-1/2} \succeq (1-\gamma)I$, we have $P \preceq (1-\gamma)^{-1}Q$, hence $-\eta_E Q \preceq \Delta Q \preceq \eta_E Q$ and

$$(1-\eta_E)Q \preceq Q - \Delta Q \preceq (1+\eta_E)Q.$$

Observe that $(A+\Delta A)^\top P(A+\Delta A) - P = -(Q-\Delta Q)$, while $(A+\Delta A)^\top \widetilde{P}(A+\Delta A) - \widetilde{P} = -Q$. Applying the Loewner-monotone linear Lyapunov operator $\widetilde{\mathcal{T}}(Y) := \sum_{k=0}^{\infty}((A+\Delta A)^\top)^k Y(A+\Delta A)^k$ (which satisfies $\widetilde{\mathcal{T}}(Q) = \widetilde{P}$ and $\widetilde{\mathcal{T}}(Q-\Delta Q) = P$) yields $(1-\eta_E)\widetilde{P} \preceq P \preceq (1+\eta_E)\widetilde{P}$, proving the left sandwich in (4). Subtracting P and conjugating by $P^{-1/2}$ yields $-\frac{\eta_E}{1+\eta_E}I \preceq P^{-1/2}(\widetilde{P}-P)P^{-1/2} \preceq \frac{\eta_E}{1-\eta_E}I$, giving the spectral-norm bound. □

3 LQR Suboptimality Deficit, Loewner Concavity, and Riccati Contraction

Consider $x_{k+1} = Ax_k + Bu_k$ ($A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times m}$, $d, m \ge 1$) with cost weights $Q \in \mathbb{S}^d_{++}$, $R \in \mathbb{S}^m_{++}$. For $P \in \mathbb{S}^d_+$ and $K \in \mathbb{R}^{m\times d}$, define the affine Bellman operator, optimal gain, closed-loop matrix, and Riccati operator by

$$\mathcal{L}_K(P) := Q + K^\top RK + (A-BK)^\top P(A-BK), \qquad K(P) := (R+B^\top PB)^{-1}B^\top PA,$$

$$A_{\mathrm{cl}}(P) := A - BK(P), \qquad \mathcal{R}(P) := Q + A^\top PA - A^\top PB(R+B^\top PB)^{-1}B^\top PA.$$

Theorem 3.1 (Completing-the-Square Deficit, Two-Sided DARE Bracket, and A Priori Concavity Rate)**.** (a (Exact Suboptimality Deficit, Loewner Monotonicity, and Concavity.) *For every $P \in \mathbb{S}^d_+$ and $K \in \mathbb{R}^{m\times d}$,*

$$\mathcal{L}_K(P) = \mathcal{R}(P) + \big(K - K(P)\big)^\top (R+B^\top PB)(K-K(P)). \tag{5}$$

Consequently, $\mathcal{R}(P) = \mathcal{L}_{K(P)}(P) = \min_K \mathcal{L}_K(P)$ in the Loewner order, $\mathcal{R}$ is both Loewner-monotone ($0 \preceq P_1 \preceq P_2 \implies \mathcal{R}(P_1) \preceq \mathcal{R}(P_2)$) and Loewner-concave on $\mathbb{S}^d_+$, and for the finite-horizon cost $J_N(x_0;u) := \sum_{k=0}^{N-1}(x_k^\top Qx_k + u_k^\top Ru_k)$ with $P_0 = 0$, $P_{t+1} = \mathcal{R}(P_t)$,

$$J_N(x_0;u) = x_0^\top P_N x_0 + \sum_{k=0}^{N-1}\big(u_k + K(P_{N-1-k})x_k\big)^\top (R+B^\top P_{N-1-k}B)\big(u_k + K(P_{N-1-k})x_k\big). \tag{6}$$

(b) (Two-Sided DARE Bracket and A Priori Concavity Contraction Rate.) *Suppose (A,B) is stabilizable via $K_0 \in \mathbb{R}^{m\times d}$ ($\rho(A-BK_0) < 1$), and let $P_{K_0} \in \mathbb{S}^d_{++}$ solve $(A-BK_0)^\top P_{K_0}(A-BK_0) - P_{K_0} = -(Q+K_0^\top RK_0)$. Define the lower sequence $P_0 := 0$, $P_{t+1} := \mathcal{R}(P_t)$ and upper sequence $\overline{P}_0 := P_{K_0}$, $\overline{P}_{t+1} := \mathcal{R}(\overline{P}_t)$. Then $\mathcal{R}$ admits a unique fixed point $P^* \in \mathbb{S}^d_+$, satisfying $P^* \succeq Q \succ 0$ and $\rho(A^*_{\mathrm{cl}}) < 1$ ($A^*_{\mathrm{cl}} := A - BK^*$, $K^* := K(P^*)$), and for every integer $t \ge 0$:*

$$0 \preceq P_t \preceq P_{t+1} \preceq P^* \preceq \overline{P}_{t+1} \preceq \overline{P}_t \preceq P_{K_0}, \tag{7}$$

$$\big((A^*_{\mathrm{cl}})^t\big)^\top P^*(A^*_{\mathrm{cl}})^t \preceq P^* - P_t \preceq (1-\alpha)^t P^* \preceq (1-\underline{\alpha})^t P_{K_0}, \tag{8}$$

$$0 \preceq \overline{P}_t - P^* \preceq \big((A^*_{\mathrm{cl}})^t\big)^\top (P_{K_0} - P^*)(A^*_{\mathrm{cl}})^t \preceq \gamma_*^t c_{\mathrm{hi}}\, P^*, \tag{9}$$

where $\alpha := \lambda_{\min}((P^*)^{-1/2}Q(P^*)^{-1/2}) \ge \underline{\alpha} := \lambda_{\min}(P_{K_0}^{-1/2}QP_{K_0}^{-1/2}) \in (0,1]$, $\gamma_* := 1 - \lambda_{\min}((P^*)^{-1/2}(Q+(K^*)^\top RK^*)(P^*)^{-1/2}) \in [0,1)$, and $c_{\mathrm{hi}} := \lambda_{\max}((P^*)^{-1/2}(P_{K_0}-P^*)(P^*)^{-1/2}) \ge 0$.

Proof. Part (a). Set $S := R + B^\top PB \succ 0$ and $K_P := K(P) = S^{-1}B^\top PA$. Expanding $\mathcal{L}_K(P) = Q + A^\top PA + K^\top SK - K^\top B^\top PA - A^\top PBK$ and substituting $B^\top PA = SK_P$ directly yields $\mathcal{L}_K(P) = Q + A^\top PA - K_P^\top SK_P + (K - K_P)^\top S(K - K_P)$, which is (5). For $0 \preceq P_1 \preceq P_2$, (5) gives $\mathcal{R}(P_1) \preceq \mathcal{L}_{K(P_2)}(P_1) \preceq \mathcal{L}_{K(P_2)}(P_2) = \mathcal{R}(P_2)$. For $\theta \in [0,1]$ and $P_\theta := \theta P_1 + (1-\theta)P_2$, affineness of $\mathcal{L}_K$ in P gives $\mathcal{R}(P_\theta) = \mathcal{L}_{K(P_\theta)}(P_\theta) = \theta\mathcal{L}_{K(P_\theta)}(P_1) + (1-\theta)\mathcal{L}_{K(P_\theta)}(P_2) \succeq \theta\mathcal{R}(P_1) + (1-\theta)\mathcal{R}(P_2)$, proving Loewner concavity. Completing the square on each stage $x_k^\top Qx_k + u_k^\top Ru_k + (Ax_k + Bu_k)^\top P_{N-1-k}(Ax_k + Bu_k) = x_k^\top P_{N-k}x_k + (u_k + K(P_{N-1-k})x_k)^\top (R + B^\top P_{N-1-k}B)(u_k + K(P_{N-1-k})x_k)$ and telescoping $k = 0, \dots, N-1$ yields (6).

Part (b). Because $P_0 = 0 \preceq P_1 = Q$ and $\mathcal{R}(P_{K_0}) \preceq \mathcal{L}_{K_0}(P_{K_0}) = P_{K_0}$, Loewner monotonicity of $\mathcal{R}$ gives by induction $0 \preceq P_t \preceq P_{t+1} \preceq \overline{P}_{t+1} \preceq \overline{P}_t \preceq P_{K_0}$ for all $t \geq 0$. Every monotone bounded sequence in $\mathbb{S}_+^d$ converges, so $P^* := \lim_{t\to\infty} P_t \in \mathbb{S}_{++}^d$ exists with $Q \preceq P^* \preceq P_{K_0}$ and satisfies $P^* = \mathcal{R}(P^*)$ by continuity of $\mathcal{R}$ on $\mathbb{S}_+^d$ [4]. By (5), $(A_{\mathrm{cl}}^*)^\top P^* A_{\mathrm{cl}}^* - P^* = -(Q + (K^*)^\top RK^*) \preceq -Q \prec 0$, so Theorem 2.1(a) immediately implies $\rho(A_{\mathrm{cl}}^*) < 1$. If $\widetilde{P} \in \mathbb{S}_+^d$ is any fixed point of $\mathcal{R}$, then $0 \preceq \widetilde{P} \implies P_t = \mathcal{R}^t(0) \preceq \widetilde{P} \implies P^* \preceq \widetilde{P}$; conversely, $\widetilde{P} = \mathcal{R}^t(\widetilde{P}) \preceq \mathcal{L}_{K^*}^t(\widetilde{P}) \to P^*$ as $t \to \infty$ (since $\rho(A_{\mathrm{cl}}^*) < 1$ makes the affine map $\mathcal{L}_{K^*}$ converge to its unique fixed point P^*), forcing $\widetilde{P} = P^*$. Monotonicity from $P^* = \mathcal{R}(P^*)$ gives (7).

For the left side of (8), $P^* - P_0 = P^*$ and $P^* - P_{t+1} = \mathcal{L}_{K^*}(P^*) - \mathcal{R}(P_t) \succeq \mathcal{L}_{K^*}(P^*) - \mathcal{L}_{K^*}(P_t) = (A_{\mathrm{cl}}^*)^\top (P^* - P_t)A_{\mathrm{cl}}^*$, yielding $((A_{\mathrm{cl}}^*)^t)^\top P^*(A_{\mathrm{cl}}^*)^t \preceq P^* - P_t$ by induction. For the concavity upper bound in (8), note $\mathcal{R}(0) = Q \succeq \alpha P^*$ ($\alpha \in (0,1]$). Induct on t that $P_t \succeq (1 - (1-\alpha)^t)P^*$: base $t = 0$ holds since $P_0 = 0$; for the step, letting $\beta_t := (1-\alpha)^t \in [0,1]$, monotonicity and Loewner concavity of $\mathcal{R}$ give

$$P_{t+1} = \mathcal{R}(P_t) \succeq \mathcal{R}\big((1-\beta_t)P^* + \beta_t \cdot 0\big) \succeq (1-\beta_t)\mathcal{R}(P^*) + \beta_t\mathcal{R}(0) = (1-\beta_t)P^* + \beta_t Q \succeq \big(1 - (1-\alpha)\beta_t\big)P^*.$$

Rearranging gives $P^* - P_t \preceq (1-\alpha)^t P^*$. Because $P^* \preceq P_{K_0}$, $Q^{-1/2}P^*Q^{-1/2} \preceq Q^{-1/2}P_{K_0}Q^{-1/2}$, so $\alpha^{-1} = \lambda_{\max}(Q^{-1/2}P^*Q^{-1/2}) \leq \lambda_{\max}(Q^{-1/2}P_{K_0}Q^{-1/2}) = \underline{\alpha}^{-1}$, giving $\underline{\alpha} \leq \alpha$ and $(1-\alpha)^t P^* \preceq (1-\underline{\alpha})^t P_{K_0}$. Finally, $\overline{P}_{t+1} - P^* = \mathcal{R}(\overline{P}_t) - \mathcal{L}_{K^*}(P^*) \preceq \mathcal{L}_{K^*}(\overline{P}_t) - \mathcal{L}_{K^*}(P^*) = (A_{\mathrm{cl}}^*)^\top (\overline{P}_t - P^*)A_{\mathrm{cl}}^*$; induction gives the middle inequality in (9), and combining $P_{K_0} - P^* \preceq c_{\mathrm{hi}} P^*$ with $(A_{\mathrm{cl}}^*)^\top P^* A_{\mathrm{cl}}^* \preceq \gamma_* P^*$ (Theorem 2.1(a)) yields the rightmost bound. □

4 Global Explicit-Constant Quadratic Kleinman–Hewer Policy Iteration

Starting from any stabilizing gain $K_0 \in \mathbb{R}^{m\times d}$ ($\rho(A - BK_0) < 1$), discrete Kleinman–Hewer policy iteration [3] generates $(X_j, K_{j+1})_{j\geq 0}$ via:

$$A_j^\top X_j A_j - X_j = -(Q + K_j^\top RK_j), \qquad K_{j+1} := (R + B^\top X_j B)^{-1} B^\top X_j A, \tag{10}$$

where $A_j := A - BK_j$.

Theorem 4.1 (Exact Algebraic Gain-Error Identity and Global Quadratic Contraction)**.** *Under $Q \in \mathbb{S}_{++}^d$, $R \in \mathbb{S}_{++}^m$, and $\rho(A - BK_0) < 1$, set $S_j := R + B^\top X_j B$ and $\Delta K_j := K_j - K_{j+1}$. For every integer $j \geq 0$:*

(a) (Stability Preservation, Monotonicity, and Exact Error Identities.) *The improved closed-loop matrix $A_{j+1} = A - BK_{j+1}$ is strictly Schur-stable ($\rho(A_{j+1}) < 1$), $P^* \preceq X_{j+1} \preceq X_j$, and the following three exact identities hold:*

$$A_{j+1}^\top (X_j - X_{j+1})A_{j+1} - (X_j - X_{j+1}) = -\Delta K_j^\top S_j \Delta K_j, \tag{11}$$

$$K_{j+1} - K^* = (R + B^\top X_j B)^{-1} B^\top (X_j - P^*)A_{\mathrm{cl}}^*, \tag{12}$$

$$A_{j+1}^\top (X_{j+1} - P^*)A_{j+1} - (X_{j+1} - P^*) = -(K_{j+1} - K^*)^\top (R + B^\top P^* B)(K_{j+1} - K^*). \tag{13}$$

(b) (Explicit Global Non-Asymptotic Quadratic Contraction from $j = 0$.) *For every $j \geq 0$ (without any local-neighborhood hypothesis),*

$$0 \preceq X_{j+1} - P^* \preceq \frac{\|A_{\mathrm{cl}}^*\|_2^2 \, \|BR^{-1}B^\top\|_2}{\lambda_{\min}(Q)} \, \|X_j - P^*\|_2^2 \, X_j, \tag{14}$$

and in spectral norm, $\|X_{j+1} - P^*\|_2 \leq \frac{\lambda_{\max}(X_0) \, \|A_{\mathrm{cl}}^*\|_2^2 \, \|BR^{-1}B^\top\|_2}{\lambda_{\min}(Q)} \, \|X_j - P^*\|_2^2$.

Proof. Part (a). We verify stability and $X_j \succ 0$ inductively. Given $\rho(A_j) < 1$, Theorem 2.1(a) yields a unique $X_j \succ 0$ satisfying $\mathcal{L}_{K_j}(X_j) = X_j$. Because $K_{j+1} = K(X_j)$, (5) gives

$$X_j = \mathcal{L}_{K_j}(X_j) = \mathcal{L}_{K_{j+1}}(X_j) + \Delta K_j^\top S_j \Delta K_j = Q + K_{j+1}^\top R K_{j+1} + A_{j+1}^\top X_j A_{j+1} + \Delta K_j^\top S_j \Delta K_j.$$

Rearranging gives

$$A_{j+1}^\top X_j A_{j+1} - X_j = -(Q + K_{j+1}^\top R K_{j+1} + \Delta K_j^\top S_j \Delta K_j) \preceq -Q \prec 0, \tag{15}$$

so Theorem 2.1(a) proves $\rho(A_{j+1}) < 1$ and existence of $X_{j+1} \succ 0$ solving $A_{j+1}^\top X_{j+1} A_{j+1} - X_{j+1} = -(Q + K_{j+1}^\top R K_{j+1})$. Subtracting this from (15) yields (11), whose Lyapunov series immediately proves $X_j - X_{j+1} \succeq 0$. Next, applying (5) at (P^*, K_{j+1}) gives $Q + K_{j+1}^\top R K_{j+1} + A_{j+1}^\top P^* A_{j+1} = P^* + (K_{j+1} - K^*)^\top (R + B^\top P^* B)(K_{j+1} - K^*)$; subtracting the defining Lyapunov equation for X_{j+1} proves (13), whose Lyapunov series immediately proves $X_{j+1} - P^* \succeq 0$ (and at $j = 0$, $X_0 = P_{K_0} \succeq P^*$ by Theorem 3.1(b)). For (12), using $S_j K_{j+1} = B^\top X_j A$ and $(R + B^\top P^* B) K^* = B^\top P^* A$,

$$S_j(K_{j+1} - K^*) = B^\top X_j A - (R + B^\top X_j B) K^* = B^\top (X_j - P^*) A - B^\top (X_j - P^*) B K^* = B^\top (X_j - P^*) A_{\mathrm{cl}}^*,$$

and multiplying by S_j^{-1} establishes (12).

Part (b). Let $\mathcal{T}_{j+1}(Y) := \sum_{k=0}^{\infty} (A_{j+1}^\top)^k Y A_{j+1}^k$ be the Loewner-monotone Lyapunov operator of A_{j+1}. Applying $\mathcal{T}_{j+1}$ to (15) yields $\mathcal{T}_{j+1}(Q) \preceq X_j$. Setting $Y_j := (K_{j+1} - K^*)^\top (R + B^\top P^* B)(K_{j+1} - K^*) \succeq 0$, since $P^* \preceq X_j$ implies $R + B^\top P^* B \preceq S_j$, substituting (12) gives

$$Y_j \preceq (K_{j+1} - K^*)^\top S_j (K_{j+1} - K^*) = (A_{\mathrm{cl}}^*)^\top (X_j - P^*) B S_j^{-1} B^\top (X_j - P^*) A_{\mathrm{cl}}^*.$$

Because $S_j = R + B^\top X_j B \succeq R \succ 0$ implies $S_j^{-1} \preceq R^{-1}$ and $B S_j^{-1} B^\top \preceq B R^{-1} B^\top$, we obtain $\|Y_j\|_2 \leq \|A_{\mathrm{cl}}^*\|_2^2 \|BR^{-1}B^\top\|_2 \|X_j - P^*\|_2^2$, whence $Y_j \preceq \frac{\|Y_j\|_2}{\lambda_{\min}(Q)} Q$. Applying $\mathcal{T}_{j+1}$ to (13) gives $0 \preceq X_{j+1} - P^* = \mathcal{T}_{j+1}(Y_j) \preceq \frac{\|Y_j\|_2}{\lambda_{\min}(Q)} \mathcal{T}_{j+1}(Q) \preceq \frac{\|Y_j\|_2}{\lambda_{\min}(Q)} X_j$, proving (14). Since $X_j \preceq X_0$, taking spectral norms gives the scalar bound. □

5 Controllability Gramian Steering and Gain-Free Deadbeat DARE Ceiling

For any integer $N \geq 1$, define the N-step controllability matrix $\mathcal{C}_N := [A^{N-1}B, \ldots, AB, B] \in \mathbb{R}^{d \times Nm}$ and controllability Gramian $W_N := \mathcal{C}_N \mathcal{C}_N^\top = \sum_{j=0}^{N-1} A^j B B^\top (A^\top)^j \in \mathbb{S}_+^d$.

Theorem 5.1 (Minimum-Energy Gramian Steering and Gain-Free Closed-Form DARE Ceiling)**.** *Suppose (A, B) is N-step controllable ($W_N \in \mathbb{S}_{++}^d$) for some integer $N \geq 1$, and let $Q \in \mathbb{S}_{++}^d$, $R \in \mathbb{S}_{++}^m$.*

(a) (Exact Minimum-Energy Steering Identity.) *For every $x_0, x_{\mathrm{f}} \in \mathbb{R}^d$, the unique control sequence $u^\star = (u_0^\star, \ldots, u_{N-1}^\star)$ minimizing $\sum_{k=0}^{N-1} \|u_k\|_2^2$ subject to $x_N = x_{\mathrm{f}}$ is*

$$u_k^\star = B^\top (A^\top)^{N-1-k} W_N^{-1} (x_{\mathrm{f}} - A^N x_0) \quad (0 \leq k \leq N-1), \qquad \sum_{k=0}^{N-1} \|u_k^\star\|_2^2 = (x_{\mathrm{f}} - A^N x_0)^\top W_N^{-1} (x_{\mathrm{f}} - A^N x_0). \tag{16}$$

(b) (Gain-Free Closed-Form Deadbeat Loewner Ceiling on P^*.) *Without requiring any prior stabilizing feedback gain K_0, the unique DARE solution $P^* \in \mathbb{S}^d_{++}$ satisfies*

$$P^* \preceq Q + 2\sum_{k=1}^{N-1}(A^k)^\top Q A^k + \left(\lambda_{\max}(R) + 2\lambda_{\max}(Q)\sum_{k=1}^{N-1}\lambda_{\max}(W_k)\right)(A^N)^\top W_N^{-1} A^N, \quad (17)$$

where empty sums vanish at $N = 1$ (yielding $P^* \preceq Q + \lambda_{\max}(R)A^\top(BB^\top)^{-1}A$, attained when B is square invertible, $R = rI$, and $A^\top P^* A \gg Q$ in the $r \to 0^+$ deadbeat limit, and exact at $A = 0$ where $P^* = Q$).

Proof. Part (a). Stacking $u = (u_0^\top, \dots, u_{N-1}^\top)^\top \in \mathbb{R}^{Nm}$, the terminal constraint $x_N = x_{\mathrm{f}}$ is $\mathcal{C}_N u = \delta := x_{\mathrm{f}} - A^N x_0$. The vector $u^\star := \mathcal{C}_N^\top W_N^{-1}\delta$ satisfies $\mathcal{C}_N u^\star = W_N W_N^{-1}\delta = \delta$ and $\|u^\star\|_2^2 = \delta^\top W_N^{-1}\delta$. For any other feasible u, write $w := u - u^\star \in \ker(\mathcal{C}_N)$; then $(u^\star)^\top w = \delta^\top W_N^{-1}\mathcal{C}_N w = 0$, so $\|u\|_2^2 = \|u^\star\|_2^2 + \|w\|_2^2 \geq \|u^\star\|_2^2$, with equality iff $w = 0$.

Part (b). Fix any $x_0 \in \mathbb{R}^d$ and steer to $x_{\mathrm{f}} = 0$ in N steps via (16), then set $u_k^\star := 0$ (hence $x_k = 0$) for all $k \geq N$. First bound $J_N(x_0; u^\star)$: the control cost obeys $\sum_{k=0}^{N-1}(u_k^\star)^\top R u_k^\star \leq \lambda_{\max}(R)\|u^\star\|_2^2 = \lambda_{\max}(R)\, x_0^\top (A^N)^\top W_N^{-1} A^N x_0$. At $k = 0$, the state cost is $x_0^\top Q x_0$. For each $1 \leq k \leq N-1$, $x_k = A^k x_0 + v_k$ where $v_k := \mathcal{C}_k u^\star_{0:k-1}$ obeys $\|v_k\|_2^2 \leq \|\mathcal{C}_k\|_2^2 \|u^\star_{0:k-1}\|_2^2 \leq \lambda_{\max}(W_k)\|u^\star\|_2^2$. Since $(a+b)^\top Q(a+b) \leq 2a^\top Q a + 2b^\top Q b$ (their difference is $(a-b)^\top Q(a-b) \geq 0$),

$$x_k^\top Q x_k \leq 2x_0^\top (A^k)^\top Q A^k x_0 + 2\lambda_{\max}(Q)\lambda_{\max}(W_k)\|u^\star\|_2^2.$$

Summing $k = 0, \dots, N-1$ and adding the control cost yields $J_N(x_0; u^\star) \leq x_0^\top C_N x_0$, where $C_N \in \mathbb{S}^d_{++}$ denotes the right-hand side of (17). For every horizon $M \geq N$, Theorem 3.1(a) gives $x_0^\top P_M x_0 \leq J_M(x_0; u^\star) = J_N(x_0; u^\star) \leq x_0^\top C_N x_0$ for all $x_0 \in \mathbb{R}^d$, hence $0 \preceq P_M \preceq P_{M+1} \preceq C_N$. Therefore the monotone bounded sequence $\{P_M\}$ converges to a fixed point $P^* = \mathcal{R}(P^*) \in \mathbb{S}^d_{++}$ satisfying $Q \preceq P^* \preceq C_N$; by (5) and Theorem 2.1(a), $\rho(A^*_{\mathrm{cl}}) < 1$ (certifying stabilizability directly without any prior gain K_0), and uniqueness of $P^* \in \mathbb{S}^d_+$ follows identically to Theorem 3.1(b). □

6 Computational Verification and Numerical Illustrations

All four theorems were verified deterministically in IEEE-754 `float64` arithmetic via the MIT-licensed companion repository `lluiseriksson/lqr-riccati-loewner-geometry`. Across 99 structured checks (plus 4 randomized multi-dimension pytest suites across $d \in \{2, 3, 4, 5\}$, seeds `20261129`–`20261202`)—covering 25 perturbation magnitudes up to 85% of the P-metric radius $1 - \sqrt{\gamma}$, 25 iterations of lower/upper Riccati brackets on an open-loop unstable 3-state system ($\rho(A) = 1.08$, $\|P^*\|_2 = 21.95$), 6 Kleinman–Hewer policy iterations converging from $\|X_0 - P^*\|_2 = 11.83$ to 4.86×10^{-14} in 4 steps, and gain-free deadbeat Gramian ceilings across $N \in \{3, \dots, 8\}$—Table 1 and Figures 1–2 confirm zero violations.

Table 1: Deterministic verification across 99 benchmark checks (`pytest -q`: 4/4 passed in 1.13 s). All matrix identities hold to machine precision (10^{-16}–10^{-14}) and all Loewner inequalities hold with zero violations.

Theorem	Checks	Max Identity / Exactness Error	Tightest Observed Metric / Convergence
Thm. 2.1 (Lyapunov Sensitivity)	25	0 Loewner violations up to $0.85(1-\sqrt{\gamma})$	$\max(\text{obs. err}/(\eta_E/(1-\eta_E))) = 0.0303 < 1$
Thm. 3.1 (Riccati Concavity)	50	0 violations in $P_t \preceq P^* \preceq \overline{P}_t$	$\|\underline{P}^* - P_{24}\|_2 = 1.06\times10^{-4}$, $\|\overline{P}_{24} - P^*\|_2 = 5.97\times10^{-5}$
Thm. 4.1 (Kleinman–Hewer)	18	5.18×10^{-16} (gain id.), 3.93×10^{-14} (Lyap.)	$\|X_0 - P^*\|_2 = 11.83 \to \|X_4 - P^*\|_2 = 4.86\times10^{-14}$
Thm. 5.1 (Gramian Ceiling)	6	0 Loewner violations ($3 \leq N \leq 8$)	$\min \lambda_{\min}(C_N - P^*) = 2.723 > 0$ ($\|P^*\|_2 = 21.95$)

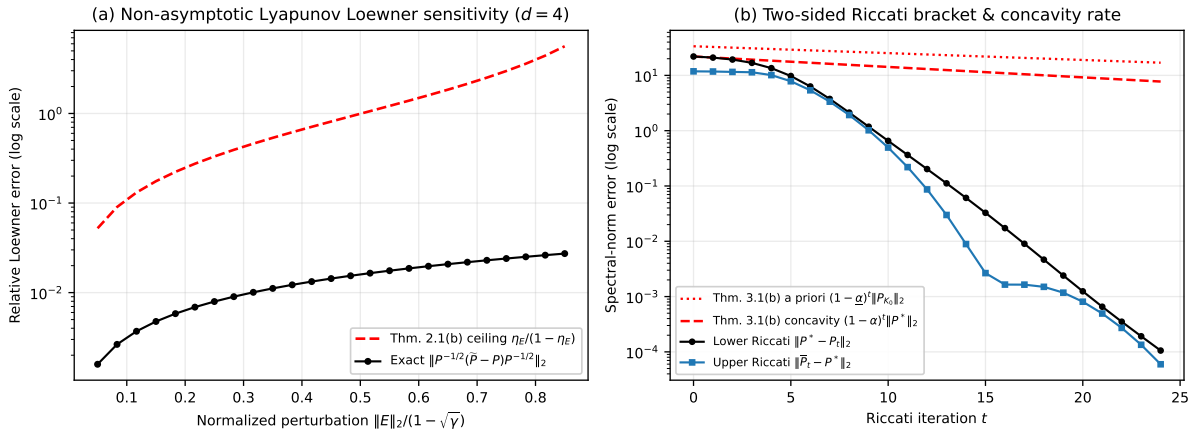

Figure 1: **(a)** Non-asymptotic Lyapunov Loewner sensitivity bound $\eta_E/(1-\eta_E)$ (4) versus observed relative error across $\|E\|_2/(1-\sqrt{\gamma}) \in [0.05, 0.85]$. **(b)** Lower and upper Riccati iterates bracketing P^* alongside the concavity contraction rates $(1-\alpha)^t\|P^*\|_2$ and a priori $(1-\underline{\alpha})^t\|P_{K_0}\|_2$ (8).

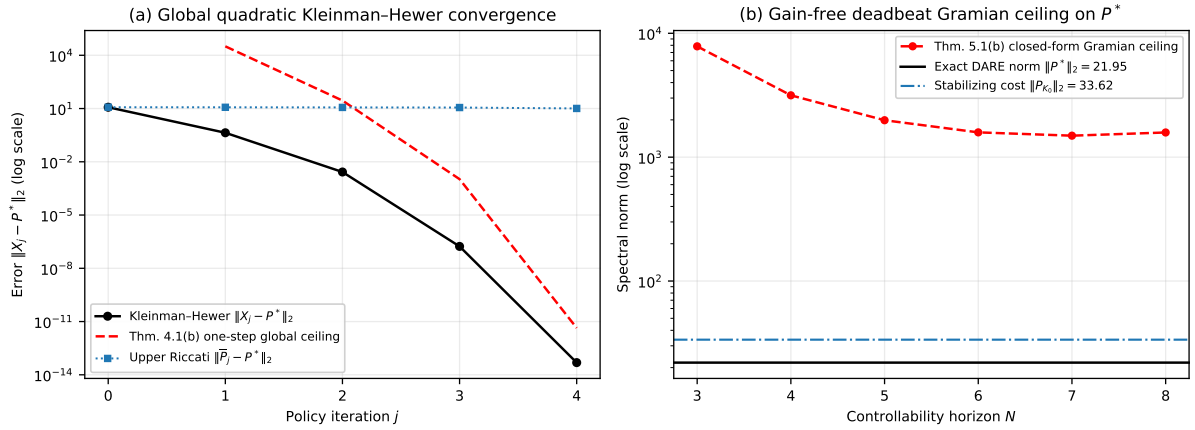

Figure 2: **(a)** Global quadratic Kleinman–Hewer convergence (14) reaching 4.86×10^{-14} in 4 iterations versus upper Riccati value iteration. **(b)** Closed-form gain-free deadbeat Gramian ceilings (17) across horizons $N \in \{3, \ldots, 8\}$ strictly enclosing $\|P^*\|_2 = 21.95$.

7 Scope and Limitations

Three structural boundaries merit explicit note:

1. *Strict positive definiteness of Q.* Theorems 2.1–5.1 assume $Q \succ 0$, which directly furnishes $\alpha > 0$ and $\lambda_{\min}(Q) > 0$; extending to detectable pairs $(A, Q^{1/2})$ with singular $Q \succeq 0$ requires replacing single-step Q by an N-step observability Gramian $\sum_{k=0}^{N-1} (A^k)^\top Q A^k \succ 0$.
2. *Conservatism of worst-case operator norms.* The perturbation ceiling $\eta_E/(1-\eta_E)$ in (4) and the deadbeat Gramian ceiling (17) (which grows with $\|A^N\|_2^2$ on unstable modes) trade tightness for complete closed-form generality; structured perturbations or optimal finite-horizon steering achieve sharper constants.
3. *Exact system matrices and deterministic dynamics.* The bounds treat noiseless linear dynamics with exact (A, B, Q, R); sample-complexity analyses for unknown systems combine Theorems 2.1 and 4.1 with finite-sample system identification error bounds [5].

References

[1] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, Vol. I, 4th ed., Athena Scientific, Belmont, MA, 2017.

[2] M. Fazel, R. Ge, S. M. Kakade, and M. Mesbahi, *Global convergence of policy gradient methods for the linear quadratic regulator*, Proc. ICML **80** (2018), 1467–1476.

[3] G. A. Hewer, *An iterative technique for the computation of the steady state gains for the discrete optimal regulator*, IEEE Trans. Automat. Control **16** (1971), 382–384.

[4] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*, Oxford University Press, Oxford, 1995.

[5] H. Mania, S. Tu, and B. Recht, *Certainty equivalence is efficient for linear quadratic control*, Adv. Neural Inf. Process. Syst. **32** (2019), 10154–10164.