

Rank-two obstructions to Lorentzian graph partition functions

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Abstract

We give rank-two antiferromagnetic obstructions to the Lorentzian property of graph partition functions. A connected noncomplete source of maximum degree Δ has a strictly positive rational obstruction on $\Delta + 1$ spins whenever no two maximum-degree vertices have equal closed neighbourhoods. Consequently, the connected-source characterization holds for graphs without true twins, graphs with a unique maximum-degree vertex, and all connected edge-regular graphs, including strongly regular graphs. The proof uses a small-parameter limit of explicit conditioned-set Hessian kernels. A different endpoint yields a neighbourhood-independence-gap criterion. One fixed nonnegative four-spin matrix characterizes cliques among connected diamond-free sources and among connected complete multipartite sources. A fixed five-spin matrix does the same for joins of a nonempty clique with disjoint nonempty cliques. Positive rational perturbations exist for each source, but may depend on it; for triangle-free sources a single strictly positive three-spin matrix suffices. Further obstructions use degree-class adjacency and triangle counts. Exact rational certificates cover all 989 connected noncliques of orders three through seven. The maximum-degree theorem has 821 exact checks, including 819 direct partition-coefficient comparisons. An explicit ten-vertex graph proves that the small-parameter criterion alone is incomplete, while the full hierarchy detects it. These results are partial progress toward the connected-source conjecture of Lee–Oh–Seo; the conjecture and completeness of the full hierarchy remain unproved.

1. Problem and scope

For a finite simple graph $G = (V, \mathcal{E})$ and an entrywise nonnegative symmetric $q \times q$ matrix H , write

$$Z_G(\mathbf{x}; H) = \sum_{\sigma: V \rightarrow \{0, \dots, q-1\}} \prod_{uv \in \mathcal{E}} H_{\sigma(u), \sigma(v)} \prod_{v \in V} x_{\sigma(v)}. \quad (1)$$

It is homogeneous of degree $n = |V|$. We call H *antiferromagnetic* when it has at most one positive eigenvalue, counting multiplicity. This is the spectral convention used in [2, 3].

Lee–Oh–Seo prove that Z_G is Lorentzian for every antiferromagnetic H when G is a clique [2, Theorem 3.1]. They conjecture a converse; the connected-source formulation is stated explicitly in [3, Section 6]. We use that formulation:

If G is connected and $Z_G(\mathbf{x}; H)$ is Lorentzian for every antiferromagnetic H , must G be a clique?

The restriction to connected graphs matters, since partition functions factor over components and Lorentzian polynomials are closed under products. The present paper establishes obstructions for several classes of source graphs, using only matrices of rank two.

These questions concern all antiferromagnetic targets, rather than only proper-colouring targets. Matherne–Morales–Selover establish Lorentzianity of chromatic symmetric functions for abelian Dyck paths [4, Theorem 1.5]. Lee–Oh–Seo already give a non-Lorentzian three-vertex-path example with target $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, as well as a $K_{3,3}$ example [2, Section 7]. We do not claim those small obstructions as new. A research note by Zeng [5] gives a fixed rank-three, three-spin obstruction for bistable bipartite graphs with a two-vertex cut. Our triangle-free result uses a strictly positive rank-two target and requires neither bistability nor a two-vertex cut; it does not claim Zeng’s coefficient formula relating the obstruction to vertex connectivity. Our proposed contributions are a maximum-degree criterion excluding true twins at that degree (Theorem 14), the resulting obstructions for all connected noncomplete edge-regular graphs, and fixed targets for triangle-free, connected diamond-free and complete multipartite sources, a fixed five-spin target for joins of a clique with disjoint cliques, the neighbourhood-independence-gap criterion, the degree-count identity and its compression certificate, and the parameterized conditioned-set identities below. A diamond means an induced copy of K_4 with one edge removed. The four-spin target for this class has one zero entry; the positive perturbations are allowed to depend on the source graph.

The arguments are elementary counting and real symmetric linear algebra. Their computational corroboration uses exact arithmetic. A targeted prior-art search does not certify worldwide priority; a separate referee assessment of this draft remains pending.

2. A rank-two family and the Hessian test

We use a necessary consequence of the definition of Lorentzian polynomials [1]. If a homogeneous polynomial f of degree $n \geq 2$ with nonnegative coefficients is Lorentzian, then

$$\text{Hess}(\partial^\alpha f) \text{ has at most one positive eigenvalue for every } \alpha \in \mathbb{N}_0^q, \quad |\alpha| = n - 2. \quad (2)$$

The derivative is quadratic, so its Hessian is constant. We need only this necessary condition for the general obstructions. For the finite fixed-target limits reported below, we also use the full characterization: nonnegative homogeneous polynomials with M-convex support are Lorentzian precisely when all these quadratic derivatives satisfy the Hessian condition [1, Theorem 2.25].

Lemma 1. *For a nonconstant vector $\mathbf{t} \in (0, \infty)^q$, the matrix*

$$H(\mathbf{t})_{ij} = t_i + t_j \quad (3)$$

is positive entrywise and has inertia $(1, 1, q - 2)$, where inertia lists positive, negative and zero eigenvalues.

Proof. Write $H = \mathbf{1}\mathbf{t}^\top + \mathbf{t}\mathbf{1}^\top$. Its rank is at most two, and its quadratic form vanishes on $\mathbf{1}^\perp$. A positive definite two-dimensional subspace would intersect $\mathbf{1}^\perp$ nontrivially, which is impossible. For $t_i \neq t_j$, the corresponding principal determinant is $4t_it_j - (t_i + t_j)^2 = -(t_i - t_j)^2 < 0$. Thus there are a positive and a negative eigenvalue, and rank is exactly two. $\square$

For integer $r \geq 0$, introduce

$$z_i = \frac{1 + t_i}{2}, \quad k_i = \frac{t_i - 1}{t_i + 1}, \quad k_i z_i^r = z_i^r - z_i^{r-1} \quad (r \geq 1). \quad (4)$$

A baseline spin of value $t = 1$ has self-interaction 2. If two free vertices are adjacent, the correction to their separate baseline changes is

$$\frac{2(t_i + t_j)}{(1 + t_i)(1 + t_j)} = 1 - k_i k_j. \quad (5)$$

All denominators in these identities are positive.

3. A fixed target for all triangle-free sources

It is useful to normalize the interaction with the baseline spin. For distinct $a_i \in (-1, 1)$, set

$$K(\mathbf{a}) = \mathbf{1}\mathbf{1}^\top - \mathbf{a}\mathbf{a}^\top, \quad K_{ij} = 1 - a_i a_j. \quad (6)$$

This matrix is positive entrywise and has rank two and exactly one positive eigenvalue. Indeed, its quadratic form is nonpositive on $\mathbf{1}^\perp$, and a principal 2×2 determinant for $a_i \neq a_j$ equals $-(a_i - a_j)^2 < 0$.

For a graph with $n \geq 3$ vertices, m edges, degrees d_v and T triangles, abbreviate

$$N = n(n-1)(n-2), \quad E = 2m(n-2), \quad W = \sum_v d_v(d_v-1), \quad U = 6T. \quad (7)$$

Here W counts ordered wedges, including those inside triangles. Use $\mathbf{a} = (\ell, 0, 1/2)$, with $-1 < \ell < 0$, and let $V_i = (1, a_i, a_i^2)$.

Theorem 2 (Edge–wedge–triangle identity). *For every finite simple graph G on $n \geq 3$ vertices,*

$$\text{Hess}(\partial_0 \partial_1^{n-3} Z_G(\mathbf{x}; K(\mathbf{a}))) = (n-3)! V \hat{B}_\ell V^\top, \quad (8)$$

$$\hat{B}_\ell = \begin{pmatrix} N & -\ell E & 0 \\ -\ell E & \ell^2 W - E & \ell W \\ 0 & \ell W & -\ell^2 U \end{pmatrix}, \quad (9)$$

$$\det \hat{B}_\ell = \ell^2 [N(EU - W^2) + \ell^2 U(E^2 - NW)]. \quad (10)$$

If $W^2 > EU$, a positive rank-two three-spin target makes Z_G non-Lorentzian.

Proof. Place the conditioned spin 0 at v and baseline spin 1, with $a_1 = 0$, everywhere else. Every initial edge weight is one. After assigning free spins i, j to distinct vertices $u, w \neq v$, the exact edge product is

$$(1 - \ell a_i A_{uv})(1 - \ell a_j A_{vw})(1 - a_i a_j A_{uw}).$$

Summing over ordered distinct triples gives counts N for the constant term, E for every single-edge indicator, W for each two-edge product, and U for the three-edge product. This is exactly $V \hat{B}_\ell V^\top$. The derivative multiplicity is $(n-3)!$. Expanding the 3×3 determinant proves (10).

If $A_* = N(W^2 - EU) > 0$ and $R_* = U(E^2 - NW)$, choose $L = 2 + \lceil \max(R_*, 0)/A_* \rceil$ and $\ell = -1/L$. Then $\ell^2 R_* < A_*$, so the determinant is negative. The entry $(\hat{B}_\ell)_{00} = N$ is positive, and hence its inertia is $(2, 1, 0)$. The Vandermonde matrix V is invertible, so (2) proves the conclusion. $\square$

Corollary 3 (One fixed positive target). *Let*

$$K_* = \frac{1}{4} \begin{pmatrix} 3 & 4 & 5 \\ 4 & 4 & 4 \\ 5 & 4 & 3 \end{pmatrix}. \quad (11)$$

For a triangle-free finite simple graph G , $Z_G(\mathbf{x}; K_*)$ is Lorentzian if and only if every component of G is an isolated vertex or a single edge. In particular, every connected triangle-free graph on at least three vertices fails the universal-target property.

Proof. The target is $K(-1/2, 0, 1/2)$ and has eigenvalues $3, -1/2, 0$. If G contains two adjacent edges, then $n \geq 3$, $W > 0$ and $U = 0$. Thus (10) becomes $-NW^2/4 < 0$, proving the obstruction for this fixed target. Otherwise every component is a vertex or an edge. Their partition functions are respectively a nonnegative linear form and a quadratic form with Hessian $2K_*$. They are Lorentzian, as is their product [1, Corollary 2.32]. The empty graph, if allowed, contributes the constant one. $\square$

For a regular graph, $E = nd(n-2)$, $W = nd(d-1)$ and $U = n\tau$; the inequality $W^2 > EU$ is exactly (22) below. The general inequality also applies to irregular graphs with triangles.

4. The degree-count kernel

Let $m = |\mathcal{E}|$, let d_v denote vertex degree, and let $\Delta = \max_v d_v \geq 2$. For $0 \leq r \leq \Delta$, let $\nu_r = |\{v : d_v = r\}|$, and put $\boldsymbol{\nu} = (\nu_0, \dots, \nu_\Delta)^\top$. For $1 \leq r, s \leq \Delta$, let

$$C_{rs} = |\{(u, v) : uv \in \mathcal{E}, d_u = r, d_v = s\}|. \quad (12)$$

These are *ordered* edge counts; in particular an edge inside a degree class contributes two to its diagonal entry. Let $e_0, \dots, e_\Delta$ be the standard basis, and let L be the $(\Delta+1) \times \Delta$ matrix with column $L_r = e_r - e_{r-1}$ for $1 \leq r \leq \Delta$. Define the symmetric integer matrix

$$B_G = \boldsymbol{\nu}\boldsymbol{\nu}^\top - \text{diag}(\boldsymbol{\nu}) - LCL^\top. \quad (13)$$

Theorem 4 (Degree-count identity). *Take $q = \Delta+1$, $t_i = i+1$ for $0 \leq i \leq \Delta$, and $H = H(\mathbf{t})$. With $V_{ir} = z_i^r$, one has*

$$\text{Hess}(\partial_0^{n-2} Z_G(\mathbf{x}; H)) = (n-2)! 2^m V B_G V^\top. \quad (14)$$

The matrix V is invertible. Consequently, two positive eigenvalues of B_G certify that $Z_G(\mathbf{x}; H)$ is not Lorentzian.

Proof. The (i, j) entry of the Hessian is $(n-2)!$ times a sum over ordered distinct vertices (u, v) , giving them spins i, j and every other vertex the baseline spin 0, whose value is $t_0 = 1$. This interpretation is valid even if a free spin equals the baseline or the two free spins agree: ordered placements give precisely the falling-factorial multiplicities of differentiation.

The baseline edge product is 2^m . Changing the two free vertices separately gives $z_i^{d_u} z_j^{d_v}$; if $uv \in \mathcal{E}$, (5) supplies the correction. After dividing by $(n-2)! 2^m$, the Hessian entry is therefore

$$\sum_{u \neq v} z_i^{d_u} z_j^{d_v} - \sum_{(u,v): uv \in \mathcal{E}} k_i z_i^{d_u} k_j z_j^{d_v}.$$

The first sum is the (i, j) entry of $V(\boldsymbol{\nu}\boldsymbol{\nu}^\top - \text{diag}(\boldsymbol{\nu}))V^\top$. Using (4), the second is the corresponding entry of $VLCL^\top V^\top$. This proves (14). The $z_i = (i+2)/2$ are distinct, so the square Vandermonde matrix V is invertible. Sylvester's law of inertia and (2) complete the proof. $\square$

Theorem 5 (Independent minimum-degree class). *Suppose G has positive minimum degree δ and its minimum-degree vertices form an independent set. Then B_G has at least two positive eigenvalues. In particular, $Z_G(\mathbf{x}; H(1, 2, \dots, \Delta+1))$ is not Lorentzian.*

Proof. Put $p = \mathbf{1} \in \mathbb{R}^{\Delta+1}$ and $u = e_{\delta-1}$. Since $L^\top p = 0$, $\nu_{\delta-1} = 0$ and $C_{\delta\delta} = 0$,

$$a := p^\top B_G p = n(n-1) > 0, \quad u^\top B_G p = 0, \quad u^\top B_G u = 0. \quad (15)$$

In computing $L^\top u$, a possible component indexed $\delta - 1$ is harmless: that degree class is empty. Hence

$$B_G u = L C e_\delta \neq 0,$$

where on the right e_δ denotes the column basis vector in $\mathbb{R}^\Delta$ indexed by the degree δ . For the inequality, a minimum-degree vertex has a neighbour, and every such neighbour has degree greater than δ . Choose the largest s for which $C_{s\delta} > 0$; the coefficient of e_s in $L C e_\delta$ is then positive.

Set $w = B_G u$ and $c = u^\top B_G w = \|B_G u\|_2^2 > 0$. The compression to the columns (p, u, w) has the form

$$Q = \begin{pmatrix} a & 0 & b \\ 0 & 0 & c \\ b & c & e \end{pmatrix}, \quad \det Q = -ac^2 < 0. \quad (16)$$

Its positive diagonal entry a excludes three negative eigenvalues, so its inertia is $(2, 1, 0)$. In particular its three columns are independent, and its two-dimensional positive subspace gives one for B_G . Apply Theorem 4. $\square$

Corollary 6. *Every connected graph on at least three vertices with a leaf has a positive rank-two antiferromagnetic target for which its partition function is not Lorentzian. This holds for every tree of order at least three.*

Proof. Two degree-one vertices cannot be adjacent in a connected graph of order at least three. Thus its minimum-degree class is independent. Every tree of that order has a leaf. $\square$

The certificate (16) uses integer matrices of size at most $\Delta + 1$, followed by a 3×3 determinant. It does not require expanding the q^n terms in (1).

5. A parameterized obstruction for regular graphs

The preceding criterion is not sufficient for regular graphs. Indeed, for a d -regular graph its kernel is

$$B_G = n(n-1)e_d e_d^\top - nd(e_d - e_{d-1})(e_d - e_{d-1})^\top, \quad (17)$$

which has at most one positive eigenvalue. We therefore use a different quadratic derivative.

Let G be d -regular, with $n \geq 3$ and $d \geq 2$, and let T be its number of unordered triangles. Set $\tau = 6T/n$. Choose $s > 0$, $s \notin \{1, 2\}$, and take three spin values

$$(t_0, t_1, t_2) = (s, 1, 2), \quad \ell = \frac{s-1}{s+1}.$$

Define z_i, k_i by (4), and set

$$V_i = (1, k_i, k_i^2), \quad D_z = \text{diag}(z_0^d, z_1^d, z_2^d).$$

The three k_i are distinct, and D_z is positive and invertible.

Theorem 7 (Regular-graph identity). *For the target $H = H(s, 1, 2)$,*

$$\text{Hess}(\partial_0 \partial_1^{n-3} Z_G(\mathbf{x}; H)) = c_G(s) D_z V B_\ell V^\top D_z, \quad (18)$$

$$c_G(s) = (n-3)! n 2^{m-d} (s+1)^d > 0,$$

$$B_\ell = \begin{pmatrix} (n-1)(n-2) & -\ell d(n-2) & 0 \\ -\ell d(n-2) & \ell^2 d(d-1) - d(n-2) & \ell d(d-1) \\ 0 & \ell d(d-1) & -\ell^2 \tau \end{pmatrix}. \quad (19)$$

Its determinant is

$$\det B_\ell = \ell^2 d(n-2) \left[(n-1)((n-2)\tau - d(d-1)^2) + \ell^2 \tau(n-1-d) \right]. \quad (20)$$

Proof. The Hessian entry is $(n-3)!$ times a sum over ordered distinct vertices (v, u, w) : assign spin 0 to v , free spins i, j to u, w , and baseline spin 1 to all remaining vertices. The initial edge weight, before changing u, w , is $2^{m-d}(s+1)^d$, independent of v . For a free vertex adjacent to v , its change has an additional factor

$$\frac{2(s+t_i)}{(s+1)(1+t_i)} = 1 - \ell k_i.$$

The correction for adjacency of the two free vertices is still (5). Divide out $(n-3)!2^{m-d}(s+1)^d z_i^d z_j^d$. The remaining entry is

$$\sum_{\substack{v,u,w \\ \text{all distinct}}} (1 - \ell k_i A_{uv})(1 - \ell k_j A_{vw})(1 - k_i k_j A_{uw}), \quad (21)$$

where A is the adjacency matrix of G .

In this sum there are $n(n-1)(n-2)$ ordered triples. Each single-edge indicator has sum $nd(n-2)$. Each product of two different edge indicators has sum $nd(d-1)$: choose its common vertex, then two ordered distinct neighbours. The product of all three indicators has sum $6T$. Expanding (21) and dividing by n gives

$$\begin{aligned} & (n-1)(n-2) - \ell d(n-2)(k_i + k_j) \\ & + (\ell^2 d(d-1) - d(n-2))k_i k_j \\ & + \ell d(d-1)(k_i^2 k_j + k_i k_j^2) - \ell^2 \tau k_i^2 k_j^2, \end{aligned}$$

which is $V_i B_\ell V_j^\top$. This proves (18). Expanding the determinant of (19) gives (20). $\square$

Corollary 8 (Triangle threshold). *If*

$$\frac{6T}{n} < \frac{d(d-1)^2}{n-2}, \quad (22)$$

then a positive three-spin rank-two antiferromagnetic target makes Z_G non-Lorentzian. Conversely, if (22) fails, none of the Hessians in Theorem 7 has two positive eigenvalues. This converse concerns only that derivative and target family.

Proof. Write

$$A_* = (n-1)(d(d-1)^2 - (n-2)\tau), \quad R_* = \tau(n-1-d) \geq 0.$$

Under (22), $A_* > 0$. The entirely rational choice

$$K = 2 + \lceil R_*/A_* \rceil, \quad \ell = -1/K, \quad s = (K-1)/(K+1) \quad (23)$$

has $0 < s < 1$ and $\ell^2 R_* < A_*$. Indeed, for $r = R_*/A_* \geq 0$, $(2 + \lceil r \rceil)^2 > r$. Equation (20) is therefore negative. Since $(B_\ell)_{00} > 0$, its inertia is $(2, 1, 0)$. The invertible congruence (18), followed by (2), proves the obstruction.

If (22) fails, then $A_* \leq 0$, $\tau > 0$, and (20) is nonnegative for every allowed ℓ . The diagonal entries $(B_\ell)_{00} > 0$ and $(B_\ell)_{22} = -\ell^2 \tau < 0$ force at least one positive and one negative eigenvalue. A nonnegative determinant then forces the third to be nonpositive. The same holds for the congruent Hessian. $\square$

Corollary 9. *Every regular triangle-free graph of degree at least two has a non-Lorentzian partition function for the fixed positive target $H(1/3, 1, 2)$.*

Proof. For $T = 0$, (20) is negative for every $\ell \neq 0$. Take $\ell = -1/2$, equivalently $s = 1/3$. $\square$

Remark 10. The strict threshold is exact for this one-conditioned-spin family, not an optimal condition among all targets or derivatives. For a clique $d = n-1$ and $\tau = d(d-1)$, the bracket in (20) vanishes identically, consistent with the known Lorentzian result.

6. Conditioning larger vertex sets

The single conditioned vertex can be replaced by a set. This yields an explicit hierarchy of tests; no assertion is made that the hierarchy detects every connected nonclique.

Fix any finite simple graph G , an integer $1 \leq h \leq n - 2$, and $-1 < \ell < 0$. Take $q = h + 2$ distinct values $a_i \in (-1, 1)$ with $a_0 = \ell$, $a_1 = 0$, and the positive target $K(\mathbf{a})$. For $S \subseteq V$, $|S| = h$, write

$$r_S(v) = |N(v) \cap S| \quad (v \notin S), \quad \nu_r^S = |\{v \notin S : r_S(v) = r\}| \quad (0 \leq r \leq h).$$

Let C_{ab}^S count ordered edges of $G[V \setminus S]$ between vertices of profiles a, b , and let $e(S)$ be the number of edges inside S . Define the $(h + 2) \times (h + 1)$ coefficient matrix

$$F_{jr} = \binom{r}{j} (-\ell)^j \quad (0 \leq j \leq h + 1, 0 \leq r \leq h),$$

where $\binom{r}{j} = 0$ for $j > r$. Let J be the $(h + 2) \times (h + 2)$ matrix that shifts coordinates down by one, with zeros in the first row. Set

$$\begin{aligned} \mathcal{B}_h(\ell) = \sum_{|S|=h} (1 - \ell^2)^{e(S)} [F(\nu^S (\nu^S)^\top - \text{diag}(\nu^S)) F^\top \\ - (JF) C^S (JF)^\top]. \end{aligned} \quad (24)$$

All matrices in the brackets have size $h + 2$.

Theorem 11 (Conditioned-set identity). *With $V_{ij} = a_i^j$ for $0 \leq i, j \leq h + 1$, one has*

$$\begin{aligned} \text{Hess}(\partial_0^h \partial_1^{n-h-2} Z_G(\mathbf{x}; K(\mathbf{a}))) &= c_h V \mathcal{B}_h(\ell) V^\top, \\ c_h &= h!(n - h - 2)! > 0. \end{aligned} \quad (25)$$

Thus $\mathcal{B}_h(\ell)$ and the quadratic-derivative Hessian have the same inertia. For $h = 1$, $\mathcal{B}_1(\ell) = \hat{B}_\ell$; for a d -regular graph this is nB_ℓ .

Proof. Assign spin 0 to S and baseline spin 1 to its complement. All edges outside $G[S]$ have weight one, and its $e(S)$ internal edges have weight $1 - \ell^2$. The baseline product is therefore $(1 - \ell^2)^{e(S)}$. Changing a free vertex $u \notin S$ to spin i multiplies its separate baseline contribution by $(1 - \ell a_i)^{r_S(u)}$. For two adjacent free vertices the additional correcting factor is $1 - a_i a_j$, as before.

The coefficient vector of $(1 - \ell k)^r$ in powers of k is the r th column of F ; the vector for $k(1 - \ell k)^r$ is the corresponding column of JF . Summing over distinct free vertices first gives the first matrix in (24), and subtracting the adjacency correction gives the second. Summing the baseline weights over S gives $\mathcal{B}_h(\ell)$. The derivative multiplicity is $h!(n - h - 2)!$: it counts ordered placements of the h conditioned spins and the remaining baseline spins. This also accounts for coincidences between free and conditioned spin labels. These observations prove (25). The a_i are distinct, so the congruence is invertible. For $h = 1$, expanding the profiles gives precisely the triple counts in Theorem 2. $\square$

For fixed h , the kernel can be assembled by examining the $\binom{n}{h}$ conditioned sets and their degree profiles; no expansion over all $(h + 2)^n$ spin assignments is needed. The formulas extend continuously to $\ell = -1$, where $(1 - \ell^2)^{e(S)}$ retains exactly the independent sets S (with exponent zero interpreted as weight one).

For comparison with the additive targets, (4) gives $H(\mathbf{t}) = 2 \text{diag}(\mathbf{z}) K(\mathbf{k}) \text{diag}(\mathbf{z})$. When G is d -regular,

$$Z_G(\mathbf{x}; H(\mathbf{t})) = 2^m Z_G((z_i^d x_i)_i; K(\mathbf{k})). \quad (26)$$

Thus the additive-target version of (25) has the additional positive factor $2^{m-hd}(s+1)^{hd}$ and the diagonal congruence D_z , when $t_0 = s$ and $t_1 = 1$. Such a source-independent diagonal change is not asserted for irregular graphs.

7. Structural consequences of the profile identity

A change of polynomial basis makes the boundary of the hierarchy particularly useful. Retain the notation r_S , ν^S and C^S from Theorem 11. Let $\bar{\nu}^S$ append a zero to ν^S , and let D_h be the $(h+2) \times (h+1)$ matrix with column r equal to $e_r - e_{r+1}$, for $0 \leq r \leq h$. For $0 \leq \theta \leq 1$ define

$$Q_h(\theta) = \sum_{|S|=h} (1-\theta)^{e(S)} [\theta(\bar{\nu}^S(\bar{\nu}^S)^\top - \text{diag}(\bar{\nu}^S)) - D_h C^S D_h^\top]. \quad (27)$$

At $\theta = 1$, the convention $0^0 = 1$ leaves precisely the independent h -sets.

Lemma 12 (Profile-basis identity). *Let $\theta = \ell^2$, with $-1 < \ell < 0$. For any $q \geq 2$, take q distinct parameters in $(-1, 1)$ with $a_0 = \ell$, $a_1 = 0$, and the target $K(\mathbf{a})$. Put $U_{ir} = (1 - \ell a_i)^r$ for $0 \leq i < q$ and $0 \leq r \leq h+1$. Then*

$$\text{Hess}(\partial_0^h \partial_1^{n-h-2} Z_G(\mathbf{x}; K(\mathbf{a}))) = \frac{h!(n-h-2)!}{\theta} U Q_h(\theta) U^\top. \quad (28)$$

The matrix U is invertible when $q = h+2$; for other q the identity is a rectangular matrix factorization. It extends to $\ell = -1$ when $a_0 = -1$ and the remaining a_i stay distinct in $(-1, 1)$.

Proof. In the polynomial basis $1, y, \dots, y^{h+1}$, where $y = 1 - \ell k$, the profile factor is y^r , and

$$k(1 - \ell k)^r = \frac{y^r - y^{r+1}}{\ell}.$$

Thus the adjacency correction in the proof of Theorem 11 has coefficient matrix $D_h C^S D_h^\top / \ell^2$. The other term is the embedded profile-count matrix in (27). This proves (28), with the same derivative multiplicity. The counting argument does not require the evaluation matrix to be square. The nodes $1 - \ell a_i$ are distinct, so U is an invertible Vandermonde matrix when $q = h+2$. All coefficients are continuous at $\ell = -1$, proving the boundary statement; in the square case invertibility also persists there. $\square$

7.1. The small-parameter limit and true twins

The other endpoint of the profile kernel gives a different structural criterion. Define the unweighted edge-profile matrix

$$T_h = \sum_{|S|=h} C^S, \quad 1 \leq h \leq n-2. \quad (29)$$

The sum here includes all h -sets, whether or not they are independent.

Lemma 13 (A small-parameter obstruction). *If T_h has a negative eigenvalue, a strictly positive rational antiferromagnetic target of rank two on $h+2$ spins makes Z_G non-Lorentzian.*

Proof. Choose f with $f^\top T_h f < 0$. The matrix D_h has full column rank, so there is $y \in \mathbb{R}^{h+2}$ with $D_h^\top y = f$. Let $p = \mathbf{1} \in \mathbb{R}^{h+2}$ and $R = n - h \geq 2$. Polynomial continuation of (27) to zero gives

$$Q_h(0) = -D_h T_h D_h^\top, \quad Q_h(0)p = 0, \quad d_0 := y^\top Q_h(0)y = -f^\top T_h f > 0.$$

For $\theta \geq 0$, direct evaluation gives

$$\begin{aligned} p^\top Q_h(\theta)p &= \theta A(\theta), & A(\theta) &= R(R-1) \sum_{|S|=h} (1-\theta)^{e(S)}, \\ p^\top Q_h(\theta)y &= \theta B(\theta), & B(\theta) &= (R-1) \sum_{|S|=h} (1-\theta)^{e(S)} (\bar{\nu}^S)^\top y. \end{aligned} \quad (30)$$

In particular $A(0) = \binom{n}{h} R(R-1) > 0$. The determinant of the compression of $Q_h(\theta)$ to columns p, y is

$$\theta A(\theta) y^\top Q_h(\theta) y - \theta^2 B(\theta)^2 = \theta A(0) d_0 + O(\theta^2).$$

For all sufficiently small $\theta > 0$, this determinant and the first diagonal entry are positive. Hence the compression is positive definite, and $Q_h(\theta)$ has at least two positive eigenvalues.

Choose $\ell = -1/m$ for a sufficiently large integer m , so that $\theta = 1/m^2$, and put $q = h + 2$. Take $a_0 = \ell$, $a_1 = 0$ and, for example, $a_i = (i-1)/(q+1)$ for $2 \leq i < q$. These rational parameters are distinct and lie in $(-1, 1)$. For each such nonzero ℓ , the matrix U in Lemma 12 is invertible. Thus the Hessian in (28) has two positive eigenvalues, and (2) proves the claim. The argument does not assume that U remains invertible at $\ell = 0$. $\square$

Two distinct vertices are *true twins* if their closed neighbourhoods are equal. Such vertices are adjacent and have equal degree.

Theorem 14 (Maximum degree and true twins). *Let G be a connected noncomplete graph with maximum degree Δ . If no two vertices of degree Δ are true twins, a strictly positive rational antiferromagnetic target of rank two on $\Delta + 1$ spins has a non-Lorentzian partition polynomial on G .*

Proof. Put $h = \Delta - 1$. Connectedness and noncompleteness imply $\Delta \geq 2$, and hence $1 \leq h \leq n - 2$. If an ordered edge uv outside an h -set S contributes to $(T_h)_{hh}$, both endpoints are adjacent to every vertex of S and to one another. The degree bound forces

$$d_u = d_v = \Delta, \quad N(u) \setminus \{v\} = S = N(v) \setminus \{u\}.$$

Then $N[u] = N[v]$, contrary to the hypothesis. Therefore $(T_h)_{hh} = 0$. Take a vertex u of degree Δ and a neighbour v , and use $S = N(u) \setminus \{v\}$. The edge uv contributes to $(T_h)_{hj}$ with $j = |N(v) \cap S| < h$, by the same argument. Thus $(T_h)_{hj} > 0$. The principal matrix on indices h, j has determinant $-(T_h)_{hj}^2 < 0$, so T_h has a negative eigenvalue. Apply Lemma 13, with $h + 2 = \Delta + 1$. $\square$

This proof is constructive. For $c = (T_h)_{hj} > 0$ and $a = (T_h)_{jj}$, the rational vector $f = e_j - (a+1)e_h/(2c)$ satisfies $f^\top T_h f = -1$. Set $y_{h+1} = 0$ and $y_r = \sum_{k=r}^h f_k$, so that $D_h^\top y = f$. Testing $\ell = -2^{-k}$ with exact arithmetic eventually gives a positive definite compression in (30).

Corollary 15. *The conclusion of Theorem 14 holds for every connected noncomplete graph without true twins, for every connected graph of order at least three with a unique maximum-degree vertex, and for every connected noncomplete edge-regular graph. In particular it holds for all connected noncomplete strongly regular graphs.*

Proof. The first two statements follow directly. For the third, edge-regular means regular of degree d , with a constant number λ of common neighbours for each adjacent pair. A true-twin edge would give $\lambda = d - 1$, making every edge a true-twin edge. Equality of closed neighbourhoods along paths would then make the connected graph complete. Thus no such edge exists. Strong regularity implies edge regularity. $\square$

For an infinite family outside the earlier triangle threshold, take the line graph of K_m , $m \geq 6$. It has $n = m(m-1)/2$, $d = 2(m-2)$ and $\lambda = m-2$; its ordered triangle count per vertex is $\tau = d\lambda$. Since

$$\lambda(n-2) - (d-1)^2 = \frac{(m-3)(m^2 - 8m + 14)}{2} > 0,$$

the threshold (22) fails, while Corollary 15 still gives an obstruction. Any nonclique counterexample to the full connected-source conjecture must therefore contain true twins of maximum degree. Neither the number of spins nor the parameter choice in this theorem is asserted uniform over all graphs.

7.2. The independent-set endpoint

For a graph with an edge, let α denote independence number, with $\alpha(\emptyset) = 0$, and set

$$b(G) = \max_{(u,v): uv \in \mathcal{E}} \alpha(G[N(u) \setminus \{v\}]), \quad (31)$$

$$c(G) = \max_{uv \in \mathcal{E}} \alpha(G[N(u) \cap N(v)]). \quad (32)$$

The first maximum uses both orientations of each edge.

Theorem 16 (Neighbourhood independence gap). *If $b(G) > c(G)$, a strictly positive rational antiferromagnetic matrix of rank two on $c(G)+3$ spins has a non-Lorentzian partition function on G .*

Proof. Put $h = c(G) + 1$. Since $b(G) \geq h$, some edge uv and independent h -set $S \subseteq N(u) \setminus \{v\}$ exist. In particular, $h \leq n - 2$. Let

$$C_* = \sum_{\substack{|S|=h \\ S \text{ independent}}} C^S, \quad Q = Q_h(1).$$

One has $(C_*)_{hh} = 0$. Otherwise the two endpoints of an edge would be adjacent to every vertex in an independent h -set, contradicting $c(G) < h$. The chosen u, v, S gives $(C_*)_{hj} > 0$ for some $j < h$, so $C_* e_h \neq 0$.

Take $p = \mathbf{1} \in \mathbb{R}^{h+2}$ and $u_0 = e_{h+1}$. Since $D_h^\top p = 0$, $D_h^\top u_0 = -e_h$, and the last entry of $\bar{v}^S$ is zero,

$$s_0 := p^\top Q p = I_h(G)(n-h)(n-h-1) > 0, \quad (33)$$

$$p^\top Q u_0 = 0, \quad u_0^\top Q u_0 = 0, \quad Q u_0 = D_h C_* e_h \neq 0, \quad (34)$$

where $I_h(G)$ counts independent h -sets. The last inequality uses injectivity of D_h . Put $w = Q u_0$ and $c_0 = \|Q u_0\|_2^2 > 0$. The compression of Q to the columns (p, u_0, w) has the form

$$\begin{pmatrix} s_0 & 0 & b_0 \\ 0 & 0 & c_0 \\ b_0 & c_0 & e_0 \end{pmatrix}, \quad \det = -s_0 c_0^2 < 0.$$

Its positive diagonal entry excludes three negative eigenvalues, so the compression has two positive eigenvalues. Hence so does Q . Choose any distinct rational remaining $a_i \in (-1, 1)$ with $a_1 = 0$. Lemma 12 first gives an obstruction at $\ell = -1$. The two positive eigenvalues persist when the rational parameter $\ell > -1$ is sufficiently close to -1 . All a_i then lie in $(-1, 1)$, so $K(\mathbf{a})$ is strictly positive and has rank two. The Hessian test (2) proves the result. $\square$

The proof is constructive: with the remaining rational a_i fixed, exactly testing $\ell = -1 + 1/k$ for integers $k \geq 2$ eventually finds a positive target. It does not assert a uniform bound on k over all source graphs.

Corollary 17 (One target for connected diamond-free sources). *Let*

$$K_\diamond = \frac{1}{9} \begin{pmatrix} 0 & 9 & 12 & 15 \\ 9 & 9 & 9 & 9 \\ 12 & 9 & 8 & 7 \\ 15 & 9 & 7 & 5 \end{pmatrix}. \quad (35)$$

For every connected diamond-free finite simple graph G , $Z_G(\mathbf{x}; K_\diamond)$ is Lorentzian if and only if G is a clique. Thus the connected-source characterization conjecture holds on diamond-free graphs. For each nonclique, a strictly positive rational rank-two target on at most four spins also gives an obstruction.

Proof. The displayed target is $K(-1, 0, 1/3, 2/3)$; it is nonnegative, has rank two and inertia $(1, 1, 2)$. For a clique the result is [2, Theorem 3.1].

Suppose G is not complete. If G is triangle-free, then $c(G) = 0$ and $b(G) \geq 1$, because a connected nonclique contains two adjacent edges. The proof of Theorem 16, at $h = 1$ and $\ell = -1$, gives a two-positive-eigenvalue Hessian for the principal target $K(-1, 0, 1/3)$. This Hessian is a principal submatrix of the same quadratic derivative for $K_\diamond$, so the latter also fails (2).

Otherwise G contains a triangle. Diamond-freeness implies $c(G) = 1$: a common neighbourhood of an edge cannot contain an independent pair, while a triangle gives a common neighbour. There is a nonsimplicial vertex u of degree at least three. Indeed, if every vertex of degree at least three were simplicial and such a vertex v existed, its closed neighbourhood $N[v]$ would be a clique of size at least four. Every vertex of that clique has degree at least three and is simplicial. An outside neighbour of any of them would therefore also be adjacent to v , a contradiction. Hence $N[v]$ would be a whole component, making G complete. If no vertex has degree at least three, a connected graph containing a triangle is just K_3 , again complete.

Choose nonadjacent $x, y \in N(u)$ and a third neighbour v . Then $\{x, y\} \subseteq N(u) \setminus \{v\}$, so $b(G) \geq 2 > c(G)$. The proof of Theorem 16, now at $h = 2$ and $\ell = -1$, applies directly to $K_\diamond$. Finally perturb its first parameter from -1 into $(-1, 0)$; the two positive Hessian eigenvalues persist and all target entries become positive. $\square$

The fixed matrix (35) has one zero entry. Its strictly positive perturbation may depend on G ; no fixed positive four-spin target for the entire class is asserted. The gap condition in Theorem 16 is sufficient, not necessary for an obstruction or for failure of the universal-target property.

8. Clique separators with disjoint complete branches

The independent-set boundary also treats sources containing diamonds that need not satisfy the neighbourhood-gap criterion. Write $\vee$ for graph join and $\sqcup$ for disjoint union.

Proposition 18 (Two branches). *For integers $a, b, s \geq 1$, let $G = K_s \vee (K_a \sqcup K_b)$. Then $Z_G(\mathbf{x}; K_\diamond)$ is not Lorentzian. More generally, the same conclusion holds for any four-spin target $K(-1, 0, c, d)$ with $-1 < c, d < 1$ and all four parameters distinct.*

Proof. Put $A = a - 1$, $B = b - 1$, $L = A + B$ and $C = A(A - 1) + B(B - 1)$. Suppose first that $L > 0$ and use $h = 2$, $\theta = 1$ in Lemma 12. Every independent two-set contains one vertex in

each branch, so there are ab such sets. Outside each one the profile counts are $\nu_1 = L$, $\nu_2 = s$, and the ordered edge counts are $C_{11} = C$, $C_{12} = C_{21} = sL$ and $C_{22} = s(s-1)$. Consequently,

$$Q_2(1) = ab(0 \oplus M), \quad M = \begin{pmatrix} 2AB & C & sL \\ C & 2sL - C & s(s-1-L) \\ sL & s(s-1-L) & -s(s-1) \end{pmatrix}. \quad (36)$$

Direct expansion gives

$$\det M = -s(L+s-1)[L^2(L+2s-1) + 2AB(s-1)(L+s)] < 0, \quad (37)$$

$$\mathbf{1}^\top M \mathbf{1} = (L+s)(L+s-1) > 0.$$

Thus M has inertia $(2, 1, 0)$. The square evaluation matrix U is invertible for the four distinct parameters, so the Hessian in (28) has two positive eigenvalues. Here $n-h = L+s \geq 2$, as required.

If $L = 0$, then $a = b = 1$ and G is K_{s+2} with one edge removed. The counts in (7) are

$$N = s(s+1)(s+2), \quad E = s^2(s+3), \quad W = s(s^2+3s-2), \quad U = s(s+4)(s-1).$$

They satisfy $W^2 - EU = E^2 - NW = 4s^2$. The polynomial counting identity (8) extends to $\ell = -1$, and its coefficient matrix does not depend on the third spin parameter. Therefore, at $h = 1$,

$$\det \hat{B}_{-1} = N(EU - W^2) + U(E^2 - NW) = -24s^3 < 0.$$

Its first diagonal entry N is positive. The three distinct parameters $(-1, 0, c)$ give an invertible evaluation matrix, hence a principal Hessian submatrix with two positive eigenvalues. The four-spin Hessian has at least as many positive directions. In both cases (2) proves the conclusion. $\square$

Theorem 19 (One fixed five-spin target). *Let*

$$K_5 = \frac{1}{16} \begin{pmatrix} 0 & 16 & 20 & 24 & 28 \\ 16 & 16 & 16 & 16 & 16 \\ 20 & 16 & 15 & 14 & 13 \\ 24 & 16 & 14 & 12 & 10 \\ 28 & 16 & 13 & 10 & 7 \end{pmatrix} = K(-1, 0, 1/4, 1/2, 3/4). \quad (38)$$

For every $s \geq 1$, $r \geq 2$ and $a_1, \dots, a_r \geq 1$, the connected noncomplete graph

$$G = K_s \vee (K_{a_1} \sqcup \dots \sqcup K_{a_r})$$

has non-Lorentzian partition polynomial $Z_G(\mathbf{x}; K_5)$. The target is nonnegative of rank two, with exactly one zero entry. For each such G , a strictly positive rational five-spin perturbation also gives an obstruction.

Proof. The target has inertia $(1, 1, 3)$ by the argument for (6). For $r = 2$, Proposition 18 applies to the first four spins; its quadratic-derivative Hessian is a principal submatrix of the five-spin Hessian. We therefore assume $r \geq 3$.

Put $A_i = a_i - 1$, $L = \sum_i A_i$, and $C = \sum_i A_i(A_i - 1)$. If $L > 0$, use $h = r$ at the boundary $\theta = 1$ in Lemma 12, now with $q = 5$. Every independent h -set chooses one vertex from each branch, and there are $P = \prod_i a_i$ choices. The remaining branch vertices have profile 1, and separator vertices have profile h . The only nonzero part of $Q_h(1)$ is on indices $(1, 2, h, h+1)$, where it is P times

$$M = \begin{pmatrix} L(L-1) - C & C & 0 & sL \\ C & -C & sL & -sL \\ 0 & sL & 0 & s(s-1) \\ sL & -sL & s(s-1) & -s(s-1) \end{pmatrix}. \quad (39)$$

Its determinant and the principal determinant on profile indices $2, h$ are

$$\begin{aligned}\det M &= Ls^2(L + s - 1)[sL^2 - (s - 1)C] > 0, \\ \det M[\{2, h\}] &= -s^2L^2 < 0.\end{aligned}\tag{40}$$

Indeed $0 \leq C \leq L(L - 1)$, so the bracket in (40) is at least $L(L + s - 1) > 0$. The negative principal determinant gives a positive and a negative direction. The positive full determinant then forces inertia $(2, 2, 0)$.

The four active columns of U have degrees $1, 2, h, h + 1$. On the four positive nodes $1, 5/4, 3/2, 7/4$ they are linearly independent. To see this, a nonzero polynomial with four distinct powers has at most three positive roots by Descartes' sign rule; dependence would give four distinct positive roots. Thus the 5×4 active evaluation matrix has full column rank. Multiplication on its two sides preserves the nonzero inertia of M , and (28) gives two positive eigenvalues. The derivative is valid because $n - h = L + s \geq 2$.

If $L = 0$, all branches are singletons. Choose $h = r - 1 \geq 2$. Each independent h -set leaves one branch vertex of profile 0 and s vertices of profile h . There are r such sets, and $Q_h(1)$ is r times the following matrix on indices $(0, 1, h, h + 1)$:

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & s \\ 0 & 0 & s & -s \\ 0 & s & 0 & s(s - 1) \\ s & -s & s(s - 1) & -s(s - 1) \end{pmatrix}.\tag{41}$$

Here $\det M_0 = s^4 > 0$, whereas its principal determinant on indices $1, h$ is $-s^2$. Hence its inertia is again $(2, 2, 0)$. The active powers $0, 1, h, h + 1$ are distinct, and the same four-node argument gives full column rank. The Hessian therefore has two positive eigenvalues. Now $n - h = s + 1 \geq 2$.

Finally, perturb the first parameter from -1 into $(-1, 0)$. The two strictly positive Hessian eigenvalues persist by continuity. Choose a rational perturbation; all target entries become strictly positive, with rank two and one positive eigenvalue. $\square$

When $r = 1$ the source is a clique and its partition polynomial is Lorentzian by [2, Theorem 3.1]. Thus the fixed target (38) characterizes the complete members of the whole family with $s \geq 1$, $r \geq 1$ and nonempty branches. The number of target states is independent of the number and sizes of the branches. We do not assert that five is minimal, that one strictly positive perturbation works for every source, or that the four-spin target of Proposition 18 suffices for the entire family.

Remark 20 (A limit of a four-spin restriction). Full column rank in the preceding proof is essential. Take $G = K_s \vee (K_2 \sqcup K_1 \sqcup K_1)$ and $h = 3$. For $K_\diamond$ the first row and column of the boundary Hessian vanish. After removing its positive scalar factor, the remaining matrix is $VMV^\top$, where M is (39) with $L = 1, C = 0$ and V evaluates powers $1, 2, 3, 4$ at $1, 4/3, 5/3$. Its determinant is

$$\det(VMV^\top) = \frac{12800}{4782969}s^3(185s - 473) > 0 \quad (s \geq 3).$$

The original M has two positive and two negative eigenvalues, so the restriction has at most two positive eigenvalues. Its positive determinant and positive trace therefore give inertia $(1, 2, 0)$. This derivative is inconclusive for $s \geq 3$. Other derivatives may still detect these graphs; this is not a counterexample to universality of $K_\diamond$.

9. Complete multipartite sources

The same four-spin target also detects a family not contained in the diamond-free class. In this section a complete multipartite graph has nonempty parts, and two vertices are adjacent exactly when they belong to different parts.

Theorem 21 (One fixed target for complete multipartite sources). *For every connected complete multipartite graph G , the partition polynomial $Z_G(\mathbf{x}; K_\diamond)$ is Lorentzian if and only if G is a clique. For every noncomplete such G , a strictly positive rational four-spin rank-two perturbation of $K_\diamond$ also has a non-Lorentzian partition polynomial. The perturbation may depend on G .*

Proof. The clique direction follows from [2, Theorem 3.1]. For the converse, write the part sizes as $a_1, \dots, a_r$. Connectedness and noncompleteness give $r \geq 2$ and $\alpha = \max_i a_i \geq 2$. Put $h = \alpha - 1$; then $1 \leq h \leq n - 2$. Apply Lemma 12 at $\ell = -1$, $\theta = 1$, and $q = 4$. Only independent h -sets contribute.

Such a set S lies in one part i with $a_i \geq h$. There are $w_i = \binom{a_i}{h}$ choices. Outside S , the $k_i = a_i - h \in \{0, 1\}$ vertices in that part have profile 0, and the other $y_i = n - a_i$ vertices have profile h . Every pair between these two classes is an edge, there are no edges within profile 0, and the ordered edge count within profile h is

$$t_i = y_i^2 - \sum_{j \neq i} a_j^2.$$

Summing over all i with $a_i \geq h$, define

$$A = \sum_i w_i \sum_{j \neq i} a_j(a_j - 1), \quad B = \sum_i w_i k_i y_i, \quad C = \sum_i w_i t_i. \quad (42)$$

All three numbers are nonnegative. Moreover $B > 0$: a largest part has $k_i = 1$ and $y_i > 0$.

First let $h \geq 2$. Substitution into (27) shows that $Q_h(1)$ is supported on indices $(0, 1, h, h + 1)$, where it is the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & B \\ 0 & 0 & B & -B \\ 0 & B & A & C \\ B & -B & C & -C \end{pmatrix}. \quad (43)$$

Indeed, the profile-0 diagonal is zero because $k_i(k_i - 1) = 0$, and its pair term with profile h cancels the corresponding edge term. The profile- h diagonal is $y_i(y_i - 1) - t_i$, whose sum is A . All other entries follow from the two differences in the columns of D_h . Now $\det M = B^4 > 0$, while its principal determinant on indices $1, h$ is $-B^2 < 0$. Thus M has inertia $(2, 2, 0)$.

The active evaluation matrix in (28) evaluates the powers $0, 1, h, h + 1$ at $0, 1, 4/3, 5/3$. It is invertible. In fact, a vanishing linear combination has zero constant coefficient by evaluation at zero. Dividing by the variable leaves at most three nonzero monomials, which by Descartes' sign rule have at most two positive roots unless the polynomial is zero. The three distinct positive evaluation nodes therefore force all coefficients to vanish. The quadratic-derivative Hessian has two positive eigenvalues.

If $h = 1$, the middle two profile indices coincide. Direct substitution instead gives the 3×3 kernel

$$M_1 = \begin{pmatrix} 0 & 0 & B \\ 0 & A + 2B & C - B \\ B & C - B & -C \end{pmatrix}.$$

Its determinant is $-B^2(A + 2B) < 0$, and its middle diagonal entry is positive, so its inertia is $(2, 1, 0)$. The evaluation matrix on powers $0, 1, 2$ at the same four nodes has full column rank. Again the Hessian has two positive eigenvalues.

Both cases violate (2). Finally, replace the first target parameter -1 by a rational $\ell \in (-1, 0)$ sufficiently close to -1 . The two positive eigenvalues persist by continuity. All target entries become strictly positive, and its rank and inertia are those of the rank-two family already proved above. $\square$

The theorem includes arbitrarily many parts and arbitrarily large part sizes. Its proof uses only four active profile columns even when $h > 2$; a square evaluation matrix of size $h + 2$ is unnecessary. The Lorentzian characterization of complete multipartite graphs by Choudhury–Khare [6, Theorem 2.3] concerns instead the homogenized determinant of a distance-matrix blowup. It is not a statement about the spin partition polynomial (1) or the fixed target here.

10. Examples and limits

10.1. A second-level obstruction beyond the triangle threshold

Start with two disjoint copies of K_4 on vertices $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$. Delete edges $(0, 1)$ and $(4, 5)$, and add $(0, 4)$ and $(1, 5)$. The resulting graph is connected, cubic, and has four triangles. Hence $\tau = 3$, while the right side of (22) is 2. The one-conditioned-spin test cannot detect it for any of its allowed parameter values.

For $h = 2$ and the boundary value $\ell = -1$, direct evaluation gives

$$\mathcal{B}_2(-1) = \begin{pmatrix} 480 & 480 & 60 & 0 \\ 480 & 264 & -120 & -12 \\ 60 & -120 & -148 & -16 \\ 0 & -12 & -16 & -4 \end{pmatrix}.$$

Symmetric elimination without permutations has consecutive pivots

$$480, \quad -216, \quad -\frac{11}{2}, \quad \frac{106}{33}.$$

Its inertia is therefore $(2, 2, 0)$. Theorem 11 at the continuous boundary supplies a non-Lorentzian Hessian. Its two positive eigenvalues persist for $\ell > -1$ sufficiently close to -1 , giving positive rank-two targets as well. The supplement checks explicitly that $\ell = -99/101$ works, using rational elimination of $\mathcal{B}_2(-99/101)$ and direct comparison with the additively scaled target $H(1/100, 1, 2, 3)$ via (26).

10.2. Parameter variation enlarges the test

Allowing the boundary value $s = 0$ in the counting identity gives $\ell = -1$ and the more restrictive sufficient condition

$$\tau < \frac{(n-1)d(d-1)^2}{(n-1)^2 - d}. \quad (44)$$

For a concrete strict improvement, take the graph on $\{0, \dots, 11\}$ with edges

$$\begin{aligned} &(0, 1), (0, 2), (0, 8), (0, 9), (1, 4), (1, 7), (1, 9), (2, 5), \\ &(2, 8), (2, 11), (3, 6), (3, 7), (3, 8), (3, 11), (4, 5), (4, 7), \\ &(4, 10), (5, 6), (5, 9), (6, 10), (6, 11), (7, 10), (8, 9), (10, 11). \end{aligned}$$

It is connected and 4-regular, and has 7 triangles. Thus $n = 12$, $d = 4$, $\tau = 7/2$. At $s = 0$, $\det B_{-1} = 540 > 0$, so this Hessian has at most one positive eigenvalue. At $s = 2/3$, $\ell = -1/5$ and

$$\det B_{-1/5} = -\frac{2004}{125} < 0.$$

The positive target is

$$H(2/3, 1, 2) = \begin{pmatrix} 4/3 & 5/3 & 8/3 \\ 5/3 & 2 & 3 \\ 8/3 & 3 & 4 \end{pmatrix}.$$

The supplement verifies the full Hessian identity for this graph with rational arithmetic.

10.3. The degree-count test does not settle all irregular graphs

Consider the connected six-vertex graph with edge set

$$\{(0, 1), (0, 4), (0, 5), (1, 2), (1, 5), (2, 3), (2, 5), (3, 4)\}.$$

Its degree counts are $\nu_2 = 2$, $\nu_3 = 4$, and its kernel is

$$B_G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -6 & 16 \\ 0 & 2 & 16 & 2 \end{pmatrix}.$$

The characteristic polynomial is $\lambda(\lambda^3 + 6\lambda^2 - 264\lambda - 560)$. The cubic has exactly one positive root by Descartes' rule of signs and a sign change between zero and sufficiently large positive λ . Thus this kernel has only one positive eigenvalue. It does not establish Lorentzianity of the partition function. A different derivative supplies a countermodel in the finite certificates described below.

Graphs with many triangles can also fail (22). The condition $b(G) > c(G)$ also leaves cases untreated. These sufficient criteria do not prove the full connected-source conjecture, and their failure is not evidence against that conjecture.

10.4. Limits of three fixed subtargets

The same fixed four-spin target $K_\diamond$ has an exact quadratic-derivative obstruction for every connected nonclique of orders three through seven in the graph atlas: 989 graphs. This is a stronger finite assertion than allowing a different target for each graph, but it is not an arbitrary-order theorem. An additional deterministic sample of 64 graphs, with orders up to ten, also yields exact obstructions. This sample is not an exhaustive census beyond order seven.

Deleting one of the last three states can destroy this property. For the principal target on states $(0, 1, 2)$, the seven-vertex graph with edge set

$$\{(0, 1), (0, 4), (1, 2), (1, 4), (2, 3), (4, 5), (5, 6)\}$$

has a Lorentzian partition polynomial. For the principal targets on states $(0, 1, 3)$ and $(0, 2, 3)$, the six-vertex graph in Section 10.3 has that property. The supplement records every quadratic-derivative Hessian: 21 in the first case and 15 in each of the other two. Exact rational congruences give at most one positive eigenvalue in each, and the full M-convex exchange condition is verified on each support. The full characterization cited after (2) proves these finite claims. They concern three specific fixed targets; they do not show that every three-spin target fails or that four spins are necessary in general.

10.5. The small-parameter criterion is not complete

Let $G = C_5[K_2]$: replace each vertex of a five-cycle by a two-vertex clique, and join all vertices in two replacement cliques whenever the original cycle vertices are adjacent. This is a connected noncomplete 5-regular graph on ten vertices, and each replacement pair is a pair of true twins. Every matrix T_h , $1 \leq h \leq 8$, is positive semidefinite, so Lemma 13 does not detect G . Here this is an exact finite claim with a short coefficient certificate.

For an ordered edge uv , put $c = |N(u) \cap N(v)|$, $a = d_u - 1 - c$, $b = d_v - 1 - c$, and $d = n - d_u - d_v + c$. Partitioning the other vertices by their adjacency to u, v gives the contribution

$$\sum_k \binom{c}{k} \binom{a}{r-k} \binom{b}{t-k} \binom{d}{h-r-t+k} \quad (45)$$

to $(T_h)_{rt}$, with out-of-range binomial coefficients zero. For this graph, ten ordered edges have $c = 4$ and forty have $c = 2$. Thus $(T_h)_{rt}$ is the coefficient of $x^r y^t z^h$ in

$$10(1 + xyz)^4(1 + z)^4 + 40(1 + xyz)^2(1 + xz)^2(1 + yz)^2(1 + z)^2.$$

For $h = 1, 2, 3, 4$, ordinary symmetric elimination without permutations has the following positive diagonal pivots:

h	Pivots of T_h
1	120, 200/3
2	100, 224, 300/7
3	40, 480, 360, 200/9
4	10, 320, 600, 592/3, 250/37

For each fixed outside edge, complementing S in its other eight vertices replaces h by $8 - h$ and each profile r by $4 - r$. It therefore relates T_h and T_{8-h} by reversal of the active profile indices and zero padding. This proves positive semidefiniteness for $h = 5, 6, 7$; for $h = 8$ the sole nonzero entry is $(T_8)_{44} = 50$.

This failure does not extend to the full profile hierarchy. Use $q = h + 2$, $a_0 = \ell$, $a_1 = 0$ and $a_i = (i - 1)/(h + 2)$ for $i \geq 2$. At $\ell = -99/100$, its kernels for $h = 3, 4, 5$ have respective inertias $(2, 3, 0)$, $(2, 4, 0)$ and $(2, 3, 2)$, verified by exact congruences and direct partition-coefficient identities. All three targets are strictly positive. Since $\alpha(G) = 2$, the unscaled endpoint $Q_h(1)$ vanishes at these levels. The first nonzero term after dividing by the smallest power of $1 - \theta$ has two positive eigenvalues for $h = 3, 4$, but only one for $h = 5$. The nearby kernel at $h = 5$ nevertheless has two. Thus neither the zero-parameter test nor the leading endpoint term alone supplies a general completeness argument.

11. Exact checks and reproducibility

The accompanying supplement contains the source scripts, their data, and a manifest of file hashes. All checks were run in the authoring context; none is a formal proof-assistant verification or an independent model assessment. The general proofs are the counting identities and compression arguments above.

Check	Cases
Normalized three-spin identity, including irregular sources	288
Degree identity, all connected graphs of orders 3–5, original nonnegative target	29
Degree identity, strictly positive target of Theorem 4	29
Independent minimum-degree compression, connected graph atlas of orders 3–7	806
Original regular identity, exact enumeration	13
Parameterized regular identity: 18 graphs, 5 rational parameters	90
Additive-target conditioned-set identity for $h = 1, 2, 3$	26
Normalized conditioned-set identity, including irregular sources	24
Certificate census	Cases
Exact three-spin countermodels for all connected noncliques of orders 3–6 in the atlas	137
Positive rank-two hierarchy certificates for all connected noncliques of orders 3–7	989
Characteristic-polynomial audit of those kernels	989
Neighbourhood-gap compression and positive continuation	634
Fixed diamond-free target, direct identities	255
Positive perturbations of the diamond-free target, direct identities	255
Fixed four-spin certificates on all connected noncliques of orders 3–7	989
Clique-separator and fixed-target checks	Cases
Two-branch sources: direct fixed four-spin boundary identities	64
Two-branch sources: direct positive continuations	46
Clique-cluster joins: direct fixed five-spin boundary identities	64
Clique-cluster joins: direct positive continuations	27
Additional seeded fixed four-spin obstructions, orders up to ten	64
Exact Lorentzian counterexamples to three fixed subtargets	3
Four-spin restriction: direct checks of Remark 20	6
Complete multipartite sources: direct fixed four-spin identities	36
Complete multipartite sources: direct positive continuations	17

Small-parameter checks	Cases
Maximum-degree theorem: exact compression certificates	821
Maximum-degree theorem: direct positive Hessian identities	819
Unweighted profile-kernel probe, atlas and selected blowups	1001
$C_5[K_2]$: exact semidefinite kernels, all valid h	8
$C_5[K_2]$: direct positive full-hierarchy identities	3

The maximum-degree checks cover all 817 eligible connected noncliques of orders three through seven, two larger wheels, the line graph of K_6 , and the 4-by-4 rook graph. Each records a rational negative direction for T_h and a positive definite two-dimensional compression of a nearby Q_h . The 15- and 16-vertex examples use exact kernels and the proved identity; the other 819 also use direct partition enumeration. Among these 821 graphs, 244 fail the earlier edge-wedge-triangle sufficient criterion. The separate unweighted probe finds an indefinite T_h for every atlas nonclique, but fails on nine selected clique blowups beyond the atlas. Section 10.5 proves the first such limitation explicitly. Neither this probe nor the theorem asserts completeness of the unweighted criterion.

The three-spin census contains 141 connected graphs, including four complete-graph controls. Each of the 137 certificates records the edge set, integer target $H_{ij} = t_i + t_j$, derivative multi-index, integer Hessian, its negative determinant and positive trace. These signs prove inertia $(2, 1, 0)$. The finite census does not establish the conjecture at arbitrary order; absence of a detected obstruction in the clique controls is not its proof.

The larger census contains all 994 connected graphs of orders three through seven, of which five are cliques. For each of the 989 noncliques it gives a strictly positive rational rank-two target, an integer-scaled profile kernel, a rational congruence certificate, and direct agreement with a Hessian obtained by partition-coefficient enumeration. A separate audit checks the characteristic polynomial of every stored kernel. Its real roots are counted exactly using Descartes bounds for positive and negative roots together with the exact zero multiplicity; the bounds sum to the degree and agree with the congruence inertia. These are separate computations within the same authoring context, not a separate referee.

The gap check covers 634 graphs in the same atlas. All 255 connected diamond-free noncliques there satisfy the criterion; 88 are triangle-free and 167 contain triangles. For these 255 sources, both the fixed boundary target and strictly positive rational perturbations were checked by direct partition-coefficient enumeration. The finite checks corroborate the general proof of Corollary 17; they do not replace it.

The clique-separator checks expand the determinant identities symbolically and compare the predicted Hessians with direct partition-coefficient enumeration. The fixed-five sample includes $r = 2, 3, 4, 5, 6$, so the number of conditioned vertices can exceed the number of nonbaseline target states. The full proof uses the rectangular identity and the four-column rank argument, rather than extrapolating from these checks. The complete multipartite sample has largest parts from two through six. Its 36 boundary identities and 17 positive identities were checked by direct partition-coefficient enumeration; the symbolic determinants in Theorem 21 were expanded separately.

The main scripts are:

<code>verify_degree_kernel.py</code>	<code>verify_neighbourhood_gap.py</code>
<code>verify_normalized_target.py</code>	<code>verify_fixed_diamond.py</code>
<code>verify_regular_obstruction.py</code>	<code>verify_two_clique_extension.py</code>
<code>verify_parametric_obstruction.py</code>	<code>verify_fixed_five_cluster.py</code>
<code>verify_conditioning_hierarchy.py</code>	<code>probe_fixed_diamond_scope.py</code>
<code>certify_source_witnesses.py</code>	<code>probe_larger_fixed_target.py</code>
<code>certify_profile_hierarchy.py</code>	<code>probe_three_spin_targets.py</code>
<code>audit_profile_certificates.py</code>	<code>verify_four_spin_restriction.py</code>
<code>verify_multipartite_extension.py</code>	<code>verify_maximum_degree_twins.py</code>
<code>verify_blowup_layer.py</code>	<code>probe_zero_parameter_kernels.py</code>

The source-witness script reads the preserved finite-search candidates and recomputes their Hessians by exact enumeration. The parameterized script separately checks the symbolic determinant, the rational parameter selection, and the positive-target identity. Python, SymPy and NetworkX suffice for these checks; NumPy is used only by the exploratory witness search. The supplementary README gives invocation and provenance details.

12. Remaining question

The obstructions extract degree-to-degree adjacency and edge, wedge and triangle counts. The conditioned-set hierarchy uses finer adjacency profiles. Its small-parameter limit gives the maximum-degree criterion: any nonclique counterexample to the source conjecture must have true twins of maximum degree. The explicit $C_5[K_2]$ limitation shows why that limit alone is insufficient. Its independent-set boundary gives the neighbourhood-gap theorem and the diamond-free characterization. Its rectangular evaluation also gives a fixed five-spin obstruction for joins of a clique with any number of complete branches, and a fixed four-spin obstruction for all noncomplete complete multipartite sources. A general argument would have to prove that a suitable test detects every connected nonclique; no such sufficiency is established here. The distinction between bounded countermodel search and a proof for graphs of arbitrary order remains essential.

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