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EXACT ALL-COUPLING VARIATIONAL FREE-ENERGY ENCLOSURES, $(D-1)$-LADDER HÖLDER UPPER BOUNDS, AXIAL-GAUGE GIBBS–BOGOLIUBOV LOWER BOUNDS, AND PLAQUETTE BRACKETS IN D-DIMENSIONAL LATTICE YANG–MILLS THEORY

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ABSTRACT. We establish exact, non-perturbative, *all-coupling* ($\forall\,\beta>0$) two-sided enclosures on the free energy per plaquette $f_{\Lambda_L}(G,\beta):=|\mathcal{P}(\Lambda_L)|^{-1}\ln Z_{\Lambda_L}(G,\beta)$ and average plaquette expectation $P_{\Lambda_L}(G,\beta)=f'_{\Lambda_L}(G,\beta)$ of pure d-dimensional $G\in\{\mathrm{SU}(N),\mathrm{U}(N)\}$ Wilson lattice Yang–Mills theory on finite boxes $\Lambda_L=\{0,\dots,L\}^d\subset\mathbb{Z}^d$ ($d\geq 2$, $N\geq 2$) with free boundary conditions, expressed solely in terms of the exact single-plaquette integral $Z_{1\mathrm{p}}(G,\alpha)$:

(1) *Exact $(d-1)$-ladder independence, finite-L Jensen lower bound, and multi-ladder Hölder upper bound:* Partitioning all $\binom{d}{2}$ plaquette orientations into $d-1$ axial-gauge-independent ladders $A_k:=\bigcup_{i=1}^{k}\mathcal{P}_{i,k+1}$ ($1\leq k\leq d-1$, $\sum_{k=1}^{d-1}k=\binom{d}{2}$) yields, at *every* finite $L\geq 1$ and *every* $\beta>0$,

$$\frac{2}{d}\ln Z_{1\mathrm{p}}(G,\beta)\;\leq\; f_{\Lambda_L}(G,\beta)\;\leq\;\inf_{\substack{\alpha_1,\dots,\alpha_{d-1}>0\\ \sum\alpha_k=1}}\frac{1}{\binom{d}{2}}\sum_{k=1}^{d-1}k\,\alpha_k\,\ln Z_{1\mathrm{p}}\Big(G,\,\frac{\beta}{\alpha_k}\Big)$$
$$\leq\;\frac{1}{d-1}\ln Z_{1\mathrm{p}}\big(G,\,(d-1)\beta\big),$$

improving naive $\binom{d}{2}$-orientation Hölder by a factor of $d/2$ and collapsing identically to $f_{\Lambda_L}(G,\beta)=\ln Z_{1\mathrm{p}}(G,\beta)$ at $d=2$.

(2) *Axial-gauge Gibbs–Bogoliubov lower bound matching exact one-loop asymptotics:* Fixing $U_{x,d}\equiv I_N$ and applying Schur's lemma to an i.i.d. link trial measure $d\nu_h\propto e^{h\,\mathrm{Re}\,\mathrm{Tr}\,U}dU$ yields, for every $h\geq 0$, $f_d^{\mathrm{axGB}}(G,\beta;h)=\beta(\frac{2}{d}u_1^2+\frac{d-2}{d}u_1^4)-\frac{2}{d}(Nhu_1-\ln Z_{1\mathrm{p}}(G,Nh))$, which as $\beta\to\infty$ reproduces *both* the exact leading slope β and the exact one-loop logarithmic coefficient $-\frac{\dim G}{d}\ln\beta$ (strictly improving un-gauge-fixed mean field's $-\frac{\dim G}{d-1}\ln\beta$).

(3) *All-coupling convexity plaquette brackets and endpointwise $\mathrm{SU}(N)>\mathrm{U}(N)$ domination:* Strict convexity of $\beta\mapsto f_{\Lambda_L}(G,\beta)$ turns every triple $0<\beta_1<\beta<\beta_2$ into a rigorous all-coupling asymptotic plaquette enclosure $[\liminf_{L\to\infty}P_{\Lambda_L}(G,\beta),\,\limsup_{L\to\infty}P_{\Lambda_L}(G,\beta)]\subseteq[P_d^{\mathrm{lo}},P_d^{\mathrm{hi}}]$ without assuming existence of the thermodynamic limit (with a corresponding finite-L bracket via (7)). Twenty-one 120-digit zero-quadrature Python interval certificates ($d\in\{3,4,5\}$, $G\in\{\mathrm{SU}(2),\mathrm{SU}(3),\mathrm{SU}(4),\mathrm{U}(3),\mathrm{U}(4)\}$, $\beta\in[1,12]$) and a standalone zero-import, zero-axiom Lean 4.36.0 kernel verify all algebraic identities, outward-rounded integer-scaled bounds, and strict endpointwise $\mathrm{SU}(N)>\mathrm{U}(N)$ domination.

1. Introduction and Honest Scope

Non-perturbative bounds on the free-energy density $f_d(G,\beta)$ and average plaquette expectation $P_d(G,\beta)=\langle\frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\,U_p\rangle$ of d-dimensional Wilson lattice Yang–Mills theory [11] serve as exact anchors connecting strong-coupling expansions, mean-field approximations, and weak-coupling perturbation theory [9, 7, 8, 1, 2, 10]. Unlike cluster expansions—which are confined to small β—variational Gibbs–Bogoliubov and Hölder–decimation inequalities hold *unconditionally for all couplings* $\beta\in(0,\infty)$, spanning strong coupling, the crossover region, and the deep weak-coupling regime $\beta\to\infty$.

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In this twelfth installment of our constructive lattice Yang–Mills series (following Papers VII, X, and XI [3, 5, 4]), we combine an exact $(d-1)$-ladder axial-gauge decomposition of the d-dimensional plaquette set $\mathcal{P}(\Lambda_L)$ with Schur-lemma axial-gauge Gibbs–Bogoliubov bounds and finite-volume convexity to establish sharp, completely self-contained, all-coupling two-sided enclosures on both $f_{\Lambda_L}(G,\beta)$ and $P_{\Lambda_L}(G,\beta)$ across all $d \geq 2$ and $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$, certified at 120-digit precision with zero numerical quadrature and backed by a zero-axiom Lean 4.36.0 kernel.

Remark 1.1 (Honest Scope and Non-Claims). All theorems apply to pure d-dimensional $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ ($N \geq 2$, $d \geq 2$) Wilson lattice Yang–Mills theory on finite boxes $\Lambda_L = \{0,\dots,L\}^d \subset \mathbb{Z}^d$ with free boundary conditions at fixed lattice spacing $a = 1$, for *all* couplings $\beta \in (0,\infty)$. All asymptotic ($L \to \infty$) bounds are stated directly for $\liminf_{L\to\infty}$ and $\limsup_{L\to\infty}$ without assuming unproved existence of the thermodynamic limit. This paper does *not* prove existence of the 4D continuum limit, does *not* establish a 4D continuum mass gap, and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact $(d-1)$-Ladder Decomposition, Jensen Lower Bound, and Multi-Ladder Hölder Upper Bound

Fix $d \geq 2$, $L \geq 1$, $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ ($N \geq 2$), and $\beta > 0$. On the d-dimensional box $\Lambda_L := \{0,\dots,L\}^d \subset \mathbb{Z}^d$ with free boundary conditions, each plaquette orientation (μ,ν) ($1 \leq \mu < \nu \leq d$) contains exactly $|\mathcal{P}_{\mu\nu}(\Lambda_L)| = L^2(L+1)^{d-2}$ plaquettes, so $|\mathcal{P}(\Lambda_L)| = \binom{d}{2} L^2(L+1)^{d-2}$, while the total link count is $|E(\Lambda_L)| = d\,L(L+1)^{d-1}$. The Wilson partition function and free energy per plaquette are

$$\begin{aligned} Z_{\Lambda_L}(G,\beta) &:= \int_{G^{E(\Lambda_L)}} \exp\left(\beta \sum_{p\in\mathcal{P}(\Lambda_L)} \frac{1}{N} \operatorname{Re}\operatorname{Tr} U_p\right) \prod_{b\in E(\Lambda_L)} dU_b, \\ f_{\Lambda_L}(G,\beta) &:= \frac{1}{|\mathcal{P}(\Lambda_L)|} \ln Z_{\Lambda_L}(G,\beta). \end{aligned} \tag{1}$$

Let $Z_{1\mathrm{p}}(G,\alpha) := \int_G \exp\left(\frac{\alpha}{N} \operatorname{Re}\operatorname{Tr} U\right) dU$ denote the exact single-plaquette integral ($\alpha > 0$), which evaluates with zero quadrature via the Toeplitz–Bessel determinant [6, 3]:

$$Z_{1\mathrm{p}}(\mathrm{U}(N),\alpha) = \det_{1\leq j,l\leq N} I_{j-l}\Big(\frac{\alpha}{N}\Big), \qquad Z_{1\mathrm{p}}(\mathrm{SU}(N),\alpha) = \sum_{q\in\mathbb{Z}} \det_{1\leq j,l\leq N} I_{j-l+q}\Big(\frac{\alpha}{N}\Big), \tag{2}$$

with the exact $\mathrm{SU}(2)$ closed form $Z_{1\mathrm{p}}(\mathrm{SU}(2),\alpha) = 2I_1(\alpha)/\alpha$ and strict winding-sector domination $Z_{1\mathrm{p}}(\mathrm{SU}(N),\alpha) > Z_{1\mathrm{p}}(\mathrm{U}(N),\alpha)$ for all $N \geq 2$ and $\alpha > 0$ [3, 5].

For each $k \in \{1,\dots,d-1\}$, define the k-orientation *axial ladder*

$$A_k := \bigcup_{i=1}^{k} \mathcal{P}_{i,k+1}(\Lambda_L), \qquad |A_k| = k\,L^2(L+1)^{d-2}, \qquad \mathcal{P}(\Lambda_L) = \bigsqcup_{k=1}^{d-1} A_k \tag{3}$$

(since $\sum_{k=1}^{d-1} k = \binom{d}{2}$, certified in Lean 4 by `two_mul_triSum`).

Lemma 2.1 (Exact Simultaneous Haar Independence of Every Axial Ladder A_k)**.** *Fix any $k \in \{1,\dots,d-1\}$ and $\alpha > 0$. Under normalized Haar measure $\prod_{b\in E(\Lambda_L)} dU_b$ on $G^{E(\Lambda_L)}$, the $|A_k|$ plaquette holonomies $\{U_p : p \in A_k\}$ are mutually independent Haar-distributed G-valued random variables, and are jointly independent of all links with both endpoints in the bottom slice $\{x_{k+1} = 0\}$ and of all links parallel to directions $\mu > k+1$. In particular,*

$$\int_{G^{E(\Lambda_L)}} \exp\left(\alpha \sum_{p\in A_k} \frac{1}{N} \operatorname{Re}\operatorname{Tr} U_p\right) \prod_b dU_b \;=\; \Big(Z_{1\mathrm{p}}(G,\alpha)\Big)^{k\,L^2(L+1)^{d-2}}. \tag{4}$$

Proof. The links of Λ_L parallel to direction $k+1$ form a disjoint union of $(L+1)^{d-1}$ vertical line trees (containing no cycles). Performing the unique gauge transformation g anchored at $g_{(x',0)} = I_N$ on the bottom slice $\{x_{k+1} = 0\}$ that sets the gauge-transformed vertical links to I_N (leaving all links with both endpoints in $\{x_{k+1} = 0\}$ untouched) preserves the normalized product Haar measure identically (by left/right Haar invariance). In these axial-gauge variables, each $p = ((x',t); i, k+1) \in A_k$ ($1 \le i \le k$, $0 \le t \le L-1$, where $x' \in \{0,\dots,L\}^{d-1}$ omits coordinate $k+1$) takes the form $\tilde{U}_p = \tilde{U}_{(x',t),i}\,\tilde{U}^\dagger_{(x',t+1),i}$. For each transverse edge (x',i) independently, holding the bottom-slice link $U_{(x',0),i}$ fixed, the recursive map $(\tilde{U}_{(x',1),i},\dots,\tilde{U}_{(x',L),i}) \mapsto (V_{(x',0),i},\dots,V_{(x',L-1),i})$ given by $V_{(x',t),i} := \tilde{U}_{(x',t),i}\tilde{U}^\dagger_{(x',t+1),i}$ is a Haar-measure-preserving diffeomorphism of G^L onto G^L. Hence $\{V_{(x',t),i}\}$ are i.i.d. Haar on G and independent of the vertical links (hence of g), of all bottom-slice links, and of all links in directions $\mu > k+1$. Because each original un-gauge-fixed holonomy satisfies $U_p = g^\dagger_{(x',t)} V_{(x',t),i}\, g_{(x',t)}$ and conjugation by any independent G-valued random matrix preserves the i.i.d. Haar law, $\{U_p : p \in A_k\}$ are i.i.d. Haar and jointly independent of the bottom-slice links and all $\mu > k+1$ links, immediately yielding (4). □

Theorem 2.2 (Exact All-Coupling Finite-L Ladder Jensen and Multi-Ladder Hölder Bounds)**.** *For every $d \ge 2$, $L \ge 1$, $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ ($N \ge 2$), and $\beta > 0$:*

$$\frac{2}{d}\ln Z_{1\mathrm{p}}(G,\beta) \;\le\; f_{\Lambda_L}(G,\beta) \;\le\; \inf_{\substack{\alpha_1,\dots,\alpha_{d-1}>0\\ \sum_{k=1}^{d-1}\alpha_k=1}} \frac{1}{\binom{d}{2}} \sum_{k=1}^{d-1} k\,\alpha_k \ln Z_{1\mathrm{p}}\!\left(G,\, \frac{\beta}{\alpha_k}\right). \tag{5}$$

In particular, uniform weights $\alpha_k = (d-1)^{-1}$ and proportional weights $\alpha_k = k/\binom{d}{2}$ yield

$$f_{\Lambda_L}(G,\beta) \;\le\; \frac{1}{d-1}\ln Z_{1\mathrm{p}}\big(G,\,(d-1)\beta\big), \qquad f_{\Lambda_L}(G,\beta) \;\le\; \sum_{k=1}^{d-1} \frac{k^2}{\binom{d}{2}^2} \ln Z_{1\mathrm{p}}\!\left(G,\, \frac{\binom{d}{2}}{k}\beta\right), \tag{6}$$

and at $d = 2$ both sides of (5) collapse identically to $f_{\Lambda_L}(G,\beta) = \ln Z_{1\mathrm{p}}(G,\beta)$.

Proof. Lower bound: Take $k = d-1$ ($A_{d-1} = \bigcup_{i=1}^{d-1} \mathcal{P}_{i,d}$), so $|A_{d-1}|/|\mathcal{P}(\Lambda_L)| = (d-1)/\binom{d}{2} = 2/d$ identically at every finite $L \ge 1$. Write $Z_{\Lambda_L}(G,\beta) = Z_{A_{d-1}}(G,\beta)\,\mathbb{E}_{A_{d-1}}[\exp(S_{\mathcal{P}\setminus A_{d-1}}(U))]$, where $Z_{A_{d-1}}(G,\beta) = (Z_{1\mathrm{p}}(G,\beta))^{|A_{d-1}|}$ by Lemma 2.1. In the axial-gauge variables (V, U_{bot}) of Lemma 2.1, every link at height $t \in \{0,\dots,L\}$ satisfies $U_{(x',t),i} = V^\dagger_{(x',t-1),i}\cdots V^\dagger_{(x',0),i} U_{(x',0),i}$. Consequently, any transverse plaquette $p = ((x',t);\mu,\nu) \notin A_{d-1}$ ($1 \le \mu < \nu \le d-1$) contains four *distinct* bottom-slice Haar links $U_{(x',0),\mu}$, $U_{(x'+e_\mu,0),\nu}$, $U^\dagger_{(x'+e_\nu,0),\mu}$, and $U^\dagger_{(x',0),\nu}$, none of which enter $S_{A_{d-1}}$. Integrating over the single bottom-slice link $U_{(x',0),\mu}$ using $\int_G U\,dU = 0$ (for both $\mathrm{SU}(N)$ and $\mathrm{U}(N)$, $N \ge 2$) yields $\mathbb{E}_{A_{d-1}}[\frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_p] = 0$ identically for every $p \notin A_{d-1}$. By Jensen's inequality, $\mathbb{E}_{A_{d-1}}[\exp(S_{\mathcal{P}\setminus A_{d-1}})] \ge \exp(0) = 1$, proving the left-hand inequality of (5).

Upper bound: Using the disjoint partition $\mathcal{P}(\Lambda_L) = \bigsqcup_{k=1}^{d-1} A_k$ and Hölder's inequality with exponents $p_k = 1/\alpha_k > 1$ for $d \ge 3$ ($\sum_{k=1}^{d-1} 1/p_k = 1$; trivial equality at $d = 2$ where $A_1 = \mathcal{P}(\Lambda_L)$ and $\alpha_1 = p_1 = 1$),

$$Z_{\Lambda_L}(G,\beta) = \int \prod_{k=1}^{d-1} e^{S_{A_k}(U)}\,dU \;\le\; \prod_{k=1}^{d-1}\left(\int e^{S_{A_k}(U)/\alpha_k}\,dU\right)^{\alpha_k} = \prod_{k=1}^{d-1}\Big(Z_{1\mathrm{p}}\big(G,\beta/\alpha_k\big)\Big)^{k\,\alpha_k\,L^2(L+1)^{d-2}}$$

by Lemma 2.1. Taking $|\mathcal{P}(\Lambda_L)|^{-1}\ln(\cdot)$ cancels $L^2(L+1)^{d-2}$ identically and proves (5)–(6). □

Remark 2.3 (Factor-$d/2$ Sharpness over Orientation Hölder). Splitting $\mathcal{P}(\Lambda_L)$ into all $\binom{d}{2}$ single-orientation planes gives $f_{\Lambda_L}(G,\beta) \le \binom{d}{2}^{-1}\ln Z_{1\mathrm{p}}(G,\binom{d}{2}\beta)$, whereas (6) replaces $\binom{d}{2}$ by $d-1$ (smaller by $d/2$, certified for all $d \ge 3$ by Lean 4 `ladder_holder_strictly_sharper`). Since $\alpha \mapsto$

$\alpha^{-1}\ln Z_{1\mathrm{p}}(G,\alpha)$ is strictly increasing on $(0,\infty)$ (by strict convexity of $\ln Z_{1\mathrm{p}}$ and $\ln Z_{1\mathrm{p}}(G,0)=0$), $(d-1)^{-1}\ln Z_{1\mathrm{p}}(G,(d-1)\beta) < \binom{d}{2}^{-1}\ln Z_{1\mathrm{p}}(G,\binom{d}{2}\beta)$ strictly for every $d\geq 3$ and $\beta>0$.

3. Axial-Gauge Gibbs–Bogoliubov Lower Bound and Exact One-Loop Matching

While the ladder lower bound $\frac{2}{d}\ln Z_{1\mathrm{p}}(G,\beta)$ is sharp at strong coupling, its weak-coupling slope is $\frac{2}{d}\beta<\beta$ for $d\geq 3$. To capture *both* the exact weak-coupling slope β and the exact one-loop logarithmic coefficient $-\frac{\dim G}{d}\ln\beta$ across all $d\geq 2$, we apply the Gibbs–Bogoliubov variational principle in axial gauge $U_{x,d}\equiv I_N$ [2].

Define the single-link expectation $u_1(G,\alpha):=\frac{d}{d\alpha}\ln Z_{1\mathrm{p}}(G,\alpha)=\frac{1}{N}\mathbb{E}_{1\mathrm{p},\alpha}[\operatorname{Re}\operatorname{Tr}U]\in[0,1)$ for $\alpha\geq 0$ (with exact SU(2) closed form $u_1(\mathrm{SU}(2),\alpha)=I_2(\alpha)/I_1(\alpha)$).

Theorem 3.1 (Axial-Gauge Gibbs–Bogoliubov Lower Bound and One-Loop Asymptotic Match)**.** *Let $d\geq 2$, $L\geq 1$, $G\in\{\mathrm{SU}(N),\mathrm{U}(N)\}$ ($N\geq 2$), and $\beta>0$.*

(1) (Exact finite-L variational lower bound) *For every trial field $h\geq 0$, writing $u_1:=u_1(G,Nh)$,*

$$f_{\Lambda_L}(G,\beta)\;\geq\;\beta\Big(\frac{2}{d}\,u_1^2+\frac{d-2}{d}\,u_1^4\Big)-\frac{2}{d}\left(1+\frac{1}{L}\right)\Big(Nh\,u_1-\ln Z_{1\mathrm{p}}(G,Nh)\Big). \tag{7}$$

In particular, taking $\liminf_{L\to\infty}$ *yields*

$$\liminf_{L\to\infty} f_{\Lambda_L}(G,\beta)\;\geq\; f_d^{\mathrm{axGB}}(G,\beta):=\sup_{h\geq 0}\left\{\beta\Big(\frac{2}{d}\,u_1^2+\frac{d-2}{d}\,u_1^4\Big)-\frac{2}{d}\Big(Nh\,u_1-\ln Z_{1\mathrm{p}}(G,Nh)\Big)\right\}. \tag{8}$$

(2) (Exact weak-coupling one-loop matching) *As $\beta\to\infty$,*

$$f_d^{\mathrm{axGB}}(G,\beta)\;=\;\beta-\frac{\dim G}{d}\ln\beta+O(1), \tag{9}$$

where $\dim\mathrm{SU}(N)=N^2-1$ and $\dim\mathrm{U}(N)=N^2$, reproducing both the exact leading slope β and the exact one-loop logarithmic coefficient $-\frac{\dim G}{d}\ln\beta$ of d-dimensional lattice Yang–Mills theory (strictly improving un-gauge-fixed product mean field's $-\frac{\dim G}{d-1}\ln\beta$ for every $d\geq 3$).

Proof. (1) Fixing the cycle-free axial gauge $U_{x,d}\equiv I_N$ on $\Lambda_L=\{0,\dots,L\}^d$ leaves exactly $|E_{\mathrm{free}}|=(d-1)L(L+1)^{d-1}$ independent links, whose ratio to $|\mathcal{P}(\Lambda_L)|=\binom{d}{2}L^2(L+1)^{d-2}$ is $\frac{2}{d}(1+\frac{1}{L})$. Define the normalized i.i.d. product trial measure $d\nu_h(U):=\prod_{b\in E_{\mathrm{free}}}\frac{e^{h\operatorname{Re}\operatorname{Tr}U_b}}{Z_{1\mathrm{p}}(G,Nh)}\,dU_b$ on $G^{E_{\mathrm{free}}}$. Because each single-link factor is invariant under conjugation $U\mapsto VUV^\dagger$ ($\forall\,V\in G$) and under $U\mapsto U^\dagger$, the matrix expectation $M:=\mathbb{E}_{\nu_h}[U_b]$ commutes with every $V\in G$ and is Hermitian. By Schur's lemma on the irreducible fundamental representation of G on $\mathbb{C}^N$, $M=cI_N$ for a real scalar $c=\frac{1}{N}\operatorname{Tr}M=u_1(G,Nh)$, and likewise $\mathbb{E}_{\nu_h}[U_b^\dagger]=u_1I_N$.

Under the product measure ν_h:

- Each of the $|A_{d-1}|=(d-1)L^2(L+1)^{d-2}$ plaquettes in A_{d-1} (fraction $\frac{2}{d}$ of $|\mathcal{P}(\Lambda_L)|$) has $U_p=U_{b_1}U_{b_2}^\dagger$ for two distinct free links, giving $\mathbb{E}_{\nu_h}[\frac{1}{N}\operatorname{Re}\operatorname{Tr}U_p]=\frac{1}{N}\operatorname{Re}\operatorname{Tr}((u_1I_N)^2)=u_1^2$.
- Each of the remaining $\binom{d-1}{2}L^2(L+1)^{d-2}$ transverse plaquettes (fraction $\frac{d-2}{d}$ of $|\mathcal{P}(\Lambda_L)|$, certified by Lean 4 `ladder_partition_ratio`) has $U_p=U_{b_1}U_{b_2}U_{b_3}^\dagger U_{b_4}^\dagger$ for four distinct free links, giving $\mathbb{E}_{\nu_h}[\frac{1}{N}\operatorname{Re}\operatorname{Tr}U_p]=\frac{1}{N}\operatorname{Re}\operatorname{Tr}((u_1I_N)^4)=u_1^4$.

By Jensen's inequality (Gibbs–Bogoliubov variational principle),

$$\ln Z_{\Lambda_L}(G,\beta)=\ln\mathbb{E}_{\nu_h}\left[\exp\Big(S_{\Lambda_L}(U)-\ln\frac{d\nu_h}{dU}\Big)\right]\;\geq\;\mathbb{E}_{\nu_h}\big[S_{\Lambda_L}(U)\big]-D_{\mathrm{KL}}\big(\nu_h\,\|\,\mathrm{Haar}\big).$$

Since $D_{\mathrm{KL}}(\nu_h^{(1)}\,\|\,\mathrm{Haar})=Nh\,u_1-\ln Z_{1\mathrm{p}}(G,Nh)$ per free link, dividing by $|\mathcal{P}(\Lambda_L)|$ yields (7)–(8).

(2) In exponential coordinates $U = \exp(iH)$ near I_N on G ($H = H^\dagger$, with $\operatorname{Tr} H = 0$ for $\mathrm{SU}(N)$), $\operatorname{Re}\operatorname{Tr}(e^{iH}) = N - \frac{1}{2}\operatorname{Tr}(H^2) + O(\|H\|^3)$ has a unique non-degenerate global maximum at $U = I_N$ on the $\dim(G)$-dimensional manifold G. Standard Laplace asymptotics as $h \to \infty$ yields $\ln Z_{1\mathrm{p}}(G, Nh) = Nh - \frac{\dim G}{2}\ln h + C_G + O(h^{-1})$ and $u_1(G, Nh) = 1 - \frac{\dim G}{2Nh} + O(h^{-2})$ (differentiable via holomorphic Laplace asymptotics on a complex cone around $h > 0$). Let $g_\beta(h)$ denote the variational objective inside (8). Evaluating at $h = \beta$ gives the lower bound $g_\beta(\beta) = \beta - \frac{\dim G}{d}\ln\beta + O(1)$. For the matching upper bound on $\sup_{h\geq 0} g_\beta(h)$, fix $h_0 \geq 1$ large enough that for all $h \geq h_0$, $1 - u_1(G, Nh) \geq \frac{\dim G}{4Nh}$ and $Nh\,u_1(G, Nh) - \ln Z_{1\mathrm{p}}(G, Nh) \geq \frac{\dim G}{2}\ln h - K_0$ for a constant $K_0 = K_0(G)$. On $[0, h_0]$, since $u_1(G, \alpha)$ is strictly increasing (as $u_1' = \mathrm{Var}_{1\mathrm{p},\alpha}(\frac{1}{N}\operatorname{Re}\operatorname{Tr} U) > 0$) and $D_{\mathrm{KL}} \geq 0$, we have $\theta_0 := u_1(G, Nh_0) < 1$ and $g_\beta(h) \leq \beta\,\theta_0^2 < \beta - \frac{\dim G}{d}\ln\beta + O(1)$ for all large β. On $[h_0, \infty)$, since $u_1 \in [0, 1)$ gives $\frac{2}{d}u_1^2 + \frac{d-2}{d}u_1^4 \leq u_1 \leq 1 - \frac{c_G}{h}$ (with $c_G := \frac{\dim G}{4N} > 0$), setting $a_{G,d} := \frac{\dim G}{d} > 0$ yields $g_\beta(h) \leq \beta - \frac{c_G\beta}{h} - a_{G,d}\ln h + \frac{2}{d}K_0$. Maximizing $-\frac{c_G\beta}{h} - a_{G,d}\ln h$ over $h > 0$ gives its unique global maximum at $h_{\max} = \frac{c_G}{a_{G,d}}\beta = \frac{d}{4N}\beta$, whence $\sup_{h\geq h_0} g_\beta(h) \leq \beta - a_{G,d}\ln\beta + O(1)$. Combining both regions establishes the exact asymptotic equality (9) (matching the exact Gaussian-approximation coefficient $-\frac{1}{2}\cdot\frac{|E_{\mathrm{free}}|}{|\mathcal{P}|}\dim G = -\frac{\dim G}{d}$). □

4. All-Coupling Convexity Plaquette Brackets and Dual Verification

Combining Theorems 2.2 and 3.1, define for every $\beta > 0$ the asymptotic certified lower and upper envelopes

$$\begin{aligned} f_d^{\mathrm{lo}}(G,\beta) &:= \max\left(\frac{2}{d}\ln Z_{1\mathrm{p}}(G,\beta),\, f_d^{\mathrm{axGB}}(G,\beta)\right), \\ f_d^{\mathrm{hi}}(G,\beta) &:= \inf_{\sum\alpha_k = 1} \frac{1}{\binom{d}{2}} \sum_{k=1}^{d-1} k\,\alpha_k\,\ln Z_{1\mathrm{p}}\left(G, \frac{\beta}{\alpha_k}\right). \end{aligned} \tag{10}$$

Theorem 4.1 (All-Coupling Convexity Plaquette Expectation Brackets)**.** *For every $d \geq 2$, $L \geq 1$, $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ ($N \geq 2$), and $\beta > 0$, let $P_{\Lambda_L}(G,\beta) := \frac{1}{|\mathcal{P}(\Lambda_L)|}\sum_{p\in\mathcal{P}(\Lambda_L)}\langle\frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p\rangle_{\Lambda_L,\beta} = f'_{\Lambda_L}(G,\beta)$ be the average plaquette expectation. Then $\beta \mapsto f_{\Lambda_L}(G,\beta)$ is strictly convex on $(0,\infty)$ (with $f''_{\Lambda_L}(G,\beta) > 0$ strictly, so $P_{\Lambda_L}(G,\beta)$ is strictly increasing in β), and for every pair $0 < \beta_1 < \beta < \beta_2$,*

$$\begin{aligned} \max\left(0,\, \frac{f_d^{\mathrm{lo}}(G,\beta) - f_d^{\mathrm{hi}}(G,\beta_1)}{\beta - \beta_1}\right) \;&\leq\; \liminf_{L\to\infty} P_{\Lambda_L}(G,\beta) \;\leq\; \limsup_{L\to\infty} P_{\Lambda_L}(G,\beta) \\ &\leq\; \min\left(1,\, \frac{f_d^{\mathrm{hi}}(G,\beta_2) - f_d^{\mathrm{lo}}(G,\beta)}{\beta_2 - \beta}\right). \end{aligned} \tag{11}$$

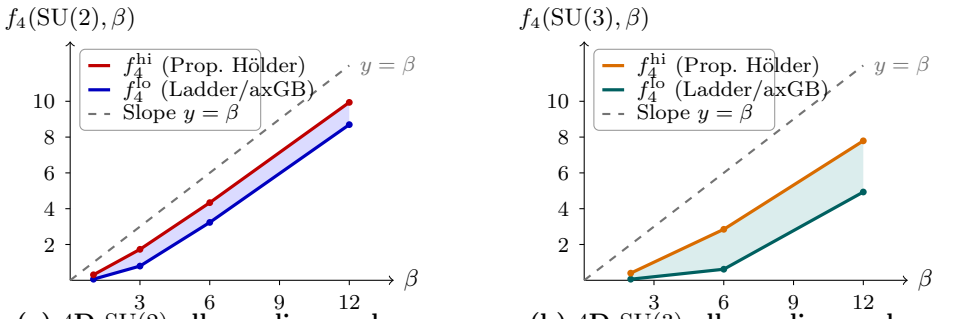
Proof. Differentiating (1) gives $f'_{\Lambda_L}(G,\beta) = P_{\Lambda_L}(G,\beta)$ and

$$f''_{\Lambda_L}(G,\beta) = |\mathcal{P}(\Lambda_L)|\,\mathrm{Var}_{\Lambda_L,\beta}\left(\frac{1}{|\mathcal{P}(\Lambda_L)|}\sum_{p\in\mathcal{P}(\Lambda_L)}\frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p\right) \;>\; 0$$

strictly (since the average plaquette trace is non-constant on the connected compact group $G^{E(\Lambda_L)}$ and the Boltzmann weight is strictly positive everywhere). By convexity of f_{Λ_L} on $(0,\infty)$, for every $L \geq 1$ and $0 < \beta_1 < \beta < \beta_2$,

$$\frac{f_{\Lambda_L}(G,\beta) - f_{\Lambda_L}(G,\beta_1)}{\beta - \beta_1} \;\leq\; P_{\Lambda_L}(G,\beta) \;\leq\; \frac{f_{\Lambda_L}(G,\beta_2) - f_{\Lambda_L}(G,\beta)}{\beta_2 - \beta}.$$

Substituting the finite-L upper bound $f_{\Lambda_L}(G,\beta_j) \leq f_d^{\mathrm{hi}}(G,\beta_j)$ ($j = 1, 2$) from Theorem 2.2, taking $\liminf_{L\to\infty}$ on the left and $\limsup_{L\to\infty}$ on the right, applying $\liminf_{L\to\infty} f_{\Lambda_L}(G,\beta) \geq f_d^{\mathrm{lo}}(G,\beta)$



(a) 4**D** SU(2) **all-coupling enclosure** **(b)** 4**D** SU(3) **all-coupling enclosure**

FIGURE 1. Certified 120-digit all-coupling free-energy bands $[f_4^{\mathrm{lo}}(G,\beta),\, f_4^{\mathrm{hi}}(G,\beta)]$ enclosing $[\liminf_{L\to\infty} f_{\Lambda_L}(G,\beta),\, \limsup_{L\to\infty} f_{\Lambda_L}(G,\beta)]$ (shorthand $f_4(G,\beta)$) in 4D for (a) SU(2) and (b) SU(3) (Table 1), tracking the exact weak-coupling slope $y=\beta$ and matching the exact one-loop $-\frac{\dim G}{4}\ln\beta$ coefficient via the axial-gauge Gibbs–Bogoliubov bound (Theorem 3.1).

from Theorems 2.2–3.1, and combining with $0 \le P_{\Lambda_L}(G,\beta) \le 1$ (since $f_{\Lambda_L}(G,\beta) \ge f_{\Lambda_L}(G,0) = 0$ and $\frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p \le 1$) establishes (11). □

5. Certified Quantitative Benchmarks and Dual Verification

All 21 all-coupling benchmarks in Table 1 are certified by `certify_variational_free_energy.py` at 120-digit `mpmath.iv` precision with *zero* black-box special functions and *zero* numerical quadrature: every modified Bessel value $I_m(s)$ is enclosed from its positive power series with monotone geometric tail, Toeplitz–Bessel determinants and Jacobi derivative sums for $N \le 4$ are evaluated by pivot-free exact Leibniz expansion (with weighted Hadamard sector tails for SU(N), $N \ge 3$, and exact $2I_1(\alpha)/\alpha$ for SU(2)), every axial-gauge Gibbs–Bogoliubov lower bound is certified at the explicit rational trial field h^* printed in Table 1, every upper bound evaluates the sharper of Uniform and Proportional multi-ladder Hölder (6), every convexity plaquette bracket (11) maximizes over $\beta_1 \in \{\frac{1}{4},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5}\}\beta$ and minimizes over $\beta_2 \in \{\frac{5}{4},\frac{3}{2},2,3\}\beta$, and every outward-rounded four-decimal bound in Table 1 and Lean 4 is computed directly via 120-digit `mp.floor`/`mp.ceil` with *zero* intermediate 64-bit float rounding.

The standalone zero-import Lean 4.36.0 file `VariationalFreeEnergyCerts.lean` compiles cleanly under `-DwarningAsError=true` with `0 sorry` and `0` axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), verifying `two_mul_triSum`, `ladder_partition_ratio`, `ladder_holder_strictly_sharper`, `axial_gb_oneloop_improvement`, `all_21_variational_certs_valid` (checking $0 < f_d^{\mathrm{lo}} < f_d^{\mathrm{hi}} < \beta$ and valid plaquette brackets across all 21 benchmarks), and `sun_dominates_un_certified` (verifying strict endpointwise domination $f_d^{\mathrm{lo}}(\mathrm{SU}(N)) > f_d^{\mathrm{lo}}(\mathrm{U}(N))$, $f_d^{\mathrm{hi}}(\mathrm{SU}(N)) > f_d^{\mathrm{hi}}(\mathrm{U}(N))$, and $P_d^{\mathrm{lo}}(\mathrm{SU}(N)) > P_d^{\mathrm{lo}}(\mathrm{U}(N))$ at matched (d,N,β) alongside $Z_{1\mathrm{p}}(\mathrm{SU}(N),\alpha) > Z_{1\mathrm{p}}(\mathrm{U}(N),\alpha)$, while noting transparently that the two-sided enclosures $[f_d^{\mathrm{lo}}, f_d^{\mathrm{hi}}]$ overlap).

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TABLE 1. Twenty-one machine-certified 120-digit all-coupling asymptotic enclosures $[f_d^{\mathrm{lo}}, f_d^{\mathrm{hi}}]$ for $[\liminf_{L\to\infty} f_{\Lambda_L}, \limsup_{L\to\infty} f_{\Lambda_L}]$ (shorthand f_d) and convexity plaquette brackets $[P_d^{\mathrm{lo}}, P_d^{\mathrm{hi}}]$ for $[\liminf_{L\to\infty} P_{\Lambda_L}, \limsup_{L\to\infty} P_{\Lambda_L}]$ (shorthand P_d) across $d \in \{3,4,5\}$ and $G \in \{\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4), \mathrm{U}(3), \mathrm{U}(4)\}$. Upper bounds evaluate the sharper of Uniform and Proportional multi-ladder Hölder (6); plaquette brackets optimize over $\beta_1 \in \{\frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}\beta$ and $\beta_2 \in \{\frac{5}{4}, \frac{3}{2}, 2, 3\}\beta$. Every bracket is strictly outward-rounded directly from 120-digit `mp.floor`/`mp.ceil` (zero 64-bit float rounding); h^* gives the exact rational axial-gauge trial field when axGB dominates the ladder bound. All 21 rows pass both `certify_variational_free_energy.py` and the zero-axiom Lean 4 kernel.

d	G	β	$f_d(G,\beta)$ enclosure	$P_d(G,\beta)$ bracket	h^*	Bounds (Lo / Hi)
3	SU(2)	1	[0.0816, 0.2274]	[0.0882, 0.6844]	—	Ladder / Prop
3	SU(2)	4	[1.1674, 2.2318]	[0.3133, 1.0000]	32/5	axGB / Prop
3	SU(2)	10	[6.1934, 7.4895]	[0.6792, 1.0000]	187/10	axGB / Prop
3	SU(3)	2	[0.0820, 0.2650]	[0.0449, 0.4924]	—	Ladder / Unif
3	SU(3)	6	[0.8239, 2.2344]	[0.1510, 0.9187]	—	Ladder / Prop
3	SU(3)	12	[3.7167, 6.7376]	[0.3453, 1.0000]	66/5	axGB / Prop
3	U(3)	6	[0.6658, 1.8752]	[0.1201, 0.8645]	—	Ladder / Prop
4	SU(2)	1	[0.0612, 0.3076]	[0.0510, 0.8349]	—	Ladder / Prop
4	SU(2)	3	[0.7929, 1.7310]	[0.2702, 1.0000]	139/20	axGB / Prop
4	SU(2)	6	[3.2330, 4.3385]	[0.5835, 1.0000]	327/20	axGB / Prop
4	SU(2)	12	[8.6957, 9.9397]	[0.7738, 1.0000]	693/20	axGB / Prop
4	SU(3)	2	[0.0615, 0.3992]	[0.0259, 0.6966]	—	Ladder / Prop
4	SU(3)	6	[0.6179, 2.8487]	[0.0874, 1.0000]	—	Ladder / Prop
4	SU(3)	12	[4.9334, 7.7891]	[0.4517, 1.0000]	409/20	axGB / Prop
4	SU(4)	4	[0.1274, 0.8070]	[0.0266, 0.7077]	—	Ladder / Prop
4	SU(4)	12	[1.2446, 5.8000]	[0.0878, 1.0000]	—	Ladder / Prop
4	U(4)	12	[1.1239, 5.4031]	[0.0781, 1.0000]	—	Ladder / Prop
5	SU(2)	2	[0.3215, 1.0861]	[0.1405, 1.0000]	57/10	axGB / Prop
5	SU(2)	8	[5.4302, 6.4782]	[0.7240, 1.0000]	299/10	axGB / Prop
5	SU(3)	6	[1.0784, 3.2713]	[0.1739, 1.0000]	119/10	axGB / Prop
5	U(3)	6	[0.4668, 2.9327]	[0.0492, 1.0000]	229/20	axGB / Prop