Complete contraction of unital qubit channels and order-constrained depolarization

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Abstract

We determine the complete equal-reference contraction coefficients of the minimal quantum Fisher metric, its quadratic chi-square divergence and the finite triangular Petz divergence for every unital qubit channel. For a faithful normalized Choi state J, all three equal $1-16/\operatorname{Tr}(J^{-1})$; for singular J they equal one. An explicit qubit–qutrit lower family matches a harmonic-weight upper bound and gives the exact stabilization envelope, including a maximal factor of nine between coefficient suprema. For depolarization in dimension d, a commuting Bell family matches the order bound $\lambda^2\kappa/[1+\lambda(\kappa-1)]$ for each normalized nonaffine operator-convex Petz generator when $1<\kappa\le d^2/2$. A principal-rich rank-two ladder extends SLD and triangular sharpness to $\kappa\le d(2d-1)/2$, with the budget taken against the actual reference marginal. A separate logarithmic analysis gives the complete equal-reference BKM and finite relative-entropy lower bound $\lambda^2P/[1+\lambda(P-1)]$, $P=d(d-1)$; for $d\ge 3$ it improves the endpoint of the specified Bell family. In the qubit case we identify exact classical-reference and separable/PPT restrictions over the full physical noise range. A localized rational counterexample addresses one printed Doeblin equivalence. These results do not determine unrestricted BKM, relative-entropy, actual-mutual-information or capacity coefficients; the earlier ladder tangents are not unitary phase models. A separate norm-one local Hamiltonian with reference dimension d realizes the BKM/Umegaki lower at fixed finite phase, and certified pairs exceed the entire common-Bell-diagonal class at both noise signs. Its exact classical benchmark is reconstructed throughout the physical noise range with prior-work limits explicit. The latter family spends order approaching d^2 and loses absolute information in the supremum limit. An explicit dimension-scaled family gives a linearly growing relative Bell separation with vanishing information and polynomial-in-dimension conditioning. An explicit joint binary measurement retains the leading output relative entropy in that family. Faithful limits, adverse controls, prior-work dependencies and model-review limits are explicit.

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1 Quantities and scope

Let A and R be finite-dimensional systems. All local maps below are tensored with id_R, whether or not this is written. A state is faithful if it is positive definite. We use the Hilbert–Schmidt inner product $\langle X,Y\rangle=\operatorname{Tr}X^\dagger Y$ on the *complex* Hilbert space of matrices. Define $L_a(X)=aX$ and $R_b(X)=Xb$. For faithful a,b, these are commuting positive invertible superoperators. This

statement concerns positivity on Hilbert–Schmidt space, not preservation of the cone of positive matrices.

For a faithful state σ and a Hermitian tangent Z, set

$$g_\sigma^{\mathrm{SLD}}(Z) = \langle Z, \left[(L_\sigma + R_\sigma)/2\right]^{-1} Z\rangle. \tag{1}$$

This is the minimal monotone, or symmetric logarithmic derivative (SLD), quantum Fisher quadratic form. Its associated finite chi-square divergence is

$$\chi^2_{\min}(\rho\|\sigma) = g_\sigma^{\mathrm{SLD}}(\rho - \sigma). \tag{2}$$

For a nonaffine operator-convex $f : (0,\infty) \to \mathbb{R}$, $f(1) = 0$, the Petz divergence is

$$D_f(\rho\|\sigma) = \langle \sigma^{1/2}, f(L_\rho R_\sigma^{-1})\sigma^{1/2}\rangle. \tag{3}$$

We will also use the distinct finite divergence

$$f_\triangle(x) = \frac{(x-1)^2}{x+1}, \qquad D_\triangle(\rho\|\sigma) = D_{f_\triangle}(\rho\|\sigma). \tag{4}$$

Although (2) and (4) have proportional local forms, they are not the same finite divergence. Nor do we identify either with the Bures distance or its geodesic contraction.

For a local channel $\mathcal{N}$, define the complete equal-reference coefficient

$$\eta_f^c(\mathcal{N}) = \sup_{\substack{|R|<\infty,\ \rho,\sigma>0 \\ \rho\neq\sigma,\ \mathrm{Tr}_A\,\rho = \mathrm{Tr}_A\,\sigma}} \frac{D_f(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))}{D_f(\rho\|\sigma)}. \tag{5}$$

Define its SLD analogue by optimizing $g^{\mathrm{SLD}}_{\mathcal{N}(\sigma)}(\mathcal{N}(Z))/g^{\mathrm{SLD}}_\sigma(Z)$ over faithful σ, nonzero Hermitian Z with $\mathrm{Tr}_A\, Z = 0$, and all finite R. It is also the coefficient of (2): every such tangent is realized by a sufficiently small faithful pair.

These restrictions matter. Actual mutual information compares a state with the product of *its own* marginals. Conditional coefficients can instead subtract a divergence between unequal reference marginals. Fixed-point entropy decay compares a state with a conditional expectation. We do not replace one optimization by another.

2 Known representation and a cancellation bound

The positive resolvent representation of operator-convex functions, and the resulting representation of Petz divergences, are established facts; see Hiai–Ruskai [1], Proposition 2.1 and equations (12)–(13). After removing an affine term, any nonaffine operator-convex f as above has

$$f(x) = f'(1)(x-1) + c(x-1)^2 + \int_{[0,\infty)} \frac{(x-1)^2}{x+t}\, d\mu(t), \tag{6}$$

where $c \geq 0$, μ is positive and $\int (1+t)^{-1}\, d\mu(t) < \infty$. Writing $Z = \rho - \sigma$, for states of the same trace this gives

$$D_f(\rho\|\sigma) = c\ \mathrm{Tr}\, Z^2\sigma^{-1} + \int_{[0,\infty)} \langle Z, (L_\rho + tR_\sigma)^{-1} Z\rangle\, d\mu(t). \tag{7}$$

We do not claim this representation as a contribution.

Lemma 1. *For positive definite Q, the inverse quadratic form $F(Q,Z) = \langle Z, Q^{-1}Z\rangle$ is jointly convex in Q, Z, for arbitrary complex vectors Z.*

Proof. Use the variational identity

$$F(Q,Z) = \sup_Y \{2\,\mathrm{Re}\langle Z, Y\rangle - \langle Y, QY\rangle\}.$$

Each expression inside the supremum is affine in Q, Z. The same identity also shows that for any trial Y, with $e = Z - QY$,

$$F(Q,Z) = 2\,\mathrm{Re}\langle Z, Y\rangle - \langle Y, QY\rangle + \langle e, Q^{-1}e\rangle. \tag{8}$$

□

Theorem 2 (Harmonic cancellation)**.** *Let $U_1, \ldots, U_m$ be local unitaries, $m \geq 2$, and let*

$$\mathcal{E} = \frac{1}{m}\sum_{j=1}^m \mathrm{Ad}_{U_j}, \qquad \mathcal{N}_w = \sum_{j=1}^m w_j\,\mathrm{Ad}_{U_j}, \qquad w_j > 0, \quad \sum_j w_j = 1.$$

For faithful states ρ, σ satisfying $\mathcal{E}(\rho) = \mathcal{E}(\sigma)$, and every nonaffine operator-convex f with $f(1) = 0$,

$$D_f(\mathcal{N}_w(\rho)\|\mathcal{N}_w(\sigma)) \leq h(w) D_f(\rho\|\sigma), \qquad h(w) = 1 - \frac{m^2}{\sum_j w_j^{-1}}. \tag{9}$$

The same $h(w)$ applies for every finite R. It also bounds the SLD metric ratio for tangents Z with $\mathcal{E}(Z) = 0$.

Proof. For a resolvent term in (7), let $Q = L_\rho + tR_\sigma$, $Z = \rho - \sigma$, and write $Q_j = \mathrm{Ad}_{U_j} Q\,\mathrm{Ad}_{U_j}^\dagger$, $Z_j = U_j Z U_j^\dagger$. The constraint gives $\sum_j Z_j = 0$. For every real a,

$$\mathcal{N}_w(Z) = \sum_j (w_j - a) Z_j = \sum_j w_j \frac{w_j - a}{w_j} Z_j.$$

Since the output superoperator is $\sum_j w_j Q_j$, Lemma 1, quadratic homogeneity and unitary covariance give

$$F\left(\sum_j w_j Q_j, \mathcal{N}_w(Z)\right) \leq \sum_j \frac{(w_j - a)^2}{w_j} F(Q,Z).$$

The scalar factor is $1 - 2ma + a^2 \sum_j w_j^{-1}$; its minimum occurs at $a = m/\sum_j w_j^{-1}$ and equals $h(w)$. The convex weights here are the positive w_j. Negative tangent multipliers do not constitute negative convex weights on states.

For the quadratic endpoint in (7), use $Q = R_\sigma$; its inverse quadratic form is $\mathrm{Tr}\, Z^2\sigma^{-1}$. Integrate the nonnegative resolvent inequalities. Affine terms vanish by equal traces, and the positive representation justifies this passage. For the SLD metric, use $Q = (L_\sigma + R_\sigma)/2$ directly. No argument depends on a bound on $|R|$. □

Cauchy–Schwarz gives $\sum_j w_j^{-1} \geq m^2$, so $0 \leq h(w) < 1$. A useful exact control is an involution $U = U^\dagger$, $U^2 = I$, with both eigenspaces nonzero. For $\mathcal{E} = (\mathrm{id} + \mathrm{Ad}_U)/2$ and $\mathcal{N} = w\,\mathrm{id} + (1-w)\,\mathrm{Ad}_U$, the bound is $(2w-1)^2$. It is sharp for each of the Petz generators under consideration: take $\sigma = I/n$, a nonzero Hermitian Z connecting the two eigenspaces, and $\rho_\epsilon = \sigma + \epsilon Z$. Then $\mathcal{N}(Z) = (2w-1)Z$, and the scalar second-order expansion of (3) gives the stated limiting ratio. At $w = 1/2$, the two outputs coincide.

Remark 3 (Relative entropy at the boundary). For $f(x) = x \log x$, (9) also holds for nonfaithful pairs with finite relative entropy. Mix both states with the same maximally mixed state. The equal-$\mathcal{E}$ constraint is preserved. Joint convexity and lower semicontinuity imply convergence of both finite divergences, and the faithful inequality passes to the limit before division. We do not assert a boundary extension for every f without specifying its boundary conventions.

3 Exact contraction of every unital qubit channel

For the Pauli matrices P_j, $j = 0, x, y, z$, write $\mathcal{N}_w = \sum_j w_j \operatorname{Ad}_{P_j}$, $w_j \geq 0$, $\sum_j w_j = 1$. The uniform Pauli twirl is $\mathcal{E}_A(X) = I_2/2 \otimes \operatorname{Tr}_A X$.

Theorem 4 (Complete unital-qubit classification)**.** *The complete equal-reference coefficients of SLD, $\chi^2_{\min}$ and the finite triangular Petz divergence coincide. For positive Pauli weights their common value is*

$$h(w) = 1 - \frac{16}{\sum_{j=0,x,y,z} w_j^{-1}}. \tag{10}$$

If any Pauli weight is zero, the common value is one. A reference of dimension three suffices for each supremum. No assertion of minimal reference dimension or of a faithful maximizing pair is made.

Consequently, for an arbitrary unital qubit CPTP channel with normalized Choi state $J = (\mathcal{N} \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)$, $|\Omega\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, the common value is

$$\eta^c(\mathcal{N}) = \begin{cases} 1 - 16/\operatorname{Tr}(J^{-1}), & J > 0, \\ 1, & J \text{ singular.} \end{cases} \tag{11}$$

For every normalized nonaffine operator-convex Petz generator this is an upper bound. Equality for each individual generator is not asserted.

Proof. Theorem 2 proves the upper for positive weights. We give the matching lower, including its faithful realization. Put

$$A = w_0 + w_z,\ B = w_x + w_y = 1 - A,\ u = w_0 - w_z,\ v = w_x - w_y,\ z = A - B,$$

$$U = u^2/A, \quad V = v^2/B, \quad k = 1 - U - V = 4w_0 w_z/A + 4w_x w_y/B > 0.$$

In the basis $(00, 01, 02, 10, 11, 12)$, let

$$a = (1, 0, 0, 0, e, 0)^T, \quad b = (0, e, 0, 0, 0, 1)^T, \quad \sigma_e = \frac{aa^* + bb^*}{2(1 + e^2)}, \qquad e > 0.$$

Define the real Hermitian tangent by

$$D_{04} = D_{40} = 1, \quad D_{44} = 2e, \quad D_{11} = -2e, \quad D_{15} = D_{51} = -1,$$

with all other entries zero. Then $\operatorname{Tr}_A D = 0$, $P = 2\sigma_e$ is a projection and $(I - P)D(I - P) = 0$. The support/kernel limit of the SLD form is

$$g^{\mathrm{SLD}}_{\sigma_e}(D) = 4 \operatorname{Tr} D^2 - 2 \operatorname{Tr}(PDP)^2 = 16(1 + e^2). \tag{12}$$

Here and below a boundary form denotes this finite limit, not the ordinary inverse of a singular matrix.

The two output blocks on coordinates $(0,4,2)$ and $(3,1,5)$ are W and JWJ, where J reverses three coordinates and

$$W = \frac{M}{2(1+e^2)}, \quad M = \begin{pmatrix} A & ue & 0 \\ ue & e^2 & ve \\ 0 & ve & B \end{pmatrix}.$$

Their tangent blocks are T and $-JTJ$, with

$$T = \begin{pmatrix} 0 & u & 0 \\ u & 2ez & -v \\ 0 & -v & 0 \end{pmatrix}.$$

The central Schur complement is e^2k, so both output blocks are positive definite. Let $Y = Y^T$ solve $MY + YM = 4(1+e^2)T$, and put $x = Y_{12}$, $y = Y_{23}$, $c = eY_{22}$. Eliminating the other three entries gives

$$K_e \begin{pmatrix} x \\ y \\ c \end{pmatrix} = 4(1+e^2) \begin{pmatrix} u \\ -v \\ z \end{pmatrix}, \quad K_e = \begin{pmatrix} A + e^2(1 - u^2/A - v^2) & -e^2uv & u \\ -e^2uv & B + e^2(1 - v^2/B - u^2) & v \\ u & v & 1 \end{pmatrix}.$$

Since $\det K_0 = ABk > 0$, the scaled unknowns converge to the nonsingular limiting solution

$$c_0 = 4(z - U + V)/k, \quad x_0 = u(4 - c_0)/A, \quad y_0 = v(-4 - c_0)/B.$$

Both blocks contribute equally to the SLD form. Thus

$$\lim_{e \downarrow 0} \frac{g^{\mathrm{SLD}}_{\mathcal{N}_w(\sigma_e)}(\mathcal{N}_w(D))}{16(1+e^2)} = U + V + \frac{(z - U + V)^2}{k} = 1 - \frac{16}{\sum_j w_j^{-1}}. \tag{13}$$

The last equality is a rational identity using $A + B = 1$.

For each fixed $e > 0$, replace the input by $\sigma_{e,\xi} = (1-\xi)\sigma_e + \xi I_6/6$, $0 < \xi < 1$. The kernel–kernel tangent vanishes, so its input form tends to (12); the output is already faithful and its inverse is continuous. Take $\xi \downarrow 0$ before $e \downarrow 0$. Every finite tangent is realized by $\rho_t = \sigma_{e,\xi} + tD$ for sufficiently small nonzero real t. For triangular Petz take $t \to 0$ first; its local form is $g^{\mathrm{SLD}}/2$. Minimal chi-square is exactly quadratic in t. These ordered limits prove all three lower suprema. The uniform-weight channel has zero output tangent and coefficient zero; no division by that coefficient is used.

For boundary weights set $w_j(\gamma) = (1-\gamma)w_j + \gamma/4$. Since $\mathcal{N}_\gamma = (1-\gamma)\mathcal{N}_w + \gamma\mathcal{E}_A$, joint convexity and equal reference give, for each of the three quantities, output at $\mathcal{N}_\gamma$ at most $(1-\gamma)$ times output at $\mathcal{N}_w$. Consequently $\eta^c(\mathcal{N}_w) \geq h(w(\gamma))/(1-\gamma) \to 1$. The upper one is data processing. Faithful inputs have faithful outputs under any Pauli mixture.

Finally use the prior signed singular-value normal form of unital qubit channels [9]: pre/post unitary conjugations reduce the Bloch matrix to a physical Pauli diagonal map. Both rotations must lie in $SO(3)$; erasing all singular-value signs is not legitimate. These conjugations preserve the equal-reference constraint, the three ratios and the normalized Choi spectrum. This proves (11). The normal-form reduction itself is not new. For the unnormalized Choi matrix $\Gamma = 2J$, the coefficient would instead read $1 - 8/\operatorname{Tr}\Gamma^{-1}$. □

4 The exact stabilization envelope

Let $s = \|T\|_\infty \in [0,1]$ for the real Bloch matrix of a unital qubit channel. The ordinary coefficient for these metrics/divergences is s^2, by the prior result of Hiai–Ruskai [1, Theorem 6.1].

Theorem 5. *At fixed s, the possible complete coefficients in Theorem 4 fill precisely $[s^2, F(s)]$, where*

$$F(s) = \begin{cases} 3s^2/(1-2s), & 0 \le s \le 1/3, \\ 1, & 1/3 \le s \le 1. \end{cases}$$

For $s > 0$ the ratio between complete and ordinary coefficients is at most nine, and nine is attained as a ratio of channel coefficient suprema. It is undefined at $s = 0$, where both coefficients vanish.

Proof. After the signed normal form write $T = \mathrm{diag}(t_1, t_2, t_3)$. The Pauli weights are the four affine functions $(1 \pm t_1 \pm t_2 \pm t_3)/4$ with an even number of minus signs. For $s < 1/3$, the cube $[-s, s]^3$ is inside the positive-weight CP tetrahedron. The reciprocal sum $S = \sum_j 1/w_j$ is convex, so its maximum occurs at a vertex. The two values there are

$$S_+ = \frac{4}{1+3s} + \frac{12}{1-s}, \qquad S_- = \frac{4}{1-3s} + \frac{12}{1+s},$$

and $S_- - S_+ = 192s^3/[(1-9s^2)(1-s^2)] \ge 0$. Since $1 - 16/S$ increases with S, the upper is achieved by $T = -sI$ and equals $F(s)$. For $s \ge 1/3$, DPI gives one, achieved by the CP boundary map $\mathrm{diag}(-s, -(1-s)/2, -(1-s)/2)$.

The map $\mathrm{diag}(-s, 0, 0)$ realizes the lower s^2. A straight segment to the corresponding upper example remains CP and keeps norm s. The formula in Theorem 4 extends continuously to the simplex boundary, with value one there. The intermediate value theorem therefore gives every value of the interval; no monotonicity along the segment is assumed. Dividing $F(s)$ by s^2 proves the factor-nine statement. □

At $T = -I/3$, the channel chooses a Pauli axis uniformly, measures it and prepares the opposite eigenstate. It is therefore entanglement breaking. Nevertheless its complete coefficient is one, while its ordinary coefficient is $1/9$. This does not say that all entanglement-breaking channels have complete coefficient one, nor that a faithful input attains the supremum exactly.

5 Exact order-constrained depolarization in arbitrary dimension

Let $\mathcal{E}_A(X) = I_d/d \otimes \mathrm{Tr}_A X$ and $\mathcal{N}_\lambda = \lambda\,\mathrm{id} + (1-\lambda)\mathcal{E}_A$, where $-1/(d^2-1) \le \lambda \le 1$, the known physical range [8, Section 8.1]. For $\kappa > 1$ restrict the finite pairs to

$$\rho_R = \sigma_R, \quad \rho, \sigma > 0, \quad \rho \ne \sigma, \quad \rho \preceq \kappa\mathcal{E}_A(\rho), \quad \sigma \preceq \kappa\mathcal{E}_A(\sigma). \tag{14}$$

For the metric impose the base order bound and $\mathrm{Tr}_A D = 0$; one may also require a two-sided small neighborhood satisfying (14). The suprema proved below are unchanged. The budget uses the *actual* reference marginal, not an optimized conditional min-entropy reference.

Proposition 6 (Commuting Bell lower and its range)**.** *For every $d \ge 2$, $1 < \kappa \le d^2/2$ and every nonaffine operator-convex Petz generator normalized by $f(1) = 0$, the complete pair coefficient under (14) is $\lambda^2\kappa/[1+\lambda(\kappa-1)]$ throughout the physical depolarizing range. The SLD and minimal-chi-square coefficients have the same value. A reference of dimension d suffices for the lower supremum.*

Proof. The order upper is proved below. Write $n = d^2$, $B = I_n/n$, and take two orthogonal maximally entangled Bell projectors P_0, P_1. For $1 < K < \kappa \le n/2$, set

$$a = K/n, \quad b = (1-2a)/(n-2), \quad \sigma_K = bI_n + (a-b)(P_0+P_1), \quad D = P_0 - P_1.$$

Here $a > b > 0$, $\mathcal{E}_A(\sigma_K) = B$, $\mathrm{Tr}_A D = 0$, and the actual order constant is K. The input and output commute; their eigenvalues on the two changed rays are a and $a_\lambda = \lambda a + (1-\lambda)/n > 0$. Hence the SLD ratio is exactly $\lambda^2 a/a_\lambda$.

For $0 < |t| < \min(a, b, (\kappa - K)/n)$, $\rho_t = \sigma_K + tD$ and σ_K are faithful with identical reference and both order bounds. Their Petz divergences are

$$a\left[f(1+t/a) + f(1-t/a)\right], \qquad a_\lambda\left[f(1+\lambda t/a_\lambda) + f(1-\lambda t/a_\lambda)\right].$$

For a normalized nonaffine operator-convex generator, $f''(1) > 0$, as follows from the nonzero positive representation in (6). The Taylor ratio as $t \to 0$ is therefore $\lambda^2 a/a_\lambda = \lambda^2 K/[1 + \lambda(K-1)]$. Then let $K \uparrow \kappa$. At the endpoint $\kappa = n/2$, each earlier base remains faithful. The cases $\lambda = 0, 1$ are immediate. The quadratic chi-square equality follows directly from the metric ratio. □

This is a reparametrization of the Bell family already present in our earlier component work, not an independent invention of Bell-assisted contraction. In particular, when $d \geq 3$, the interval $\kappa \leq d+1$ is contained in $\kappa \leq d^2/2$; the rank-d ladder previously considered on that interval is an alternative proof. A conditional-expectation index alone does not imply sharpness: the pinching $\mathcal{E} = (\mathrm{id} + \mathrm{Ad}_Z)/2$ has index two, but its equal-$\mathcal{E}$ contraction is λ^2, rather than the larger raw order comparison, for $0 < \lambda < 1$.

Theorem 7 (Principal-rich rank-two order profile)**.** *For $d \geq 2$, set $M_d = d(2d-1)/2$. For $1 < \kappa \leq M_d$, the complete SLD, minimal-chi-square and finite triangular-Petz contraction coefficients under* (14) *equal*

$$C_\kappa(\lambda) = \frac{\lambda^2 \kappa}{1 + \lambda(\kappa - 1)}. \tag{15}$$

The complete supremum is over finite reference systems; a reference of dimension $2d-1$ supplies the lower. The same expression is an upper for each normalized nonaffine operator-convex Petz generator, sharp uniformly over that class. Equality for each generator outside the Bell range, or for every fixed reference dimension, is not asserted.

Proof. For each admissible state, the prior order comparison gives $\mathcal{N}_\lambda(\sigma) \succeq c\sigma$, where $c = \lambda + (1-\lambda)/\kappa > 0$. Indeed $1 + \lambda(M_d - 1) \geq d/[2(d^2-1)] > 0$. An equal-reference difference obeys $\mathcal{N}_\lambda(D) = \lambda D$. Inverse antitonicity for $(L_\sigma + R_\sigma)/2$, or for $L_\rho + uR_\sigma$ using both state budgets, gives the factor λ^2/c. Include the quadratic endpoint and the positive integral in (7). The new lower, rather than this comparison machinery, is the point of the construction.

Put $a = d-1$, $r = 2a+1$, $q = a + e^2$, $e > 0$, and use the reference labels $0, \ldots, 2a$. Define

$$v_0 = \sum_{j=0}^{a-1} |j+1, j\rangle + e|0, 2a\rangle, \qquad v_1 = \sum_{j=0}^{a-1} |j, a+j\rangle + e|d-1, 2a\rangle.$$

Let $b_0 = |0, 2a\rangle$, $b_1 = |d-1, 2a\rangle$, and

$$\sigma_e = \frac{v_0 v_0^* + v_1 v_1^*}{2q}, \qquad D = v_0 b_0^* + b_0 v_0^* - v_1 b_1^* - b_1 v_1^*.$$

The supports are disjoint, $\|v_i\|^2 = q$, and $Q = 2\sigma_e$ is a projection. All principal–center cross terms change the A index. The two diagonal center derivatives cancel in the same reference label. Thus $\mathrm{Tr}_A D = 0$.

The actual reference marginal has mass $1/(2q)$ on its $2a$ principal labels and e^2/q on the center. Consequently $B_e = \mathcal{E}_A(\sigma_e) > 0$ and

$$\frac{[v_0, v_1]^* B_e^{-1} [v_0, v_1]}{2q} = d(a + 1/2) I_2 = M_d I_2.$$

This proves that the whitened state has two nonzero eigenvalues M_d. Also $(I - Q)D(I - Q) = 0$; its boundary SLD form, evaluated spectrally without inverting the singular base, is $16q$.

Set $s_{e,p} = (1-p)B_e + p\sigma_e$. Each support has an invariant plane spanned by its uniform principal vector and center coordinate. In orthonormal coordinates the two base blocks are

$$W_p = \frac{1}{2dq} \begin{pmatrix} A_p & dpe\sqrt{a} \\ dpe\sqrt{a} & C_p e^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 2e \end{pmatrix}, \tag{16}$$

where $A_p = 1 + (da - 1)p$, $C_p = 2 + (d - 2)p$; the tangent blocks are $T, -T$. Each support has $a - 1$ additional principal modes with eigenvalue $(1 - p)/(2dq)$ and zero tangent. Outside these supports the remaining $d(2d - 1) - 2d$ coordinates are diagonal replacement modes with zero tangent. This accounts for the full dimension. The aligned determinant is

$$\det W_p = \frac{e^2(1-p)[1 + (M_d - 1)p]}{2d^2 q^2} > 0 \qquad \left(-\frac{1}{M_d - 1} < p < 1 \right).$$

Both $p = \theta$ and $p = \lambda\theta$ lie in this domain for $0 < \theta < 1$ and the entire physical range of λ.

For $W_p Y + Y W_p = 2T$, the two-dimensional identity

$$Y = \frac{T + \det(W_p) W_p^{-1} T W_p^{-1}}{\operatorname{Tr} W_p}$$

follows by Cayley–Hamilton and direct substitution. Inserting (16) and taking the limit at fixed admissible p gives

$$\lim_{e \downarrow 0} \operatorname{Tr}(TY) = \frac{8aM_d}{1 + (M_d - 1)p}. \tag{17}$$

The full ambient SLD form is twice this quantity. At fixed $\theta < 1$, its retention therefore tends to $\lambda^2[1 + (M_d - 1)\theta]/[1 + \lambda(M_d - 1)\theta]$.

Choose $0 < \theta < (\kappa - 1)/(M_d - 1)$. Both $s_{e,\theta}$ and $\kappa B_e - s_{e,\theta}$ are positive definite, so sufficiently small bilateral perturbations $\rho_t = s_{e,\theta} + tD$ preserve faithfulness, the actual marginal and both budgets. For triangular divergence take $t \to 0$ first; for minimal chi-square the ratio is exactly quadratic. Then take $e \downarrow 0$ at fixed θ, finally $\theta \uparrow (\kappa - 1)/(M_d - 1)$. This reaches (15) as a supremum. The endpoints $\lambda = 0, 1$ are immediate. No exchange of singular limits or finite faithful maximizer is needed. □

The earlier rank-two allocation achieved the smaller cap $L_d = d(d+1)/2$ with reference dimension $d + 1$. The principal-rich allocation raises the cap by $d(d - 2)/2$, at the cost of reference dimension $2d - 1$; for $d = 2$ they coincide. This is a development of the same mechanism, not an independently invented theory or a free reference-size improvement.

Remark 8 (The empty endpoint and the domain of the order bound). At $\kappa = 1$, positivity and equal traces force $\rho = \mathcal{E}_A(\rho)$ and $\sigma = \mathcal{E}_A(\sigma)$; equal reference then forces equality. There are no distinct finite pairs, so (15) is not assigned to that empty optimization. The base-only SLD problem with unrestricted tangents at a product base is a different problem. Likewise the expression λ^2/c must never be used when $c \le 0$; for example $\lambda = -1/3, \kappa = 5$ gives a negative raw expression in dimension two, not a contraction bound.

6 Exact qubit resource restrictions

For $d = 2$, Theorems 4 and 7 combine to give, for every $\kappa > 1$,

$$C_\kappa(\lambda) = \frac{\lambda^2 K}{1 + \lambda(K-1)}, \qquad K = \min(\kappa, 3), \qquad -1/3 \le \lambda \le 1.$$

For $\kappa \ge 3$, the upper is the unrestricted isotropic case of Theorem 4, $3\lambda^2/(1+2\lambda)$; the same lower family has order arbitrarily close to three. At the channel endpoints use DPI or the identity directly.

For a qubit and any finite reference,

$$2\mathcal{E}_A(\sigma) - \sigma = (\sigma_y \otimes I)\sigma^{T_A}(\sigma_y \otimes I).$$

Thus PPT bases obey the order-two bound, as do separable bases. The exact complete SLD/minimal-chi-square/triangular coefficient restricted to PPT or separable models is $2\lambda^2/(1+\lambda)$. For finite Petz both states must belong to the class. To prove the matching lower, use the qubit ladder above (equivalent under a local basis swap to the earlier allocation) with $s_{e,\theta}$ and $\theta < 1/2$. Its partial transpose has two blocks with determinant $e^2(1-\theta-2\theta^2)/[8(1+e^2)^2] > 0$ and two positive singletons. Small bilateral perturbations stay PPT. In dimension 2×3, the prior Horodecki theorem [10] converts these pairs to separable ones. This uses PPT=SEP only for the lower in 2×3, not for arbitrary references or larger local dimensions.

For *each* normalized nonaffine operator-convex Petz generator, the SEP/PPT pair coefficient is also exactly $2\lambda^2/(1+\lambda)$. Indeed, the order-two comparison gives the upper, while the qubit Bell family of Proposition 6, with $K < 2$, has all Bell probabilities p_j strictly below $1/2$. The partial-transpose eigenvalues are $1/2 - p_j$. Both the base and sufficiently small finite perturbations are strictly PPT, hence separable in 2×2, and approach the stated lower. The same gives the budget profile inside SEP/PPT for $1 < \kappa \le 2$. This restricted-pair statement includes Umegaki but does not determine the unrestricted coefficient or actual MI, whose second state has an additional product-marginal requirement.

A classical-reference *family* has both states and the tangent block diagonal in one fixed reference basis, with fixed marginal weights. The forms split into weighted ordinary-qubit forms, whose coefficient is λ^2 by [1]; $R = 1$ gives equality. Classicality of the base alone is insufficient. For example at $\lambda = -1/3$, the classical base $\sigma = \mathrm{diag}(9,1,1,9)/20$ with $D_{03} = D_{30} = 1$ gives retention $3/11 > 1/9$; the direction creates reference coherences.

At $\lambda = -1/3$ the exact levels are therefore

$$\text{classical-reference family} : \frac{1}{9}, \qquad \text{SEP/PPT family} : \frac{1}{3}, \qquad \text{unrestricted quantum family} : 1.$$

SLD retention (or minimal-chi-square retention with base σ) above $2\lambda^2/(1+\lambda)$ is a sufficient certificate that the input base is NPT. For a general finite Petz pair, the proved implication is only that exceeding this threshold excludes both input states being PPT; it does not identify which state is NPT. This mathematical implication is a contrapositive of the prior reduction criterion plus the quantitative upper; it is not an experimentally demonstrated protocol.

7 A complete equal-reference relative-entropy lower bound

Write $P_d = d(d-1)$, distinct from $M_d = d(2d-1)/2$. Define

$$g_s^{\mathrm{BKM}}(D) = \mathrm{Tr}\, D\, \mathrm{D}(\log)_s[D], \qquad D_{\mathrm{U}}(\rho\|\sigma) = \mathrm{Tr}\, \rho(\log\rho - \log\sigma).$$

Their complete equal-reference coefficients use exactly the state/tangent restrictions of Section 1. They do not optimize actual mutual information. The BKM kernel, faithful local expansion and relative-entropy/BKM comparison mechanisms are prior [1, 4, 12].

Theorem 9 (Principal-rich BKM and Umegaki lower)**.** *For depolarization on dimension $d \geq 2$ throughout its physical range,*

$$\eta^{c}_{\mathrm{BKM}}(\mathcal{N}_\lambda) \geq C_{P_d}(\lambda), \qquad \eta^{c}_{\mathrm{U}}(\mathcal{N}_\lambda) \geq C_{P_d}(\lambda).$$

With both actual-marginal order budgets at $\kappa > 1$, put

$$K_*(\kappa) = 1 + (P_d - 1)\min\left\{\frac{\kappa - 1}{M_d - 1}, 1\right\}.$$

Both constrained coefficients are at least

$$\max\{C_{\min(\kappa, d^2/2)}(\lambda),\ C_{K_*(\kappa)}(\lambda)\}.$$

These are lower bounds, not proposed exact optima. The second construction uses a reference of dimension $2d - 1$.

Proof. Use the bases and tangent in (16). Let $w_+ \geq w_- > 0$ be the eigenvalues of W_p, and Z the matrix of T in that eigenbasis. At fixed $-1/(M_d - 1) < p < 1$,

$$w_+ = \frac{A_p}{2da} + O(e^2), \qquad w_- = \frac{(1-p)[1 + (M_d - 1)p]}{daA_p}e^2 + O(e^4).$$

The leading coefficient of w_- is positive. The eigenvector rotation is $dp\sqrt{a}\, e/A_p + O(e^3)$, whence

$$Z_{++} = \frac{2dpa}{A_p}e + O(e^3), \quad Z_{--} = \frac{2(1-p)}{A_p}e + O(e^3), \quad Z_{+-} = \sqrt{a} + O(e^2).$$

The BKM form for one plane is

$$H_e(p) = \frac{Z_{++}^2}{w_+} + \frac{Z_{--}^2}{w_-} + 2|Z_{+-}|^2\frac{\log w_+ - \log w_-}{w_+ - w_-}.$$

The diagonal terms are bounded, whereas the logarithmic term yields

$$H_e(p) = \frac{8da^2}{1 + (P_d - 1)p}\log(1/e) + O(1). \tag{18}$$

The full form is $2H_e(p)$; all remaining modes have zero tangent. Therefore, with $\theta < 1$ fixed,

$$\lim_{e \downarrow 0} \lambda^2 \frac{H_e(\lambda\theta)}{H_e(\theta)} = \lambda^2 \frac{1 + (P_d - 1)\theta}{1 + \lambda(P_d - 1)\theta}.$$

Taking $\theta \uparrow 1$ gives the unrestricted lower; taking it up to the budget endpoint gives C_{K_*}. The domain also contains the negative physical endpoint. The cases $\lambda = 0, 1$ are direct. No uniform remainder in $\theta \uparrow 1$ is asserted.

For each fixed faithful e, θ, choose the same admissible small perturbations as in Theorem 7. The prior local identity is

$$D_{\mathrm{U}}(s_{e,\theta} + tD \| s_{e,\theta}) = \tfrac{1}{2}t^2 g^{\mathrm{BKM}}_{s_{e,\theta}}(D) + o(t^2).$$

Take $t \to 0$ first, then $e \downarrow 0$, then θ to its allowed endpoint. This proves the finite-pair lower in the stated orientation. The first bound inside the maximum follows from the Bell family of Proposition 6, since $x \log x$ is operator convex; for larger budgets retain its endpoint lower by approximation from below. It can be stronger than C_{K_*} at smaller budgets and must not be discarded. □

For $d \geq 3$, $P_d > d^2/2$ and $\partial_K C_K = \lambda^2(1-\lambda)/[1+\lambda(K-1)]^2$. Thus, except at $\lambda = 0, 1$, the new unrestricted lower strictly exceeds the endpoint supplied by the specific preceding Bell family. At the negative physical endpoint this reads

$$C_{P_d}\left(-\frac{1}{d^2-1}\right) = \frac{1}{d+1}, \qquad C_{d^2/2}\left(-\frac{1}{d^2-1}\right) = \frac{1}{d^2-1}.$$

The ratio of these two lower bounds is $d-1$. This dimension-dependent comparison is not a claim of a worldwide-best bound, an exact coefficient or an operational advantage. For $d = 2$, $P_d = 2$: it recovers the Bell lower and does not settle unrestricted qubit relative-entropy contraction.

Finite certified comparisons. For $e = 10^{-30}$, $t = 10^{-90}$, $\theta = 99/100$, the actual finite pairs above, with budget M_d, yield the following rigorous interval locations of their gaps over the specified Bell endpoint:

d	λ	gap interval (outward decimal bounds)
3	$-1/8$	(0.11255711780015, 0.11255711780017)
3	$1/2$	(0.00593846351695, 0.00593846351696)
4	$-1/15$	(0.11934942272408, 0.11934942272410)
4	$1/2$	(0.00238915978275, 0.00238915978276)

The exact binary/rational interval endpoints, rather than rounded midpoints, are retained in the supplement. Audit1044 reproduced positivity and both budgets exactly, and enclosed all four calculations with a separate integer interval implementation. Subblocks must not be normalized separately: their opposite linear trace terms cancel only in the global divergence. The input divergences are of order 10^{-177}, with cancelling block terms of order 10^{-120}. Such severe conditioning precludes an inference of laboratory robustness from these mathematical certificates.

8 A fixed local Hamiltonian and finite phase

The nonunitary ladder tangent used above has a genuine obstruction to a Hamiltonian interpretation. The following different family removes that particular restriction while paying a larger order budget. Its base state and Schmidt-support decomposition have direct antecedents in Frey–Collins–Gerlach [14]; phase estimation with noisy probes and inactive ancillas, and the SLD calculation of a coherence block, are prior [11]. We claim neither a new preparation family nor optimal absolute metrological information.

Put $a = d-1$, $q = a + e^2$, $e > 0$, and on $A_d R_d$ set

$$v = \sum_{j=1}^{d-1} |j, j\rangle + e|0, 0\rangle, \quad s = vv^*/q, \quad r = \mathrm{Tr}_A\, s = \mathrm{diag}(e^2, 1, \ldots, 1)/q, \quad B = I_d/d \otimes r.$$

Here s is a pure preparation state, whereas the states used below are faithful mixtures. Define $s_p = (1-p)B + ps$ and the fixed local Hamiltonian

$$H = \mathrm{diag}(0, 1, \ldots, 1), \quad U_\phi = e^{-i\phi H} \otimes I_R, \quad \rho_\phi = U_\phi s_\theta U_\phi^*, \qquad 0 < \theta < 1.$$

Both the norm and the spectral diameter of H are one, independently of e, θ and the channel parameter.

Theorem 10 (Finite-phase realization)**.** *For every $d \geq 2$ and physical depolarizing parameter $-1/(d^2-1) \leq \lambda \leq 1$, the pair (ρ_ϕ, s_θ) is faithful with common reference marginal r, and each state's exact actual-marginal order is*

$$K_\theta = 1 + (d^2-1)\theta.$$

Let W_p be the subnormalized two-dimensional block

$$W_p = \frac{1}{dq}\begin{pmatrix} A_p & dpe\sqrt{a} \\ dpe\sqrt{a} & F_p e^2 \end{pmatrix}, \qquad A_p = 1 + (da-1)p, \quad F_p = 1 + ap,$$

and let $\ell(W) = (\log w_+ - \log w_-)/(w_+ - w_-)$ for its positive eigenvalues, with its continuous extension when they coincide. For $\phi \notin 2\pi\mathbb{Z}$, write $c = \theta e\sqrt{a}/q$. Then

$$D_{\mathrm{U}}(\rho_\phi \| s_\theta) = 2c^2 \ell(W_\theta)(1 - \cos\phi), \tag{19}$$

$$D_{\triangle}(\rho_\phi \| s_\theta) = \frac{4c^2(1-\cos\phi)}{\operatorname{Tr} W_\theta}. \tag{20}$$

Consequently the finite Umegaki retention is exactly

$$\lambda^2 \frac{\ell(W_{\lambda\theta})}{\ell(W_\theta)},$$

equal to the BKM retention of the physical phase tangent. Triangular retention equals the SLD retention

$$\lambda^2 \frac{\operatorname{Tr} W_\theta}{\operatorname{Tr} W_{\lambda\theta}}.$$

For each fixed allowed λ, first letting $e \downarrow 0$ at fixed θ, and then $\theta \uparrow 1$, makes these retentions approach $C_{P_d}(\lambda)$, $P_d = d(d-1)$. In particular, the phase may remain fixed at π throughout. This establishes a realization of a lower bound, not a matching upper or an optimum over Hamiltonians.

Proof. The sole nonzero eigenvalue of $B^{-1/2} s B^{-1/2}$ is $v^* B^{-1} v/q = d^2$. Thus s_p is faithful for $-1/(d^2-1) < p < 1$, and whitening gives the asserted exact order. Since U_ϕ commutes with B, it preserves the order and marginal; local depolarization takes s_p to $s_{\lambda p}$ and commutes with the conjugation. All these states remain faithful at the negative channel endpoint because $\theta < 1$.

The uniform principal Schmidt vector and $|0,0\rangle$ span W_p. Its determinant is

$$\det W_p = \frac{e^2(1-p)[1 + (d^2-1)p]}{d^2 q^2} > 0.$$

The remaining $d-2$ Schmidt modes and d^2-d off-diagonal-coordinate modes are phase independent. The physical tangent on the active block is $(\theta e/q)T$, with

$$T = \begin{pmatrix} 0 & -i\sqrt{a} \\ i\sqrt{a} & 0 \end{pmatrix}.$$

For real symmetric W, $WT + TW = (\operatorname{Tr} W)T$. The Sylvester formula therefore gives phase SLD information $4c^2/\operatorname{Tr} W$. A real orthogonal diagonalization leaves T unchanged up to sign and gives BKM information $2c^2\ell(W)$.

Unitary invariance cancels the entropy term in $D_{\mathrm{U}}(U_\phi s_\theta U_\phi^* \| s_\theta)$. Functional calculus gives $(\log W)_{01} = \ell(W) W_{01}$, so the two changed coherences yield (19). The two-block spectral overlap formula for $f_\triangle$ gives $\operatorname{Tr}(\rho_\phi - s_\theta)^2/\operatorname{Tr} W_\theta$, which is (20). Other modes contribute zero. Blocks are never normalized separately. Substitution of $\lambda\theta$ proves the retentions.

At fixed faithful p, $w_+ = A_p/(da) + O(e^2)$ and $w_- = k_p e^2 + O(e^4)$ with $k_p > 0$. Hence $\ell(W_p) = (2da/A_p)\log(1/e) + O(1)$ and $\operatorname{Tr} W_p \to A_p/(da)$. Both retentions tend to $\lambda^2 A_\theta/A_{\lambda\theta} = C_{1+(P_d-1)\theta}(\lambda)$. The final ordered limit gives the claim; $\lambda = 0, 1$ also follow directly. □

The resource tradeoff and nonuniform limits. This realization uses reference dimension d instead of $2d-1$, but order tending to d^2 rather than M_d. At a common budget $1 < \kappa \le d^2$, its limiting lower is only $C_{1+(P_d-1)(\kappa-1)/(d^2-1)}$. It does not improve the earlier ladder's budget profile. At $\phi = \pi$, the trace distance is $2|c| = O(e)$, SLD information is $O(e^2)$ and Umegaki/BKM information is $O(e^2 \log(1/e))$. A finite angle therefore does not keep the states a fixed distance apart. The order of limits is essential: for $0 < \lambda < 1$ and fixed $e > 0$, sending $\theta \uparrow 1$ first gives Umegaki retention zero, not C_{P_d}. This family does not settle unrestricted qubit BKM or actual-MI contraction.

Proposition 11 (The entire common-Bell class, throughout the noise range)**.** *For every $-1/(d^2-1) \le \lambda \le 1$, the global contraction coefficient of each normalized nonaffine operator-convex Petz divergence, restricted to pairs diagonal in one common orthonormal maximally entangled basis of A_dR_d, is exactly*

$$b_{d^2}(\lambda) = \frac{\lambda^2}{\lambda + 2(1-\lambda)/d^2} = C_{d^2/2}(\lambda).$$

The same value holds on the union of these classes over the common basis. For $d \ge 3$ and each fixed physical $\lambda \ne 0, 1$, the phase lower $C_{P_d}(\lambda)$ is strictly larger; hence finite faithful phase pairs separating from this whole class exist. At the negative endpoint the ratio of the lower to this exact benchmark is $d-1$. This states a supremum and finite existence, not one finite parameter choice uniform in λ or saturation by one finite pair for every generator.

Proof. We give an elementary reconstruction of a classical benchmark, without claiming classical novelty. For the symmetric channel $Kr = \lambda r + c\mathbf{1}$ on $n \ge 2$ symbols, put $c = (1-\lambda)/n$, $h = \lambda + 2c > 0$. Its stochastic range is $-1/(n-1) \le \lambda \le 1$. For positive probability r and real z with $\sum z_i = 0$, the Fisher upper is

$$\lambda^2 \sum_i \frac{z_i^2}{\lambda r_i + c} \le \frac{\lambda^2}{h} \sum_i \frac{z_i^2}{r_i}.$$

For $\lambda = 0, 1$ this is direct. Otherwise rearrangement reduces it to nonnegativity of

$$\sum_i \frac{(1-2r_i)z_i^2}{r_i(\lambda r_i + c)}.$$

If all $r_i \le 1/2$, every term is nonnegative. Otherwise $r_1 > 1/2$ is unique. Set $S = 1 - r_1$ and $\varphi(x) = x(\lambda x + c)/(1-2x)$. Since $(\varphi(x)/x)' = h/(1-2x)^2 > 0$, $\sum_{i>1} \varphi(r_i) \le \varphi(S)$. Cauchy and $\sum z_i = 0$ bound the positive terms below by $z_1^2/\varphi(S)$. This cancels the sole negative term because

$$r_1(\lambda r_1 + c) - S(\lambda S + c) = (2r_1 - 1)(\lambda + c) \ge 0.$$

For the resolvent generator $(x-1)^2/(x+u)$, put $z = \rho - \sigma$ and $r = (\rho + u\sigma)/(1+u)$. The divergence is $(1+u)^{-1} \sum z_i^2/r_i$; the Fisher upper applies after K with the same factor. Include $u = 0$, the separate quadratic endpoint and the positive representation (6); affine terms vanish. This gives the upper for every stated f. Common uniform regularization of both distributions extends the inequality to the standard boundary convention when the input divergence is finite and positive. The lower follows from two equal masses $a < 1/2$, tangent $(1, -1, 0, \ldots, 0)$, first a faithful local perturbation, then $a \uparrow 1/2$: the ratio is $\lambda^2 a/(\lambda a + c)$ before the last limit. For $n = 2$ use the uniform base directly. Each $f''(1) > 0$ supplies a cancelling common Fisher factor. At the binary negative endpoint K is a permutation and the formula is one; no endpoint denominator vanishes.

For $n = d^2$, a common maximally entangled basis has shared marginal I/d, and local depolarization transforms its probabilities by this K. Petz divergences reduce to the classical expressions.

Finally $\partial_K C_K = \lambda^2(1-\lambda)/[1+\lambda(K-1)]^2 > 0$ away from $\lambda = 0, 1$, and $P_d > d^2/2$ for $d \geq 3$. Theorem 10 then gives finite existence for each fixed noise parameter. At the negative endpoint $C_{P_d} = 1/(d+1)$ and $b_{d^2} = 1/(d^2-1)$. □

The equivalence of global coefficients for nonlinear operator-convex generators is explicitly prior in Makur–Zheng [16, Proposition 6]; Dobrushin theory [15] already gives the negative-endpoint upper directly. Lee–Makur [17] is a particularly close symmetric-channel antecedent: its author-provided conference abstract announces closed KL formulas for product symmetric channels. We have not obtained its formula, complete hypotheses or proof, and therefore do not assert precise coverage of the negative range or novelty of the formula. The reconstruction above supplies a benchmark for the quantum comparison, not a claimed classical discovery. A variable reference distribution in this global classical coefficient must not be replaced by a fixed uniform source.

Certified finite separation and falsified extension. At $\phi = \pi$, $\theta = .99$ and the negative CP endpoint, audit1054 gives the following outward interval enclosures for the Umegaki retention minus the entire Bell-class benchmark:

d	e	strict gap enclosure
3	.01	(.06388352285582, .06388352285583)
3	.001	(.07668420592533, .07668420592535)
4	.01	(.06782712362496, .06782712362498)
4	.001	(.08088908969722, .08088908969723)

The certificates use directed integer intervals; the displayed wider decimal enclosures are not midpoint tests. For $d = 3, e = .01$, the actual order is 8.92, input Umegaki divergence about .0030030759 and condition number about $5.95 \cdot 10^6$. For $d = 4, e = .01$, they are 15.85, .0021105849 and $1.189 \cdot 10^7$. These are finite mathematical examples, not well-conditioned preparation claims. Fourteen of the original twenty-four diagnostic cases did not exceed their compared Bell endpoint and remain in the evidence. The comparison does not bound all commuting or separable quantum pairs, nor isolate entanglement or noncommutativity as its sole causal resource.

At positive noise $\lambda = 1/2$, audit1058 separately encloses two phase-pi examples with $e = 10^{-30}$ and $\theta = .99$: in $d = 3$, retention lies in (.41495890226843, .41495890226844), strictly above 9/22; in $d = 4$, it lies in (.44684466269731, .44684466269733), above 4/9. Their input divergences are about $2.8492 \cdot 10^{-58}$ and $1.9117 \cdot 10^{-58}$, with condition numbers about $5.95 \cdot 10^{62}$ and $1.189 \cdot 10^{63}$. The common-Bell comparison is thus finite for both noise signs, but these positive-noise examples give no robustness evidence. Their geometry and interval arithmetic have declared1051/1054 dependencies; they are not independent physical constructions.

The equality of finite Umegaki and local BKM retention is special to this block family. A certified counterexample to its extension to arbitrary unitary orbits is $R = 1, d = 3, \lambda = 1/2$, $H = \mathrm{diag}(0, 1/2, 1)$, $\phi = \pi$, and

$$s = \begin{pmatrix} 1/2 & 1/10 & 1/20 \\ 1/10 & 3/10 & 1/25 \\ 1/20 & 1/25 & 1/5 \end{pmatrix}.$$

The finite retention lies in (.2585764702077098, .2585764702077101), while its local BKM retention lies in (.2579634718941141, .2579634718941153). Audit1054 bounds the matrix-logarithm and Frechet-derivative tails rationally, rather than inferring this separation from floating point. The original ladder's nonunitary obstruction remains valid.

9 Dimension-scaled costs of the finite-phase separation

The exact phase identity permits an explicit succession of finite pairs, rather than unspecified successive perturbation limits at each fixed dimension. This is a quantitative refinement of the preceding family, not a new preparation scheme or an optimal resource frontier. The noise parameter in this section changes with dimension.

Theorem 12 (An achievable information–conditioning tradeoff)**.** *Fix $t > 0$ and $\beta > 1$ independently of the integer $d \geq 2$, and put $\nu = \beta + 3 > 4$. In Theorem 10 choose*

$$e^2 = t/d, \quad \theta = 1 - d^{-\beta}, \quad \phi = \pi, \quad \lambda_d = -1/(d^2 - 1).$$

The faithful input pair has reference dimension d, the stated norm-one local Hamiltonian, and exact actual-marginal order

$$K_d = d^2 - d^{2-\beta} + d^{-\beta}.$$

Write $I_d = D_{\mathrm{U}}(\rho_\pi \| s_\theta)$, let R_d be its output/input relative-entropy retention, and let $\mathrm{cond}(s_\theta)$ *be the ratio of largest to smallest eigenvalue. Then*

$$dR_d \longrightarrow \alpha = \frac{\nu - 4}{(1+t)\nu}, \tag{21}$$

$$\frac{d^2 I_d}{\log d} \longrightarrow 4t\nu, \qquad \frac{\mathrm{cond}(s_\theta)}{d^\nu} \longrightarrow \frac{1}{t}, \tag{22}$$

$$\frac{d}{2}\|\rho_\pi - s_\theta\|_1 \longrightarrow 2\sqrt{t}. \tag{23}$$

In particular $R_d / b_{d^2}(\lambda_d) \sim \alpha d$, although both R_d and the absolute input/output information tend to zero. The information of the output is asymptotic to $4t(\nu - 4)\log(d)/[(1+t)d^3]$.

Proof. Set $x = e^2$, $\delta = 1 - \theta$, $q = d - 1 + x$ and $\Xi = d(d-1) - 1 + (d-1)x$. Directly from the subnormalized W_p,

$$T_{\mathrm{in}} = 1 - \frac{\delta \Xi}{dq}, \qquad \Delta_{\mathrm{in}} = \frac{x\delta[d^2 - (d^2-1)\delta]}{d^2 q^2},$$

$$T_{\mathrm{out}} = \frac{d[1 + (d-1)x] + \delta\Xi}{(d^2-1)dq}, \qquad \Delta_{\mathrm{out}} = \frac{x\delta(d^2 - \delta)}{(d^2-1)d^2 q^2},$$

where T denotes trace and Δ determinant, not divergence. Substituting $x = t/d$, $\delta = d^{-\beta}$ gives

$$T_{\mathrm{in}} \to 1, \quad d^\nu \Delta_{\mathrm{in}} \to t, \quad d^3 T_{\mathrm{out}} \to 1 + t, \quad d^{\nu+2}\Delta_{\mathrm{out}} \to t.$$

These limits also hold for noninteger fixed $\beta > 1$: use $u = 1/d$ and $y = d\delta$, and evaluate the corresponding rational functions jointly at $(u, y) = (0, 0)$. Their denominators are nonzero there for fixed $t > 0$.

For a positive two-dimensional matrix with trace T and determinant Δ, if $\Delta/T^2 \to 0$ then $w_+ \sim T$ and $w_- \sim \Delta/T$. Therefore

	w_+	w_-
input	$1 + o(1)$	$td^{-\nu}(1 + o(1))$
output	$(1+t)d^{-3}(1 + o(1))$	$\frac{t}{1+t} d^{-(\nu-1)}(1 + o(1))$.

It follows that $\ell(W_\theta)/\log d \to \nu$ and $\ell(W_{\lambda_d\theta})/(d^3\log d) \to (\nu-4)/(1+t)$. Since $d^4\lambda_d^2 \to 1$, the exact finite-phase identity proves (21). Its physical coherence obeys $d^2c^2 \to t$, giving $I_d = 4c^2\ell(W_\theta)$ and the information limit. The trace distance is exactly $2|c|$.

The exact minimum eigenvalue of the full input state is

$$\lambda_{\min}(s_\theta) = \frac{\delta \min(1,x)}{dq}.$$

Indeed $s_\theta \succeq \delta B$, and an off-Schmidt coordinate with a reference label of minimum weight attains equality. Eventually $x = t/d \le 1$, so $d^\nu\lambda_{\min} \to t$. The largest eigenvalue tends to one, since it lies between θ and one. This proves the condition-number limit, including all ambient modes. The order and faithful positivity follow from Theorem 10 at each finite dimension, before taking limits. □

For any fixed $0 < \gamma < 1$, choose $t > 0$ with $\gamma(1+t) < 1$ and then

$$\nu > \frac{4}{1-\gamma(1+t)}, \qquad \beta = \nu - 3.$$

This yields $\alpha > \gamma$ with parameters independent of d. The conditioning exponent and constants depend on γ; no fixed exponent works uniformly as $\gamma \uparrow 1$, and $\alpha = 1$ is not attained with finite parameters. This approximates the scale of an existing lower, not an unrestricted optimal coefficient.

A different order cost and necessary quantifiers. The alternative explicit choice $e = 1/d$, $\theta = 1 - 1/d$ gives $dR_d \to 1/10$, $I_d \sim 20\log(d)/d^3$, condition number $\sim d^5$ and trace distance $\sim 2d^{-3/2}$. Its exact order is $d^2 - d + 1/d$. The choice $t = 1, \beta = 2$ above retains the same leading retention and condition exponent with input information larger by a factor asymptotic to d, but spends the larger order $d^2 - 1 + d^{-2}$. It is not a fixed-budget improvement.

Fixed parameters are substantive. If t is instead replaced by $t_d = 1/d$ with $\beta = 2$, the true limit is $dR_d \to 1/3$, not the $1/5$ obtained by inserting t_d into the fixed-t formula and then taking $d \to \infty$. The input information is then $24\log(d)/d^3$, not the analogous constant20. Likewise $\beta = 1$ is outside this theorem: it changes the output-root regime. No uniform substitution of variable parameters is justified.

Evidence and operational boundary. Audit1068 reconstructed the rational and logarithmic limits, including noninteger β, and checked seven full ambient constructions,57 integer-interval cases and containment of the40 supplied interval certificates. For the $e = 1/d$, $\theta = 1 - 1/d$ sequence, it retains negative gaps versus Bell at $d = 2, \ldots, 7$; positive checked points beginning at8 do not prove a threshold for every integer $d \ge 8$. Nine of the32 original fixed tradeoff points are also adverse. The limit proves eventual separation, without assigning a universal finite threshold from the table. Initial control failures and the correction of a matrix-indexing error remain documented.

This is a relative separation under $\lambda_d \to 0$, approaching the replacement channel, not a growing absolute advantage at fixed noise. If $d = 2^m$, the condition number is exponential in m and the input information is exponentially small up to a factor m. Locality means $H_A \otimes I_R$, not a proved few-body implementation on m qubits. The SLD information of the same input phase is $4c^2/T_{\text{in}} \sim 4t/d^2$ and does not grow. No preparation complexity, experimental tolerance, Pareto frontier, necessary resource exponent, optimal absolute estimation or MI/capacity consequence follows. The specific quantitative scaling is a direct refinement once the preceding exact block identity is available; its priority is not certified.

10 A binary measurement retaining the leading output information

An output measurement makes the previous separation a specified statistical experiment. Its denominator remains the *quantum* input relative entropy. It does not define a new classical channel contraction coefficient or certify efficient implementation.

Theorem 13 (Explicit joint two-outcome measurement)**.** *Use the fixed $t > 0$, $\beta > 1$ family of Theorem 12, and write $x = t/d$, $a = d-1$, $q = a + x$, $\delta = d^{-\beta}$ and $\theta = 1 - \delta$. In its active plane let*

$$|u\rangle = \frac{1}{\sqrt{a}} \sum_{j=1}^{d-1} |j, j\rangle, \qquad |k_d\rangle = \frac{\sqrt{ax}|u\rangle + |0, 0\rangle}{\sqrt{1 + ax}}, \qquad \Pi_d^0 = |k_d\rangle\langle k_d|.$$

Extend this rank-one effect by zero outside the plane. The ambient projective measurement $\{\Pi_d^0, I - \Pi_d^0\}$ has rare-event probabilities

$$q_d = \frac{\delta x}{q(1 + ax)}, \qquad\qquad p_d = q_d + \frac{4\theta ax}{(d^2 - 1)q(1 + ax)} \tag{24}$$

under the output base $\mathcal{N}_{\lambda_d}(s_\theta)$ and output phase state $\mathcal{N}_{\lambda_d}(\rho_\pi)$, respectively. Set

$$D_d^{\mathrm{bin}} = p_d \log \frac{p_d}{q_d} + (1 - p_d) \log \frac{1 - p_d}{1 - q_d}.$$

Then

$$\frac{D_d^{\mathrm{bin}}}{D_{\mathrm{U}}(\mathcal{N}_{\lambda_d}(\rho_\pi) \| \mathcal{N}_{\lambda_d}(s_\theta))} \longrightarrow 1, \qquad \frac{dD_d^{\mathrm{bin}}}{I_d} \longrightarrow \frac{\beta - 1}{(1 + t)(\beta + 3)}.$$

The finite binary divergence is strictly smaller than the quantum output divergence. Its ratio to the quantum input divergence nevertheless eventually exceeds $1/(d^2 - 1)$, the common-Bell-class upper even when arbitrary output measurements are allowed.

Proof. At $\theta = 1$, the output block is proportional to

$$\begin{pmatrix} 1 & -\sqrt{ax} \\ -\sqrt{ax} & ax \end{pmatrix},$$

so $|k_d\rangle$ is its kernel vector. This limiting matrix is used only to choose a measurement: the actual states have $0 < \theta < 1$ and remain faithful. Taking the trace with the faithful block $W_{\lambda_d \theta}$ gives q_d in (24); flipping its coherence gives $p_d - q_d$. These are probabilities in the full ambient states, with the complementary outcomes $1 - q_d, 1 - p_d$, including all inactive modes. Positivity and normalization therefore give $0 < q_d < p_d < 1$.

For the fixed parameters,

$$q_d \sim \frac{t}{1 + t} d^{-(\beta+2)}, \qquad p_d \sim \frac{4t}{1 + t} d^{-3}, \qquad \log(p_d/q_d) \sim (\beta - 1) \log d.$$

The complementary term is $q_d - p_d + O(p_d^2 + q_d^2) = O(p_d)$ and is smaller than $p_d \log d$. Hence

$$D_d^{\mathrm{bin}} \sim \frac{4t(\beta - 1)}{1 + t} \frac{\log d}{d^3}.$$

Theorem 12 gives the same leading output constant and the input constant $4t(\beta + 3)$, proving both limits.

Let Q be the plane projector. Refining the measurement to $\{\Pi_d^0, Q-\Pi_d^0, I-Q\}$ yields distributions $(p_d, T-p_d, 1-T)$ and $(q_d, T-q_d, 1-T)$ with the same plane mass $T<1$. Merging the latter two outcomes loses divergence strictly by log-sum, since $p_d \neq q_d$ and $1-T>0$. Data processing bounds the refined divergence by the quantum divergence, proving strictness for the binary measurement. For any common-Bell pair, data processing and Proposition 11 bound *every* measured output by its quantum input divided by d^2-1. The displayed positive limit, compared with this d^{-2} benchmark, proves the eventual separation. □

Spectral form, simplification and prior work. The original proposal used the exact smaller-eigenvalue projector of the output block. For its eigenvalues $h>l>0$, off-diagonal b, and plane projector Q, this is

$$\Pi_d^- = \frac{hQ - Q\mathcal{N}_{\lambda_d}(s_\theta)Q}{h-l}, \quad q_d^- = l, \quad p_d^- = l + \frac{4b^2}{h-l}.$$

It has the same leading limits. Audit1077 reconstructed that formula, then derived the simpler kernel-at-$\theta=1$ projector Π_d^0 and its exact probabilities as an additional result. The latter avoids eigenvalue diagonalization and depends on d, x, not on β. This provenance is distinct from the original1070/1073 inputs. No finite or global optimality of either measurement is claimed, and exact diagonalization is not necessary for the leading limit.

Measured relative entropy and data processing are prior. Berta–Fawzi–Tomamichel [18] establish the projective/POVM comparison and the strict gap from Umegaki for faithful noncommuting pairs. Our limit does not assert finite equality. Quantum Stein results and regularized measured-entropy achievability [19] instead concern fixed states and a growing number of copies. Here the single-copy pair itself changes with dimension. Neither those general theories nor binary hypothesis testing are claimed as discoveries; the contribution is the explicit leading-constant realization for this family.

Falsifiers and costs. The support convention matters: replacing Q by the full identity in Π_d^- gives an effect exceeding one outside the plane. Renormalizing the subnormalized block removes the cost of postselection and changes the experiment. Measuring also the input changes the denominator: for Π_d^- its classical input divergence is asymptotic to $8t/d^2$, rather than the quantum $4t(\beta+3)\log(d)/d^2$. The affine relation between these particular measured probabilities does not define a stochastic binary channel on all probabilities: its extrapolated transition can be negative. No classical SDPI is contradicted.

Fixed parameters and orientation remain essential. For the original spectral measurement, the inadmissible substitution $t_d = 1/d$, $\beta = 2$ gives measured/quantum output ratio tending to $1/2$, not one; reversing its Bernoulli orientation instead loses the logarithmic leading term. Replacing its eigenvector by the constant limiting vector $(\sqrt{t}, 1)/\sqrt{1+t}$ gives limiting fraction $\min(\beta-1, 2)/(\beta-1)$, losing information when $\beta > 3$. This does not apply to the finite-d kernel vector in the theorem.

Both measurements are joint on $A:R$, parameter dependent and generally entangled effects. Rare probabilities scale as d^{-3} and $d^{-(\beta+2)}$; the information vanishes. For the exact spectral projector, a within-plane angular error ϵ changes q_d^- by $(h-l)\sin^2\epsilon$; $\epsilon = o(d^{-(\beta-1)/2})$ is sufficient for a negligible relative error in that probability. This is not a universal apparatus tolerance, and leakage to inactive modes has another cost. No local/LOCC implementation, efficient preparation, precision fixed independently of dimension, optimal sample complexity, noncircular entanglement witness, fixed-noise advantage, capacity or MI conclusion is proved. Costs polynomial in d may be exponential in a qubit encoding.

The full1077 report retains all adverse cases, controls, source scopes and failures; no priority or universal finite threshold is inferred.

11 A localized Doeblin counterexample

We record a discrepancy in a specific printed statement, rather than infer a failure of the surrounding theory. In the convention of George et al. [3, Definition 4, Eq. (3.1)], the ordinary quantum Doeblin coefficient has the primal form

$$\alpha(\mathcal{N}) = \sup\{\operatorname{Tr} X : \ X = X^\dagger,\ I_A \otimes X \preceq \Gamma^{\mathcal{N}}_{AB}\},$$

where Γ is the unnormalized Choi matrix. Its dual in Proposition 3 admits $Y \succeq 0$ with $\operatorname{Tr}_A Y = I_B$, minimizing $\operatorname{Tr}\Gamma Y$. Requiring $X \succeq 0$ instead defines the different coefficient α_+ of Definition 7, Eq. (3.25); the two optimization problems must not be identified.

Consider $\mathcal{N} = (\mathrm{id} + \mathrm{Ad}_X + \mathrm{Ad}_Y)/3$. Its Bloch matrix is $\mathrm{diag}(1, 1, -1)/3$, so its *ordinary trace-distance* contraction coefficient is exactly $1/3$. Direct calculation gives

$$\Gamma = \frac{2}{3} I_4 - \frac{1}{3} v v^*, \qquad v = (1, 0, 0, -1)^T.$$

The primal $X = 0$ is feasible. The dual $Y = vv^*$ is positive, satisfies $\operatorname{Tr}_A Y = I_2$, and gives $\operatorname{Tr}\Gamma Y = 0$. Weak duality therefore already proves $\alpha(\mathcal{N}) = 0$. This contradicts the equivalence between positive Doeblin coefficient and strict ordinary trace contraction in Theorem 5 as printed in arXiv:2503.22823v3 and the checked journal PDF. It does not contradict the usual one-sided Doeblin upper bound or turn ordinary trace contraction into the complete SLD coefficient. The four-dimensional primal/dual certificate is logically separate from Theorem 4. We do not claim a global review of that article, priority for the observation, or that no correction exists elsewhere.

12 Finite evidence, adverse controls and operational limits

The proofs above, rather than numerical optimization, establish the proposed universal statements. The associated evidence separates exact symbolic identities, rational finite-pair certificates, bounded numerical searches and model-assisted proof inspection. Component audit1012 reconstructed the Pauli blocks, faithful limits, order profile and envelope; it checked the supplied qubit rational pairs through full Sylvester systems and added a finite triangular pair with retention above 0.999997 for the entanglement-breaking channel at $-I/3$. Its base has an eigenvalue $1/(6 \cdot 10^{18})$; this proves neither a uniform usable parameter interval nor robustness to preparation errors. A factor-nine ratio of coefficient suprema must not be described as a faithful finite-pair maximum.

The audit preserves three unsuccessful optimizer terminations and an approximately $6.3 \cdot 10^{-13}$ numerical bound excess whose separate recalc changes sign. That discrepancy is not a certified counterexample. The discovery optimizer was not restarted to obtain a favorable result. Public controls include negative parameters, Choi-normalization checks and complex noncommuting resolvent orientation tests. Bounded passing tests are not a universal numerical or priority certificate.

Some tempting literal extensions are false. Without centering $f(1)$, the generator $f(x) = (x-1)^2/(x+1) + 1$, input $\rho = \mathrm{diag}(3/4, 1/4)$, $\sigma = I_2/2$ and $\lambda = 0$ give ratio $15/17$, not zero. The requirement $f(1) = 0$ is substantive. The negative raw order bound and classical-base-only shortcut were excluded explicitly above. The original rational certificate refuting an auxiliary factor-two conditional-variance bound is also retained; it does not settle the Umegaki coefficient.

No unitary-encoding interpretation of the displayed lower. Let Q_i project onto one ladder support block. It commutes with the base and its replacement mixture, but $\operatorname{Tr} Q_i D = \pm 2e \neq 0$. For every Hermitian H, $\operatorname{Tr} Q_i(-i[H, \sigma]) = -i \operatorname{Tr}([\sigma, Q_i]H) = 0$. Thus this tangent is not a

unitary-encoding direction even for a global Hamiltonian. This rules out that interpretation of our construction, not all other unitary families or known ancilla advantages.

The NPT implication in Section 6 would require an output-QFI lower and an input-QFI upper obtained without assuming the entanglement conclusion. Two observed classical Fisher informations do not supply those two directions of inequality. Full tomography already permits a direct PPT test. No noncircular experimental advantage, fixed-Hamiltonian optimum, capacity or quantum-error-correction protocol is established here. Neither an order between metrics nor local SLD exactness determines a BKM, Umegaki or actual-mutual-information coefficient. For the earlier center-rich rank-two tangent at fixed θ, audit1025 finds the BKM limit

$$\lambda^2 \frac{1+(d-1)\theta}{1+(d-1)\lambda\theta},$$

which is strictly below its SLD limit when $0 < \lambda < 1$ and $\theta > 0$. This refutes automatic transfer for this tangent; it is not an upper bound on the optimized BKM coefficient.

Evidence availability. The published first version [13] and its exact-PDF report remain immutable. The version-pinned component supplement contains the complete1012 report, verdict, controls, source-reading scopes and incidents. Its `TRANSPORT1019.json` separates original C/D hashes from LF repository hashes; embedded original manifests describe the original dossiers. The public1017 construction in that snapshot is an earlier, weaker general-dimensional candidate, not the principal-rich range in Theorem 7. The earlier rank-two cap has a separate frozen1021 proof and exact controls, technically examined in audit1025 together with the Bell/SEP synthesis1024. The immutable expanded supplement contains the full1018/1025 technical reports, selected controls, both proofs and a portable bundle of five fixed symbolic/rational checks. All65 source-head files checked, including the unchanged canonical graph files, matched their remote bytes. The repository-transport bundle manifest has SHA256 `73b09af49fc3155f02c5e80e53489d258404ba9e47951f75b172e428743b877a`. Install `bundle/requirements.txt` and run `python bundle/replay.py --workdir ./runs --out ./receipt.json`. The tested runtime was Python3.12.14 with SymPy1.14.0; all five replayed outputs matched their expected parsed JSON. The first public-tree replay stopped before computation because LF transport changed a metadata hash. The original manifest contents, explicit transport rebinding and incident are preserved. These fixed controls are not a certified computational-campaign capsule, formal proof or novelty certificate.

13 Prior work and remaining questions

The resolvent representation, inverse-quadratic convexity and ordinary unital-qubit coefficient are prior [1]. Signed Bloch normal form and local-unitary Choi covariance are prior [9]. The 2×3 PPT criterion used in the separable lower is prior [10]; it is not extended to higher-dimensional bipartitions. Cao–Lu [2] study chi-square contraction and tensorization with stated product-reference hypotheses. Hirche–Rouzé–Stilck França [8] develop channel orders, generalized contraction and Weyl-covariant comparisons; Section 8.1 also records the full physical depolarizing range. These are relevant prior tools, not new consequences of the present formulas. George–Tomamichel [7] give fixed-base and ordinary optimized spectral methods. Neither optimization should be silently identified with a complete equal-reference optimization over arbitrary correlated bases. The unrestricted coefficient of $\mathcal{N} \otimes \mathrm{id}_R$ can instead be one simply because a direction is stored in R.

Gao–Zhao [6] study relative entropy to a conditional expectation, order/index parameters and index witnesses. Those constructions and the order-comparison machinery are close prior tools.

Theorem 7 concerns two faithful equal-reference states, their individual actual-marginal order budgets and the specified SLD/triangular quantities. A source covering those same quantifiers would affect novelty; selected reading does not certify its absence. Belzig et al. [4, Section 4.1, Lemma 4.1] prove the prior ordinary relative-entropy/BKM equivalence; Han–Li [12, Section 2.2, Lemma 1] use and attribute it. Their statements must not be repackaged as our discovery. The faithful segment/local method preserves the present convex budget class, but does not compute its optimum. Hirche et al. [5] further distinguish reference-assisted information quantities.

Ancilla-assisted metrology under depolarization is prior; Chapeau-Blondeau [11] considers unitary-phase probe families. The earlier nonunitary ladder is not such a protocol. The different local phase family in Section 8 realizes a scoped retention lower, with vanishing absolute information in its limit; no improved experimental protocol is claimed. The bibliographic comparison has been directed, with abstract-only access for one additional2017 metrology source and selected primary passages for the cited2018 work. It is not an exhaustive audit of frontier literature or a worldwide originality certificate.

Theorem 9 supplies lower bounds but no matching general BKM upper. The unrestricted higher-dimensional coefficient, optimal range beyond M_d, minimal sufficient reference dimension, nonunital classification, unrestricted BKM/Umegaki and actual-MI coefficients, tensorization and practical robustness remain open in this manuscript. Proposition 6 gives generator-specific equality only in its stated budget range; it is not transferred to the larger rank-two range. The qubit classification and order-budget results belong to one evolving contraction problem; we do not count each corollary or adverse control as a separate paper.

New component evidence. The version-pinned supplement contains the 1033/1035 derivations, fixed replay scripts and original interval certificates with transport hashes. Technical audit1044 separately reconstructed the geometry, logarithmic coefficient and ordered limits; solved an ambient rational225-by225 Sylvester system; checked Umegaki orientation without normalizing subblocks; and verified four finite gaps using directed integer intervals. Its full report, controls, incidents and primary-reading scope are retained with the working dossier and must accompany any later publication. The original interval solver and this separate implementation support bounded certificates, not universal numerical or novelty claims. The frozen1037 grid found no violation of a hypothetical upper in270 fixed points; that inconclusive diagnostic supplies no upper and is not used in a proof.

Finite-phase supplement. The version-pinned finite-phase supplement preserves full1054, original1051/1052 derivations, directed-integer intervals, adverse cases and source-reading scopes, with explicit archival/LF transport hashes. All82 changed public files were checked at the merged commit and five public endpoints matched local bytes; these checks concern publication integrity, not theorem novelty. Audit1058 and its full all-noise clarification, controls, incident and positive-noise certificates are retained in the working dossier and must accompany publication of this expansion. No technical audit supplies a manuscript score.

14 Model assessments and review context

The named author directed substantial OpenAI Codex assistance in derivation, writing and numerical work. Full review1031, corrective continuation1042 and full review1048 examined predecessor PDFs; all reports, objections and exact-PDF hashes are retained. Review1048 assigned importance6.1, global5.9 and technical8.7 only to its exact fifteen-page input (SHA256 f805a688fb62e9bb34f4c8f107f906c4f85613f831cfda424d1805b797fffe07). Those scores do not evaluate

this expanded manuscript. Technical audits1018/1025,1044,1054,1058,1068 and1077 were requested as GPT-6 Astra xhigh, with reused conversational context and a shared filesystem. Effective model identity, effort, runtime isolation, blinding and independence are unattested. No independent second voice or human peer review is claimed.

Audit1044 supports the principal-rich exact SLD/minimal-chi-square/triangular range and the separate BKM/Umegaki lower, with the distinction between M and P, complete versus fixed-reference quantifiers, increased reference cost and no unitary-tangent interpretation preserved. It supplies no manuscript grade or editorial recommendation. Its directed prior-art comparisons do not certify priority. Audit1054 supports the distinct fixed local Hamiltonian, finite-phase identities, ordered lower limit, common-Bell endpoint benchmark and four finite separations; its certified falsifier to general unitary-orbit transfer and its negative diagnostics are retained. It gives no manuscript grade. The original1051 checker eigenvector error and the1054 manifest-versus-C-only-sealer inventory incident remain documented. Audit1058 supports the full-noise common-Bell benchmark, its global and boundary conventions, and two finite positive-noise gaps; classical novelty is explicitly disclaimed and precise Lee–Makur coverage remains unread. Its auxiliary Float/Rational control failure and correction are retained. Audit1068 supports the dimension-scaled tradeoffs only at fixed parameters, with negative finite cases, vanishing absolute information, dimension-dependent noise, exponential costs in qubit count and no resource frontier preserved. Its initial ambient indexing failure and repair remain in the full report. Audit1077 supports the binary probabilities, orientation and fixed-parameter limits, preserves all12 original adverse cases and distinguishes the original spectral measurement from the simpler kernel projector derived during that audit. It does not establish finite optimality, efficient measurement or priority. At source freezing, a full exact-PDF assessment of this combined version has not yet been obtained; no predicted score, Lean proof of the analytic theorems or worldwide originality certificate is claimed. Publication as a Working paper would not constitute editorial acceptance. The author remains responsible for attribution and claims.

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