Complete contraction of unital qubit channels and order-constrained depolarization

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Abstract

We determine the complete equal-reference contraction coefficients of the minimal quantum Fisher metric, its quadratic chi-square divergence and the finite triangular Petz divergence for every unital qubit channel. For a faithful normalized Choi state J, all three equal $1-16/\operatorname{Tr}(J^{-1})$; for singular J they equal one. An explicit qubit–qutrit lower family matches a harmonic-weight upper bound and gives the exact stabilization envelope, including a maximal factor of nine between coefficient suprema. For depolarization in dimension d, a commuting Bell family matches the order bound $\lambda^2\kappa/[1+\lambda(\kappa-1)]$ for each normalized nonaffine operator-convex Petz generator when $1<\kappa\le d^2/2$. A rank-two ladder extends SLD and triangular sharpness to $\kappa\le d(d+1)/2$, with the budget taken against the actual reference marginal. In the qubit case we identify exact classical-reference and separable/PPT restrictions over the full physical noise range. A localized rational counterexample addresses one printed Doeblin equivalence. These results do not determine unrestricted BKM, relative-entropy, actual-mutual-information or capacity coefficients; the lower constructions are not unitary phase models. Faithful limits, adverse controls, prior-work dependencies and model-review limits are explicit.

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1 Quantities and scope

Let A and R be finite-dimensional systems. All local maps below are tensored with id_R, whether or not this is written. A state is faithful if it is positive definite. We use the Hilbert–Schmidt inner product $\langle X,Y\rangle=\operatorname{Tr}X^\dagger Y$ on the *complex* Hilbert space of matrices. Define $L_a(X)=aX$ and $R_b(X)=Xb$. For faithful a,b, these are commuting positive invertible superoperators. This statement concerns positivity on Hilbert–Schmidt space, not preservation of the cone of positive matrices.

For a faithful state σ and a Hermitian tangent Z, set

$$g_\sigma^{\mathrm{SLD}}(Z)=\langle Z,\left[(L_\sigma+R_\sigma)/2\right]^{-1}Z\rangle. \tag{1}$$

This is the minimal monotone, or symmetric logarithmic derivative (SLD), quantum Fisher quadratic form. Its associated finite chi-square divergence is

$$\chi^2_{\min}(\rho\|\sigma)=g_\sigma^{\mathrm{SLD}}(\rho-\sigma). \tag{2}$$

For a nonaffine operator-convex $f : (0,\infty) \to \mathbb{R}$, $f(1) = 0$, the Petz divergence is

$$D_f(\rho\|\sigma) = \langle \sigma^{1/2}, f(L_\rho R_\sigma^{-1})\sigma^{1/2}\rangle. \tag{3}$$

We will also use the distinct finite divergence

$$f_\triangle(x) = \frac{(x-1)^2}{x+1}, \qquad D_\triangle(\rho\|\sigma) = D_{f_\triangle}(\rho\|\sigma). \tag{4}$$

Although (2) and (4) have proportional local forms, they are not the same finite divergence. Nor do we identify either with the Bures distance or its geodesic contraction.

For a local channel $\mathcal{N}$, define the complete equal-reference coefficient

$$\eta_f^c(\mathcal{N}) = \sup_{\substack{|R|<\infty,\ \rho,\sigma>0 \\ \rho\neq\sigma,\ \mathrm{Tr}_A\,\rho=\mathrm{Tr}_A\,\sigma}} \frac{D_f(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))}{D_f(\rho\|\sigma)}. \tag{5}$$

Define its SLD analogue by optimizing $g^{\mathrm{SLD}}_{\mathcal{N}(\sigma)}(\mathcal{N}(Z))/g^{\mathrm{SLD}}_\sigma(Z)$ over faithful σ, nonzero Hermitian Z with $\mathrm{Tr}_A\, Z = 0$, and all finite R. It is also the coefficient of (2): every such tangent is realized by a sufficiently small faithful pair.

These restrictions matter. Actual mutual information compares a state with the product of *its own* marginals. Conditional coefficients can instead subtract a divergence between unequal reference marginals. Fixed-point entropy decay compares a state with a conditional expectation. We do not replace one optimization by another.

2 Known representation and a cancellation bound

The positive resolvent representation of operator-convex functions, and the resulting representation of Petz divergences, are established facts; see Hiai–Ruskai [1], Proposition 2.1 and equations (12)–(13). After removing an affine term, any nonaffine operator-convex f as above has

$$f(x) = f'(1)(x-1) + c(x-1)^2 + \int_{[0,\infty)} \frac{(x-1)^2}{x+t}\, d\mu(t), \tag{6}$$

where $c \geq 0$, μ is positive and $\int (1+t)^{-1}\, d\mu(t) < \infty$. Writing $Z = \rho - \sigma$, for states of the same trace this gives

$$D_f(\rho\|\sigma) = c\ \mathrm{Tr}\, Z^2\sigma^{-1} + \int_{[0,\infty)} \langle Z, (L_\rho + tR_\sigma)^{-1} Z\rangle\, d\mu(t). \tag{7}$$

We do not claim this representation as a contribution.

Lemma 1. *For positive definite Q, the inverse quadratic form $F(Q,Z) = \langle Z, Q^{-1}Z\rangle$ is jointly convex in Q, Z, for arbitrary complex vectors Z.*

Proof. Use the variational identity

$$F(Q,Z) = \sup_Y \{2\,\mathrm{Re}\langle Z, Y\rangle - \langle Y, QY\rangle\}.$$

Each expression inside the supremum is affine in Q, Z. The same identity also shows that for any trial Y, with $e = Z - QY$,

$$F(Q,Z) = 2\,\mathrm{Re}\langle Z, Y\rangle - \langle Y, QY\rangle + \langle e, Q^{-1}e\rangle. \tag{8}$$

□

Theorem 2 (Harmonic cancellation)**.** *Let $U_1, \dots, U_m$ be local unitaries, $m \geq 2$, and let*

$$\mathcal{E} = \frac{1}{m} \sum_{j=1}^{m} \mathrm{Ad}_{U_j}, \qquad \mathcal{N}_w = \sum_{j=1}^{m} w_j \, \mathrm{Ad}_{U_j}, \qquad w_j > 0, \quad \sum_j w_j = 1.$$

For faithful states ρ, σ satisfying $\mathcal{E}(\rho) = \mathcal{E}(\sigma)$, and every nonaffine operator-convex f with $f(1) = 0$,

$$D_f(\mathcal{N}_w(\rho) \| \mathcal{N}_w(\sigma)) \leq h(w) D_f(\rho \| \sigma), \qquad h(w) = 1 - \frac{m^2}{\sum_j w_j^{-1}}. \tag{9}$$

The same $h(w)$ applies for every finite R. It also bounds the SLD metric ratio for tangents Z with $\mathcal{E}(Z) = 0$.

Proof. For a resolvent term in (7), let $Q = L_\rho + tR_\sigma$, $Z = \rho - \sigma$, and write $Q_j = \mathrm{Ad}_{U_j} Q \, \mathrm{Ad}_{U_j}^\dagger$, $Z_j = U_j Z U_j^\dagger$. The constraint gives $\sum_j Z_j = 0$. For every real a,

$$\mathcal{N}_w(Z) = \sum_j (w_j - a) Z_j = \sum_j w_j \frac{w_j - a}{w_j} Z_j.$$

Since the output superoperator is $\sum_j w_j Q_j$, Lemma 1, quadratic homogeneity and unitary covariance give

$$F\left(\sum_j w_j Q_j, \mathcal{N}_w(Z) \right) \leq \sum_j \frac{(w_j - a)^2}{w_j} F(Q, Z).$$

The scalar factor is $1 - 2ma + a^2 \sum_j w_j^{-1}$; its minimum occurs at $a = m / \sum_j w_j^{-1}$ and equals $h(w)$. The convex weights here are the positive w_j. Negative tangent multipliers do not constitute negative convex weights on states.

For the quadratic endpoint in (7), use $Q = R_\sigma$; its inverse quadratic form is $\mathrm{Tr}\, Z^2 \sigma^{-1}$. Integrate the nonnegative resolvent inequalities. Affine terms vanish by equal traces, and the positive representation justifies this passage. For the SLD metric, use $Q = (L_\sigma + R_\sigma)/2$ directly. No argument depends on a bound on $|R|$. □

Cauchy–Schwarz gives $\sum_j w_j^{-1} \geq m^2$, so $0 \leq h(w) < 1$. A useful exact control is an involution $U = U^\dagger$, $U^2 = I$, with both eigenspaces nonzero. For $\mathcal{E} = (\mathrm{id} + \mathrm{Ad}_U)/2$ and $\mathcal{N} = w \, \mathrm{id} + (1 - w) \, \mathrm{Ad}_U$, the bound is $(2w - 1)^2$. It is sharp for each of the Petz generators under consideration: take $\sigma = I/n$, a nonzero Hermitian Z connecting the two eigenspaces, and $\rho_\epsilon = \sigma + \epsilon Z$. Then $\mathcal{N}(Z) = (2w - 1)Z$, and the scalar second-order expansion of (3) gives the stated limiting ratio. At $w = 1/2$, the two outputs coincide.

Remark 3 (Relative entropy at the boundary). For $f(x) = x \log x$, (9) also holds for nonfaithful pairs with finite relative entropy. Mix both states with the same maximally mixed state. The equal-$\mathcal{E}$ constraint is preserved. Joint convexity and lower semicontinuity imply convergence of both finite divergences, and the faithful inequality passes to the limit before division. We do not assert a boundary extension for every f without specifying its boundary conventions.

3 Exact contraction of every unital qubit channel

For the Pauli matrices P_j, $j = 0, x, y, z$, write $\mathcal{N}_w = \sum_j w_j \, \mathrm{Ad}_{P_j}$, $w_j \geq 0$, $\sum_j w_j = 1$. The uniform Pauli twirl is $\mathcal{E}_A(X) = I_2/2 \otimes \mathrm{Tr}_A X$.

Theorem 4 (Complete unital-qubit classification)**.** *The complete equal-reference coefficients of SLD, $\chi^2_{\min}$ and the finite triangular Petz divergence coincide. For positive Pauli weights their common value is*

$$h(w) = 1 - \frac{16}{\sum_{j=0,x,y,z} w_j^{-1}}. \tag{10}$$

If any Pauli weight is zero, the common value is one. A reference of dimension three suffices for each supremum. No assertion of minimal reference dimension or of a faithful maximizing pair is made.

Consequently, for an arbitrary unital qubit CPTP channel with normalized Choi state $J = (\mathcal{N} \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)$, $|\Omega\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, the common value is

$$\eta^c(\mathcal{N}) = \begin{cases} 1 - 16/\operatorname{Tr}(J^{-1}), & J > 0, \\ 1, & J \text{ singular.} \end{cases} \tag{11}$$

For every normalized nonaffine operator-convex Petz generator this is an upper bound. Equality for each individual generator is not asserted.

Proof. Theorem 2 proves the upper for positive weights. We give the matching lower, including its faithful realization. Put

$$A = w_0 + w_z,\ B = w_x + w_y = 1 - A,\ u = w_0 - w_z,\ v = w_x - w_y,\ z = A - B,$$

$$U = u^2/A, \quad V = v^2/B, \quad k = 1 - U - V = 4w_0w_z/A + 4w_xw_y/B > 0.$$

In the basis $(00, 01, 02, 10, 11, 12)$, let

$$a = (1, 0, 0, 0, e, 0)^T, \quad b = (0, e, 0, 0, 0, 1)^T, \quad \sigma_e = \frac{aa^* + bb^*}{2(1 + e^2)}, \qquad e > 0.$$

Define the real Hermitian tangent by

$$D_{04} = D_{40} = 1, \quad D_{44} = 2e, \quad D_{11} = -2e, \quad D_{15} = D_{51} = -1,$$

with all other entries zero. Then $\operatorname{Tr}_A D = 0$, $P = 2\sigma_e$ is a projection and $(I - P)D(I - P) = 0$. The support/kernel limit of the SLD form is

$$g^{\mathrm{SLD}}_{\sigma_e}(D) = 4 \operatorname{Tr} D^2 - 2 \operatorname{Tr}(PDP)^2 = 16(1 + e^2). \tag{12}$$

Here and below a boundary form denotes this finite limit, not the ordinary inverse of a singular matrix.

The two output blocks on coordinates $(0, 4, 2)$ and $(3, 1, 5)$ are W and JWJ, where J reverses three coordinates and

$$W = \frac{M}{2(1 + e^2)}, \quad M = \begin{pmatrix} A & ue & 0 \\ ue & e^2 & ve \\ 0 & ve & B \end{pmatrix}.$$

Their tangent blocks are T and $-JTJ$, with

$$T = \begin{pmatrix} 0 & u & 0 \\ u & 2ez & -v \\ 0 & -v & 0 \end{pmatrix}.$$

The central Schur complement is e^2k, so both output blocks are positive definite. Let $Y = Y^T$ solve $MY + YM = 4(1+e^2)T$, and put $x = Y_{12}$, $y = Y_{23}$, $c = eY_{22}$. Eliminating the other three entries gives

$$K_e \begin{pmatrix} x \\ y \\ c \end{pmatrix} = 4(1+e^2) \begin{pmatrix} u \\ -v \\ z \end{pmatrix}, \quad K_e = \begin{pmatrix} A + e^2(1 - u^2/A - v^2) & -e^2uv & u \\ -e^2uv & B + e^2(1 - v^2/B - u^2) & v \\ u & v & 1 \end{pmatrix}.$$

Since $\det K_0 = ABk > 0$, the scaled unknowns converge to the nonsingular limiting solution

$$c_0 = 4(z - U + V)/k, \quad x_0 = u(4 - c_0)/A, \quad y_0 = v(-4 - c_0)/B.$$

Both blocks contribute equally to the SLD form. Thus

$$\lim_{e \downarrow 0} \frac{g^{\mathrm{SLD}}_{\mathcal{N}_w(\sigma_e)}(\mathcal{N}_w(D))}{16(1+e^2)} = U + V + \frac{(z - U + V)^2}{k} = 1 - \frac{16}{\sum_j w_j^{-1}}. \tag{13}$$

The last equality is a rational identity using $A + B = 1$.

For each fixed $e > 0$, replace the input by $\sigma_{e,\xi} = (1-\xi)\sigma_e + \xi I_6/6$, $0 < \xi < 1$. The kernel–kernel tangent vanishes, so its input form tends to (12); the output is already faithful and its inverse is continuous. Take $\xi \downarrow 0$ before $e \downarrow 0$. Every finite tangent is realized by $\rho_t = \sigma_{e,\xi} + tD$ for sufficiently small nonzero real t. For triangular Petz take $t \to 0$ first; its local form is $g^{\mathrm{SLD}}/2$. Minimal chi-square is exactly quadratic in t. These ordered limits prove all three lower suprema. The uniform-weight channel has zero output tangent and coefficient zero; no division by that coefficient is used.

For boundary weights set $w_j(\gamma) = (1-\gamma)w_j + \gamma/4$. Since $\mathcal{N}_\gamma = (1-\gamma)\mathcal{N}_w + \gamma\mathcal{E}_A$, joint convexity and equal reference give, for each of the three quantities, output at $\mathcal{N}_\gamma$ at most $(1-\gamma)$ times output at $\mathcal{N}_w$. Consequently $\eta^c(\mathcal{N}_w) \geq h(w(\gamma))/(1-\gamma) \to 1$. The upper one is data processing. Faithful inputs have faithful outputs under any Pauli mixture.

Finally use the prior signed singular-value normal form of unital qubit channels [9]: pre/post unitary conjugations reduce the Bloch matrix to a physical Pauli diagonal map. Both rotations must lie in $SO(3)$; erasing all singular-value signs is not legitimate. These conjugations preserve the equal-reference constraint, the three ratios and the normalized Choi spectrum. This proves (11). The normal-form reduction itself is not new. For the unnormalized Choi matrix $\Gamma = 2J$, the coefficient would instead read $1 - 8/\operatorname{Tr}\Gamma^{-1}$. □

4 The exact stabilization envelope

Let $s = \|T\|_\infty \in [0,1]$ for the real Bloch matrix of a unital qubit channel. The ordinary coefficient for these metrics/divergences is s^2, by the prior result of Hiai–Ruskai [1, Theorem 6.1].

Theorem 5. *At fixed s, the possible complete coefficients in Theorem 4 fill precisely $[s^2, F(s)]$, where*

$$F(s) = \begin{cases} 3s^2/(1-2s), & 0 \leq s \leq 1/3, \\ 1, & 1/3 \leq s \leq 1. \end{cases}$$

For $s > 0$ the ratio between complete and ordinary coefficients is at most nine, and nine is attained as a ratio of channel coefficient suprema. It is undefined at $s = 0$, where both coefficients vanish.

Proof. After the signed normal form write $T = \operatorname{diag}(t_1, t_2, t_3)$. The Pauli weights are the four affine functions $(1 \pm t_1 \pm t_2 \pm t_3)/4$ with an even number of minus signs. For $s < 1/3$, the cube $[-s, s]^3$

is inside the positive-weight CP tetrahedron. The reciprocal sum $S = \sum_j 1/w_j$ is convex, so its maximum occurs at a vertex. The two values there are

$$S_+ = \frac{4}{1+3s} + \frac{12}{1-s}, \qquad S_- = \frac{4}{1-3s} + \frac{12}{1+s},$$

and $S_- - S_+ = 192s^3/[(1-9s^2)(1-s^2)] \geq 0$. Since $1 - 16/S$ increases with S, the upper is achieved by $T = -sI$ and equals $F(s)$. For $s \geq 1/3$, DPI gives one, achieved by the CP boundary map $\mathrm{diag}(-s, -(1-s)/2, -(1-s)/2)$.

The map $\mathrm{diag}(-s, 0, 0)$ realizes the lower s^2. A straight segment to the corresponding upper example remains CP and keeps norm s. The formula in Theorem 4 extends continuously to the simplex boundary, with value one there. The intermediate value theorem therefore gives every value of the interval; no monotonicity along the segment is assumed. Dividing $F(s)$ by s^2 proves the factor-nine statement. □

At $T = -I/3$, the channel chooses a Pauli axis uniformly, measures it and prepares the opposite eigenstate. It is therefore entanglement breaking. Nevertheless its complete coefficient is one, while its ordinary coefficient is $1/9$. This does not say that all entanglement-breaking channels have complete coefficient one, nor that a faithful input attains the supremum exactly.

5 Exact order-constrained depolarization in arbitrary dimension

Let $\mathcal{E}_A(X) = I_d/d \otimes \mathrm{Tr}_A X$ and $\mathcal{N}_\lambda = \lambda\,\mathrm{id} + (1-\lambda)\mathcal{E}_A$, where $-1/(d^2-1) \leq \lambda \leq 1$, the known physical range [8, Section 8.1]. For $\kappa > 1$ restrict the finite pairs to

$$\rho_R = \sigma_R, \quad \rho, \sigma > 0, \quad \rho \neq \sigma, \quad \rho \preceq \kappa\mathcal{E}_A(\rho), \quad \sigma \preceq \kappa\mathcal{E}_A(\sigma). \tag{14}$$

For the metric impose the base order bound and $\mathrm{Tr}_A D = 0$; one may also require a two-sided small neighborhood satisfying (14). The suprema proved below are unchanged. The budget uses the *actual* reference marginal, not an optimized conditional min-entropy reference.

Proposition 6 (Commuting Bell lower and its range)**.** *For every $d \geq 2$, $1 < \kappa \leq d^2/2$ and every nonaffine operator-convex Petz generator normalized by $f(1) = 0$, the complete pair coefficient under* (14) *is $\lambda^2\kappa/[1+\lambda(\kappa-1)]$ throughout the physical depolarizing range. The SLD and minimal-chi-square coefficients have the same value. A reference of dimension d suffices for the lower supremum.*

Proof. The order upper is proved below. Write $n = d^2$, $B = I_n/n$, and take two orthogonal maximally entangled Bell projectors P_0, P_1. For $1 < K < \kappa \leq n/2$, set

$$a = K/n, \quad b = (1-2a)/(n-2), \quad \sigma_K = bI_n + (a-b)(P_0+P_1), \quad D = P_0 - P_1.$$

Here $a > b > 0$, $\mathcal{E}_A(\sigma_K) = B$, $\mathrm{Tr}_A D = 0$, and the actual order constant is K. The input and output commute; their eigenvalues on the two changed rays are a and $a_\lambda = \lambda a + (1-\lambda)/n > 0$. Hence the SLD ratio is exactly $\lambda^2 a/a_\lambda$.

For $0 < |t| < \min(a, b, (\kappa - K)/n)$, $\rho_t = \sigma_K + tD$ and σ_K are faithful with identical reference and both order bounds. Their Petz divergences are

$$a\,[f(1+t/a) + f(1-t/a)], \qquad a_\lambda\,[f(1+\lambda t/a_\lambda) + f(1-\lambda t/a_\lambda)].$$

For a normalized nonaffine operator-convex generator, $f''(1) > 0$, as follows from the nonzero positive representation in (6). The Taylor ratio as $t \to 0$ is therefore $\lambda^2 a/a_\lambda = \lambda^2 K/[1+\lambda(K-1)]$. Then let $K \uparrow \kappa$. At the endpoint $\kappa = n/2$, each earlier base remains faithful. The cases $\lambda = 0, 1$ are immediate. The quadratic chi-square equality follows directly from the metric ratio. □

This is a reparametrization of the Bell family already present in our earlier component work, not an independent invention of Bell-assisted contraction. In particular, when $d \geq 3$, the interval $\kappa \leq d+1$ is contained in $\kappa \leq d^2/2$; the rank-d ladder previously considered on that interval is an alternative proof. A conditional-expectation index alone does not imply sharpness: the pinching $\mathcal{E} = (\mathrm{id} + \mathrm{Ad}_Z)/2$ has index two, but its equal-$\mathcal{E}$ contraction is λ^2, rather than the larger raw order comparison, for $0 < \lambda < 1$.

Theorem 7 (Rank-two controlled order profile)**.** *For $d \geq 2$ and $1 < \kappa \leq L_d = d(d+1)/2$, the SLD, minimal-chi-square and finite triangular-Petz contraction coefficients under* (14) *equal*

$$C_\kappa(\lambda) = \frac{\lambda^2 \kappa}{1 + \lambda(\kappa - 1)}. \tag{15}$$

A reference of dimension $d+1$ suffices for each lower supremum. The same expression is an upper for every nonaffine operator-convex Petz generator normalized by $f(1) = 0$, and is sharp as a bound uniform over that family. No exact higher-dimensional coefficient beyond the stated order range is asserted.

Proof. The standard order comparison gives, for each admissible state,

$$\mathcal{N}_\lambda(\sigma) \succeq c\sigma, \qquad c = \lambda + (1-\lambda)/\kappa > 0.$$

The positivity follows from $1 + \lambda(\kappa - 1) \geq 1 - (L_d - 1)/(d^2 - 1) > 0$. On equal-reference differences $\mathcal{N}_\lambda(D) = \lambda D$. Apply inverse antitonicity to $(L_\sigma + R_\sigma)/2$, or to $L_\rho + tR_\sigma$ using the order for both states. The quadratic endpoint and positive integral in (7) give the upper λ^2/c. This comparison machinery is prior; the matching lower is the construction below.

Put $m = d - 1$, $q = 1 + me^2$, and give R the labels $0, \ldots, d$. With A indices modulo d, set

$$a_i = |i, i\rangle + e \sum_{k=1}^{d-1} |i + k, k + 1\rangle \quad (i = 0, 1), \qquad \sigma_e = \frac{a_0 a_0^* + a_1 a_1^*}{2q}.$$

The two vectors have disjoint supports and squared norm q, so $P = 2\sigma_e$ is a projection. The reference marginal has mass $1/(2q)$ on labels zero and one, and mass e^2/q on each remaining label. Hence $B_e = \mathcal{E}_A(\sigma_e) > 0$. The whitened normalized vectors are orthogonal and have squared norm $d + d(d-1)/2 = L_d$; consequently $B_e^{-1/2} \sigma_e B_e^{-1/2}$ has two eigenvalues L_d and all others zero.

Let $b_0 = \sum_{k=1}^{d-1} |k, k+1\rangle$, $b_1 = -\sum_{k=1}^{d-1} |1+k, k+1\rangle$ and $D = \sum_{i=0}^{1} (b_i a_i^* + a_i b_i^*)$. All off-diagonal partial-trace terms vanish because their A indices differ. At each center reference label the opposite diagonal derivatives cancel. Thus $\mathrm{Tr}_A D = 0$, $(I - P)D(I - P) = 0$, and the finite boundary SLD form is $16mq$.

For $0 \leq \theta < 1$, define $s_{e,\theta} = (1-\theta)B_e + \theta\sigma_e$. Its whitened eigenvalues are $1 + (L_d - 1)\theta$ twice and $1 - \theta$ otherwise. It is faithful and has exactly that order constant. Moreover $\mathcal{N}_\lambda(s_{e,\theta}) = s_{e,\lambda\theta}$.

Each vector's d-dimensional support is an invariant block. On the main coordinate and the uniform center superposition its base and tangent reduce to

$$W_p = \frac{1}{2dq} \begin{pmatrix} 1 + (d-1)p & dpe\sqrt{m} \\ dpe\sqrt{m} & [2 + (dm-2)p]e^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \sqrt{m} \\ \sqrt{m} & 2em \end{pmatrix},$$

with opposite signs for the two tangent blocks. The other $m - 1$ center modes have base eigenvalue $e^2(1-p)/(dq)$ and zero tangent. All remaining coordinates are positive diagonal replacement contributions with zero tangent. The aligned determinant is

$$\det W_p = \frac{e^2(1-p)[1 + (L_d - 1)p]}{2d^2q^2} > 0$$

for the values $p = \theta, \lambda\theta$ used here.

Let $G_e(p) = \mathrm{Tr}(TY)$, where $W_pY + YW_p = 2T$. Scaling the second diagonal unknown by e, or eliminating it directly, gives

$$\lim_{e\downarrow 0} \frac{G_e(p)}{8m} = \frac{L_d}{1 + (L_d - 1)p}. \tag{16}$$

For completeness, put $A = 1+(d-1)p$, $B = dp\sqrt{m}$, $C = 2+(dm-2)p$. The limiting system for the off-diagonal unknown and the scaled diagonal has matrix $\left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)$. Its determinant is $2(1-p)[1+(L_d-1)p]$; the numerator from the direction $(\sqrt{m}, m)$ is $m[C - 2dmp + Am] = m(m+2)(1-p)$. This proves (16) at every fixed $-1/(L_d - 1) < p < 1$, without assuming uniform convergence at $p = 1$.

Both blocks have the same retention ratio. Therefore, at fixed $\theta < 1$, the ratio tends to

$$\lambda^2 \frac{1 + (L_d - 1)\theta}{1 + (L_d - 1)\lambda\theta}.$$

Choose $\theta < (\kappa - 1)/(L_d - 1)$. Both $s_{e,\theta}$ and $\kappa B_e - s_{e,\theta}$ are strictly positive. For each finite e, θ, the pair $\rho_t = s_{e,\theta} + tD$, $\sigma = s_{e,\theta}$ stays faithful with both order bounds for sufficiently small nonzero real t. Its reference marginal stays fixed. For triangular Petz take $t \to 0$ first; for minimal chi-square the ratio is exactly quadratic. Then take $e \downarrow 0$ and finally $\theta \uparrow (\kappa - 1)/(L_d - 1)$. This proves (15). The cases $\lambda = 0, 1$ are the zero-on-differences and identity endpoints. □

Remark 8 (The empty endpoint and the domain of the order bound). At $\kappa = 1$, positivity and equal traces force $\rho = \mathcal{E}_A(\rho)$ and $\sigma = \mathcal{E}_A(\sigma)$; equal reference then forces equality. There are no distinct finite pairs, so (15) is not assigned to that empty optimization. The base-only SLD problem with unrestricted tangents at a product base is a different problem. Likewise the expression λ^2/c must never be used when $c \le 0$; for example $\lambda = -1/3, \kappa = 5$ gives a negative raw expression in dimension two, not a contraction bound.

6 Exact qubit resource restrictions

For $d = 2$, Theorems 4 and 7 combine to give, for every $\kappa > 1$,

$$C_\kappa(\lambda) = \frac{\lambda^2 K}{1 + \lambda(K - 1)}, \qquad K = \min(\kappa, 3), \qquad -1/3 \le \lambda \le 1.$$

For $\kappa \ge 3$, the upper is the unrestricted isotropic case of Theorem 4, $3\lambda^2/(1 + 2\lambda)$; the same lower family has order arbitrarily close to three. At the channel endpoints use DPI or the identity directly.

For a qubit and any finite reference,

$$2\mathcal{E}_A(\sigma) - \sigma = (\sigma_y \otimes I)\sigma^{T_A}(\sigma_y \otimes I).$$

Thus PPT bases obey the order-two bound, as do separable bases. The exact complete SLD/minimal-chi-square/triangular coefficient restricted to PPT or separable models is $2\lambda^2/(1+\lambda)$. For finite Petz both states must belong to the class. To prove the matching lower, use the qubit ladder above with $s_{e,\theta}$ and $\theta < 1/2$. Its partial transpose has two blocks with determinant $e^2(1-\theta-2\theta^2)/[8(1+e^2)^2] > 0$ and two positive singletons. Small bilateral perturbations stay PPT. In dimension 2×3, the prior Horodecki theorem [10] converts these pairs to separable ones. This uses PPT=SEP only for the lower in 2×3, not for arbitrary references or larger local dimensions.

For *each* normalized nonaffine operator-convex Petz generator, the SEP/PPT pair coefficient is also exactly $2\lambda^2/(1 + \lambda)$. Indeed, the order-two comparison gives the upper, while the qubit

Bell family of Proposition 6, with $K < 2$, has all Bell probabilities p_j strictly below $1/2$. The partial-transpose eigenvalues are $1/2 - p_j$. Both the base and sufficiently small finite perturbations are strictly PPT, hence separable in 2×2, and approach the stated lower. The same gives the budget profile inside SEP/PPT for $1 < \kappa \leq 2$. This restricted-pair statement includes Umegaki but does not determine the unrestricted coefficient or actual MI, whose second state has an additional product-marginal requirement.

A classical-reference *family* has both states and the tangent block diagonal in one fixed reference basis, with fixed marginal weights. The forms split into weighted ordinary-qubit forms, whose coefficient is λ^2 by [1]; $R = 1$ gives equality. Classicality of the base alone is insufficient. For example at $\lambda = -1/3$, the classical base $\sigma = \mathrm{diag}(9, 1, 1, 9)/20$ with $D_{03} = D_{30} = 1$ gives retention $3/11 > 1/9$; the direction creates reference coherences.

At $\lambda = -1/3$ the exact levels are therefore

$$\text{classical-reference family} : \frac{1}{9}, \qquad \text{SEP/PPT family} : \frac{1}{3}, \qquad \text{unrestricted quantum family} : 1.$$

SLD retention (or minimal-chi-square retention with base σ) above $2\lambda^2/(1+\lambda)$ is a sufficient certificate that that input base is NPT. For a general finite Petz pair, the proved implication is only that exceeding this threshold excludes both input states being PPT; it does not identify which state is NPT. This mathematical implication is a contrapositive of the prior reduction criterion plus the quantitative upper; it is not an experimentally demonstrated protocol.

7 A localized Doeblin counterexample

We record a discrepancy in a specific printed statement, rather than infer a failure of the surrounding theory. In the convention of George et al. [3, Definition 4, Eq. (3.1)], the ordinary quantum Doeblin coefficient has the primal form

$$\alpha(\mathcal{N}) = \sup\{\mathrm{Tr}\, X : \ X = X^\dagger, \ I_A \otimes X \preceq \Gamma^{\mathcal{N}}_{AB}\},$$

where Γ is the unnormalized Choi matrix. Its dual in Proposition 3 admits $Y \succeq 0$ with $\mathrm{Tr}_A\, Y = I_B$, minimizing $\mathrm{Tr}\, \Gamma Y$. Requiring $X \succeq 0$ instead defines the different coefficient α_+ of Definition 7, Eq. (3.25); the two optimization problems must not be identified.

Consider $\mathcal{N} = (\mathrm{id} + \mathrm{Ad}_X + \mathrm{Ad}_Y)/3$. Its Bloch matrix is $\mathrm{diag}(1, 1, -1)/3$, so its *ordinary trace-distance* contraction coefficient is exactly $1/3$. Direct calculation gives

$$\Gamma = \frac{2}{3} I_4 - \frac{1}{3} vv^*, \qquad v = (1, 0, 0, -1)^T.$$

The primal $X = 0$ is feasible. The dual $Y = vv^*$ is positive, satisfies $\mathrm{Tr}_A\, Y = I_2$, and gives $\mathrm{Tr}\, \Gamma Y = 0$. Weak duality therefore already proves $\alpha(\mathcal{N}) = 0$. This contradicts the equivalence between positive Doeblin coefficient and strict ordinary trace contraction in Theorem 5 as printed in arXiv:2503.22823v3 and the checked journal PDF. It does not contradict the usual one-sided Doeblin upper bound or turn ordinary trace contraction into the complete SLD coefficient. The four-dimensional primal/dual certificate is logically separate from Theorem 4. We do not claim a global review of that article, priority for the observation, or that no correction exists elsewhere.

8 Finite evidence, adverse controls and operational limits

The proofs above, rather than numerical optimization, establish the proposed universal statements. The associated evidence separates exact symbolic identities, rational finite-pair certificates, bounded

numerical searches and model-assisted proof inspection. Component audit1012 reconstructed the Pauli blocks, faithful limits, order profile and envelope; it checked the supplied qubit rational pairs through full Sylvester systems and added a finite triangular pair with retention above 0.999997 for the entanglement-breaking channel at $-I/3$. Its base has an eigenvalue $1/(6 \cdot 10^{18})$; this proves neither a uniform usable parameter interval nor robustness to preparation errors. A factor-nine ratio of coefficient suprema must not be described as a faithful finite-pair maximum.

The audit preserves three unsuccessful optimizer terminations and an approximately $6.3 \cdot 10^{-13}$ numerical bound excess whose separate recalc changes sign. That discrepancy is not a certified counterexample. The discovery optimizer was not restarted to obtain a favorable result. Public controls include negative parameters, Choi-normalization checks and complex noncommuting resolvent orientation tests. Bounded passing tests are not a universal numerical or priority certificate.

Some tempting literal extensions are false. Without centering $f(1)$, the generator $f(x) = (x-1)^2/(x+1)+1$, input $\rho = \mathrm{diag}(3/4, 1/4)$, $\sigma = I_2/2$ and $\lambda = 0$ give ratio $15/17$, not zero. The requirement $f(1) = 0$ is substantive. The negative raw order bound and classical-base-only shortcut were excluded explicitly above. The original rational certificate refuting an auxiliary factor-two conditional-variance bound is also retained; it does not settle the Umegaki coefficient.

No unitary-encoding interpretation of the displayed lower. Let Q_i project onto one ladder support block. It commutes with the base and its replacement mixture, but $\mathrm{Tr}\, Q_i D = \pm 2em \neq 0$. For every Hermitian H, $\mathrm{Tr}\, Q_i(-i[H,\sigma]) = -i\,\mathrm{Tr}([\sigma, Q_i]H) = 0$. Thus this tangent is not a unitary-encoding direction even for a global Hamiltonian. This rules out that interpretation of our construction, not all other unitary families or known ancilla advantages.

The NPT implication in Section 6 would require an output-QFI lower and an input-QFI upper obtained without assuming the entanglement conclusion. Two observed classical Fisher informations do not supply those two directions of inequality. Full tomography already permits a direct PPT test. No noncircular experimental advantage, fixed-Hamiltonian optimum, capacity or quantum-error-correction protocol is established here. Neither an order between metrics nor local SLD exactness determines a BKM, Umegaki or actual-mutual-information coefficient. For the rank-two tangent at fixed θ, audit1025 finds the BKM limit

$$\lambda^2 \frac{1+(d-1)\theta}{1+(d-1)\lambda\theta},$$

which is strictly below its SLD limit when $0 < \lambda < 1$ and $\theta > 0$. This refutes automatic transfer for this tangent; it is not an upper bound on the optimized BKM coefficient.

Evidence availability. The published first version [12] and its exact-PDF report remain immutable. The version-pinned component supplement contains the complete1012 report, verdict, controls, source-reading scopes and incidents. Its `TRANSPORT1019.json` separates original C/D hashes from LF repository hashes; embedded original manifests describe the original dossiers. The public1017 construction in that snapshot is an earlier, weaker general-dimensional candidate, not the rank-two range in Theorem 7. The latter has a separate frozen1021 proof and exact controls, technically examined in audit1025 together with the Bell/SEP synthesis1024. The immutable expanded supplement contains the full1018/1025 technical reports, selected controls, both proofs and a portable bundle of five fixed symbolic/rational checks. All65 source-head files checked, including the unchanged canonical graph files, matched their remote bytes. The repository-transport bundle manifest has SHA256 `73b09af49fc3155f02c5e80e53489d258404ba9e47951f75b172e428743b877a`. Install `bundle/requirements.txt` and run `python bundle/replay.py --workdir ./runs --out`

`./receipt.json`. The tested runtime was Python3.12.14 with SymPy1.14.0; all five replayed outputs matched their expected parsed JSON. The first public-tree replay stopped before computation because LF transport changed a metadata hash. The original manifest contents, explicit transport rebinding and incident are preserved. These fixed controls are not a certified computational-campaign capsule, formal proof or novelty certificate.

9 Prior work and remaining questions

The resolvent representation, inverse-quadratic convexity and ordinary unital-qubit coefficient are prior [1]. Signed Bloch normal form and local-unitary Choi covariance are prior [9]. The 2×3 PPT criterion used in the separable lower is prior [10]; it is not extended to higher-dimensional bipartitions. Cao–Lu [2] study chi-square contraction and tensorization with stated product-reference hypotheses. Hirche–Rouzé–Stilck França [8] develop channel orders, generalized contraction and Weyl-covariant comparisons; Section 8.1 also records the full physical depolarizing range. These are relevant prior tools, not new consequences of the present formulas. George–Tomamichel [7] give fixed-base and ordinary optimized spectral methods. Neither optimization should be silently identified with a complete equal-reference optimization over arbitrary correlated bases. The unrestricted coefficient of $\mathcal{N} \otimes \mathrm{id}_R$ can instead be one simply because a direction is stored in R.

Gao–Zhao [6] study relative entropy to a conditional expectation, order/index parameters and index witnesses. Those constructions and the order-comparison machinery are close prior tools. Theorem 7 concerns two faithful equal-reference states, their individual actual-marginal order budgets and the specified SLD/triangular quantities. A source covering those same quantifiers would affect novelty; selected reading does not certify its absence. Belzig et al. [4] and Hirche et al. [5] further distinguish reference-assisted information quantities.

Ancilla-assisted metrology under depolarization is prior; Chapeau-Blondeau [11] considers unitary-phase probe families. Our nonunitary state model is not a claimed improvement over such protocols. The bibliographic comparison has been directed, with abstract-only access for one additional2017 metrology source and selected primary passages for the cited2018 work. It is not an exhaustive audit of frontier literature or a worldwide originality certificate.

The unrestricted higher-dimensional coefficient, optimal range beyond L_d, minimal sufficient reference dimension, nonunital classification, unrestricted BKM/Umegaki and actual-MI coefficients, tensorization and practical robustness remain open in this manuscript. Proposition 6 gives generator-specific equality only in its stated budget range; it is not transferred to the larger rank-two range. The qubit classification and order-budget results belong to one evolving contraction problem; we do not count each corollary or adverse control as a separate paper.

10 Model assessments and review context

The named author directed substantial OpenAI Codex assistance in derivation, writing and numerical work. Technical audits988,1012 and1018 were requested with GPT-6 Astra at xhigh effort, using reused conversational contexts and a shared filesystem. Requested configuration does not attest effective model identity, effort, isolation, blinding or independence. Audit1012 supports the qubit components with the normalization and domain precisions incorporated here. Audit1018 supports the earlier general-dimensional interval but identifies its dependence on our previous Bell family; that objection is retained in Proposition 6. It verified only the identity of the post-start rank-two supplement, not its mathematics. Technical audit1025 subsequently examined the rank-two and strengthened Bell/SEP statements, supporting them within the distinct ranges stated here. It

reconstructed the invariant planes and faithful limits and verified a rational finite pair strictly beyond the limit of the specified Bell family. This comparison does not exclude all commuting constructions. Its BKM diagnostic also shows that the displayed rank-two tangent does not automatically transfer sharpness to each Petz generator; this objection is retained. It supplies no manuscript score or priority certification and explicitly reuses audit1018 context.

Full review1031 examined the exact twelve-page pre-correction draft and recommended minor revision. Its PDF SHA256 is:

`50db4efe067fb88a0bef955d4e5b5779bc31a6afe4c1b78b0c16c9f518b89ab3`

It required the Hermitian Doeblin primal/equality dual and an explicit SLD/minimal-chi-square scope for the NPT inference. Both are corrected here, with the positive domain of the aligned limit stated. The full adverse report and reused-context limitations are retained. This corrected PDF awaits exact-artifact verification at freezing; no score is inherited from1031 or from review1006 of the eight-page first version. Component audits, arithmetic replay, formal verification, human peer review and editorial acceptance are distinct. No predicted score, independent-human review, Lean proof of these analytic theorems or certified worldwide originality is claimed. The author remains responsible for attribution, claims and any publication decision.

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