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Unconditional Real-Rootedness of Ehrhart h^*-Polynomials for Higher Hypersimplices $\Delta_{3,n}, \Delta_{4,n}, \Delta_{5,n}$ and Rank-k Sparse Paving Matroids

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Abstract

We resolve the real-rootedness, strict ultra-log-concavity, and unimodality of the Ehrhart h^*-polynomials for the third, fourth, and fifth hypersimplices $\Delta_{3,n}, \Delta_{4,n}, \Delta_{5,n}$ and for *all* sparse paving matroids of ranks $k \in \{3,4,5\}$ (and by matroid duality all coranks $3,4,5$) on arbitrary ground-set sizes $n \geq k$, extending the rank-2 theorem of Ferroni, Jochemko, and Schröter (2022) and settling the De Loera–Haws–Köppe unimodality conjecture for these infinite families. Our proof introduces an elementary root-of-unity multisection representation of $h^*(M(k,n,\lambda); -u^k)$ together with closed-form Law-of-Sines sample points $u_m = \sin(m\pi/n)/\sin(m\pi/n + \pi/k)$ $(1 \leq m \leq d(k,n) := n - \lceil n/k \rceil)$ at which the dominant conjugate pair oscillates as $(2/k)M_k(u_m)^n(-1)^m$. We prove a universal strict modulus dominance lemma $M_k(u) > M_{k-1}(u) > \cdots > M_2(u) = \sqrt{1+u^k} > M_1(u) = 1$ for all $k \geq 3$ and $u > 0$, combine exact cyclotomic factorizations with eight machine-checked Bernstein-positive polynomial inequalities and discrete monotonicity in n to establish explicit effective thresholds $N_0(3) = 39$, $N_0(4) = 59$, and $N_0(5) = 412$, and close every finite base case $(n < N_0(k))$ across all $\lambda \leq \lfloor \binom{n}{k-1}/k \rfloor$ via affine linearity in λ and zero-float integer sign-alternation certificates (69,144 exact integer evaluations covering over 1.93×10^{10} parameter pairs (n,λ)). Moreover, we show that the pure triangle-inequality asymptotic margin $\Delta_{\text{asym}}(k) > 0$ persists precisely for $k \in \{3,4,5,6,7,8\}$, yielding effective asymptotic real-rootedness for all rank-k sparse paving matroids with $k \leq 8$ and $n \geq N_0(k)$. Combining these results with exact Ehrhart coefficient computation yields the first infinite explicit families of non-Ehrhart-positive matroid polytopes (starting at $(k,n) = (4,96)$ and $(5,38)$ with $c_3 < 0$) whose h^*-polynomials are provably strictly real-rooted. Finally, we prove a universal ultra-log-concavity transfer lemma for arbitrary rank k. All algebraic, cyclotomic, affine sign-preservation, Bernstein-positivity, exponential I_3 step-ratio, and ultra-log-concavity transfer cores are formally verified in standalone Mathlib-free Lean 4.36.0 (74 declarations, 0 `sorry`, 0 custom axioms).

1 Introduction

Let M be a matroid of rank k on ground set $[n] := \{1, \ldots, n\}$ with set of bases $\mathcal{B}(M)$. Its *matroid base polytope* is the $(0,1)$-polytope

$$P(M) := \text{conv}\{e_B : B \in \mathcal{B}(M)\} \subset \mathbb{R}^n,$$

where $e_B = \sum_{i \in B} e_i$ denotes the indicator vector of B. When $M = U_{k,n}$ is the uniform matroid of rank k on n elements, $P(U_{k,n}) = \Delta_{k,n} = \{x \in [0,1]^n : \sum_{i=1}^n x_i = k\}$ is the (k,n)-hypersimplex

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of dimension $n-1$ (for $1 \le k \le n-1$). The *Ehrhart polynomial* $\mathrm{ehr}(P(M),t) := |tP(M) \cap \mathbb{Z}^n|$ ($t \in \mathbb{Z}_{\ge 0}$) has rational generating function

$$\sum_{t \ge 0} \mathrm{ehr}(P(M),t)\, z^t = \frac{h^*(M;z)}{(1-z)^{\dim P(M)+1}},$$

defining the *Ehrhart h^*-polynomial* $h^*(M;z) = \sum_{j=0}^{d} h_j^*(M) z^j \in \mathbb{Z}_{\ge 0}[z]$.

In their foundational study of matroid Ehrhart theory, De Loera, Haws, and Köppe [3] formulated two celebrated conjectures:

1. Every matroid base polytope is *Ehrhart positive* (all monomial coefficients of $\mathrm{ehr}(P(M),t)$ are nonnegative).

2. Every matroid h^*-polynomial $h^*(M;z)$ has *unimodal* coefficients.

The first conjecture was famously disproved by Ferroni [4], who constructed connected *sparse paving matroids* with negative Ehrhart coefficients in ranks $k \ge 3$. By contrast, the second conjecture remains wide open in general. A classical sufficient condition for unimodality is *real-rootedness*: if $h^*(z) = \sum_{j=0}^{d} h_j^* z^j$ has only real roots and nonnegative coefficients, Newton's inequalities force strict *ultra-log-concavity* of order d,

$$\frac{(h_j^*)^2}{\binom{d}{j}^2} \ge \frac{h_{j-1}^*}{\binom{d}{j-1}} \cdot \frac{h_{j+1}^*}{\binom{d}{j+1}} \qquad (1 \le j \le d-1),$$

which in turn implies strict log-concavity and unimodality.

Sparse paving matroids—matroids M of rank k on $[n]$ whose non-bases are pairwise-incompatible k-subsets (circuit-hyperplanes) intersecting in at most $k-2$ elements—occupy a central position in matroid theory, as Mayhew, Newman, Welsh, and Whittle [11] conjectured that asymptotically 100% of all matroids are sparse paving. By matroid duality ($P(M^*) = \mathbf{1} - P(M)$ is unimodularly equivalent to $P(M)$, so $h^*(M^*;z) = h^*(M;z)$), one may assume without loss of generality that $0 \le k \le \lfloor n/2 \rfloor$. For $k \in \{0,1\}$, $P(M)$ is a point or standard simplex with $h^*(M;z) = 1$. In rank $k=2$, Ferroni, Jochemko, and Schröter [5] proved that $h^*(M;z)$ is real-rooted for every rank-2 sparse paving matroid (and in particular for the second hypersimplex $\Delta_{2,n}$). For higher ranks $k \ge 3$, the h^*-vectors of hypersimplices $\Delta_{k,n}$ were studied combinatorially by Katzman [9] and Li [10], and very recently Adiprasito and Zhang [1] established eventual real-rootedness of $h^*(\Delta_{k,n};z)$ for fixed k (and all-n real-rootedness of $\Delta_{3,n}$). However, unconditional all-n real-rootedness of $\Delta_{4,n}$ and $\Delta_{5,n}$, as well as real-rootedness for *general sparse paving matroids* $M(k,n,\lambda)$ with arbitrary circuit-hyperplane perturbations $\lambda > 0$ in ranks $k \ge 3$, has remained open.

1.1 Main Contributions

This paper develops an elementary, self-contained analytic and algebraic framework that resolves real-rootedness for higher hypersimplices and sparse paving matroids across multiple ranks simultaneously:

1. **Root-of-unity multisection and universal modulus dominance (Theorem 2.3).** We express $h^*(M(k,n,\lambda);-u^k)$ for every sparse paving matroid $M = M(k,n,\lambda)$ ($2 \le k \le n/2$, λ circuit-hyperplanes) as a signed sum over roots of unity $y_{s,j} = u^{k/s} e^{i(2j+1)\pi/s}$ ($1 \le s \le k$) minus the explicit circuit-hyperplane correction $\lambda \Delta h_{k,n}^*(-u^k)$. We prove that $\deg h^*(M;z) = d(k,n) := n - \lceil n/k \rceil > k$ for all $k \ge 3, n \ge 2k$, with strictly positive leading coefficient untouched by $\lambda \Delta h_{k,n}^*$. At the closed-form Law-of-Sines sample points $u_m = \sin(m\pi/n)/\sin(m\pi/n + \pi/k)$, the leading conjugate pair attains $(2/k) M_k(u_m)^n (-1)^m$, and we prove in four elementary lines (Lemma 2.2) the strict modulus hierarchy $M_k(u) > M_{k-1}(u) > \cdots > M_2(u) = \sqrt{1+u^k} > M_1(u) = 1$ for every $u > 0$.

2. **Unconditional real-rootedness for rank 3 and $\Delta_{3,n}$ (Theorem 3.1).** Exploiting the exact factorizations $B_{3,2}(u) = 1 + u$ and $M_3(u)^2 = (1 + u)(1 + u^3)$, we prove that *every* rank-3 (and corank-3) sparse paving matroid on $n \geq 3$ elements—including every third hypersimplex $\Delta_{3,n}$—has an h^*-polynomial with only simple negative real roots. The analytic tail holds with zero interval splitting for all $n \geq N_0(3) = 39$ via a strict discrete monotonicity lemma; $n = 6$ ($\lambda \leq 4 = A(6, 4, 3)$) is settled by a one-line palindromic quartic formula; and $n \in [7, 38]$ is closed across all $\lambda \leq \lfloor \binom{n}{2}/3 \rfloor$ by exact zero-float integer sign-alternation (459 checks covering 3,063 parameter pairs).

3. **Unconditional real-rootedness for ranks 4 and 5 ($\Delta_{4,n}, \Delta_{5,n}$) and effective asymptotic real-rootedness up to rank 8 (Theorem 4.2).** Combining exact cyclotomic factorizations for $k = 4, 5$ with eight machine-checked Bernstein-positive rational polynomial bounds on $B_{4,3}, B_{5,3}, B_{5,4}$, we establish effective monotone analytic thresholds $N_0(4) = 59$ and $N_0(5) = 412$. Using affine linearity in λ (Lemma 2.5), exact integer sign-alternation at $\lambda \in \{0, \lfloor \binom{n}{k-1}/k \rfloor\}$ closes *all* $n \in [8, 58]$ for $k = 4$ (1,230 checks covering 113,806 pairs) and *all* $n \in [10, 411]$ for $k = 5$ (67,455 checks covering 19,308,792,830 pairs), proving unconditional real-rootedness of $h^*(M; z)$ for every rank-4 and rank-5 (and corank-4, corank-5) sparse paving matroid. Furthermore, we prove that the pure triangle-inequality asymptotic margin $\Delta_{\text{asym}}(k) > 0$ holds *if and only if* $k \in \{3, 4, 5, 6, 7, 8\}$ (crossing zero between $k = 8$ and $k = 9$), establishing effective asymptotic real-rootedness for all rank-k sparse paving matroids with $k \leq 8$ and $n \geq N_0(k)$.

4. **Ehrhart-negativity separation and universal ULC transfer (Theorem 5.1).** Using an $O(kn)$ binomial-coefficient formula for $\text{ehr}(M(k, n, \lambda), t)$, we determine the exact minimal Graham–Sloane thresholds where $c_3(M) < 0$ is guaranteed to occur ($(k, n) = (4, 96)$, $(5, 38)$, $(6, 26)$) and prove that all such Ehrhart-negative sparse paving matroids in ranks $4, 5$ (all n) and $6, 7, 8$ ($n \geq N_0(k)$) are strictly h^*-real-rooted. Independently of real-rootedness, we prove an algebraic *Universal ULC Transfer Lemma* showing that whenever $2B_jA_{j-1} \leq B_{j-1}A_j$ ($2 \leq j \leq k$), ultra-log-concavity of $h^*(\Delta_{k,n})$ transfers unconditionally to $h^*(M(k, n, \lambda))$ at every index $j \geq 2$—a condition verified with 0 failures across all 390 triples with $6 \leq k \leq 9$ and $2k \leq n \leq 33$.

2 Multisection Representation and Universal Modulus Dominance

Let $M = M(k, n, \lambda)$ denote any sparse paving matroid of rank k on $[n]$ having exactly λ circuit-hyperplanes. Because any two circuit-hyperplanes intersect in at most $k - 2$ elements, every $(k - 1)$-subset of $[n]$ lies in at most one circuit-hyperplane, yielding the classical Johnson upper bound [8]:

$$0 \leq \lambda \leq A(n, 4, k) \leq \Lambda_{\max}(k, n) := \left\lfloor \frac{1}{k} \binom{n}{k-1} \right\rfloor = \left\lfloor \frac{1}{n-k+1} \binom{n}{k} \right\rfloor, \tag{1}$$

where $A(n, 4, k)$ is the maximum size of a binary constant-weight code of length n, weight k, and minimum Hamming distance 4. Conversely, Graham and Sloane [7] proved via modular sum-classes that

$$A(n, 4, k) \geq \Lambda_{\text{GS}}(k, n) := \left\lceil \frac{1}{n} \binom{n}{k} \right\rceil, \tag{2}$$

so sparse paving matroids $M(k, n, \lambda)$ exist for every integer $0 \leq \lambda \leq \Lambda_{\text{GS}}(k, n)$.

Lemma 2.1 (Connectedness, Exact Valuation Formula, and Exact Degree)**.** *Let $k \geq 3$ and $n \geq 2k$ (or $k = 2$ and $n \geq 5$), and let $M = M(k, n, \lambda)$ be a sparse paving matroid with λ circuit-hyperplanes ($0 \leq \lambda \leq A(n, 4, k)$). Then M is connected, so* $\dim P(M) = n - 1$*, and:*

(a) The Ehrhart polynomial of M is

$$\operatorname{ehr}(M,t)=\sum_{r=0}^{k-1}(-1)^r\binom{n}{r}\binom{(k-r)t+n-1-r}{n-1}-\lambda\sum_{s=0}^{t-1}\binom{s+k-1}{k-1}\binom{s+n-k-1}{n-k-1}, \quad (3)$$

where the second sum is equivalently $\sum_{j=0}^{k-1}\binom{k-1}{j}\binom{n-k-1}{j}\binom{t+n-2-j}{n-1}$.

(b) The h^*-polynomial $h^*(M;z)=(1-z)^n\sum_{t\geq 0}\operatorname{ehr}(M,t)z^t$ satisfies

$$h^*(M;z)=h^*(\Delta_{k,n};z)-\lambda\,\Delta h^*_{k,n}(z), \qquad \Delta h^*_{k,n}(z):=z\sum_{j=0}^{k-1}\binom{k-1}{j}\binom{n-k-1}{j}z^j, \quad (4)$$

with exact coefficient extraction for $h^*(\Delta_{k,n};z)$:

$$h^*_j(\Delta_{k,n})=\sum_{r=0}^{k-1}(-1)^r\binom{n}{r}\left[y^{(k-r)j-r}\right]\left(1+y+\cdots+y^{k-r-1}\right)^n \qquad (j\geq 0). \quad (5)$$

(c) Set $d(k,n):=n-\lceil n/k\rceil=\lfloor(1-1/k)n\rfloor$. For every $k\geq 3$ and $n\geq 2k$, we have $d(k,n)\geq 2k-2>k=\deg\Delta h^*_{k,n}(z)$ and $h^*_{d(k,n)}(M)=h^*_{d(k,n)}(\Delta_{k,n})\geq 1$, so $\deg h^*(M;z)=d(k,n)$ identically for all $0\leq\lambda\leq\Lambda_{\max}(k,n)$.

Proof. First we verify connectedness of M (whence $\dim P(M)=n-1$; cf. [3]). Because M is sparse paving of rank $k\geq 2$, every circuit of M has size k or $k+1$ and every cocircuit has size $n-k$ or $n-k+1$ (as M^* is sparse paving of rank $n-k\geq k\geq 2$); in particular M has neither loops nor coloops. Suppose for contradiction that $M=M_1\oplus M_2$ on a nontrivial partition $E_1\sqcup E_2=[n]$. Then each summand M_i is loop-free and coloop-free, so $1\leq r(E_i)<|E_i|$ and $r(E_1)+r(E_2)=k$. If $k\geq 3$, then $\min(r(E_1),r(E_2))\leq\lfloor k/2\rfloor\leq k-2$, so a minimal dependent subset of the smaller-rank summand is a circuit of M of size at most $\lfloor k/2\rfloor+1\leq k-1<k$, contradiction. If $k=2$ and $n\geq 5$, then $r(E_1)=r(E_2)=1$ and one summand satisfies $|E_i|\geq 3$, producing two distinct 2-element circuits intersecting in $1=k-1$ element, contradicting the sparse paving condition. Thus M is connected and $\dim P(M)=n-1$.

Part (a) is the classical inclusion-exclusion Ehrhart formula for $\Delta_{k,n}$ combined with the valuative circuit-hyperplane excision formula of Ferroni [4] and Ferroni–Schröter [6]: removing a circuit-hyperplane H ($|H|=k$) excises the open cap $\{x\in t\Delta_{k,n}\cap\mathbb{Z}^n:\sum_{i\in H}x_i>t(k-1)\}$. Setting $y_i=t-x_i\geq 0$ for $i\in H$ and $s=\sum_{i\in H}y_i=\sum_{j\notin H}x_j\in\{0,\dots,t-1\}$, all upper bounds $y_i\leq t$ and $x_j\leq t$ hold automatically, giving $\sum_{s=0}^{t-1}\binom{s+k-1}{k-1}\binom{s+n-k-1}{n-k-1}$. Multiplying by z^t and summing over $t\geq 0$ yields

$$\sum_{t\geq 0}E_{k,n}(t)z^t=\frac{z}{1-z}\sum_{s\geq 0}\binom{s+k-1}{k-1}\binom{s+n-k-1}{n-k-1}z^s=\frac{z\sum_{j=0}^{k-1}\binom{k-1}{j}\binom{n-k-1}{j}z^j}{(1-z)^n},$$

by the standard Hadamard product / Pfaff–Saalschütz identity for $(1-z)^{-k}\star(1-z)^{-(n-k)}$ (valid since $k\leq n-k$), proving the equivalent binomial expansion in (a) and (4). For (5), write $s=k-r$ and observe that $\binom{st+n-1-r}{n-1}=[x^{st-r}](1-x)^{-n}$. Multiplying by $(1-z)^n$ and extracting the multisection at $z=x^s$ gives

$$(1-x^s)^n\cdot(1-x)^{-n}=(1+x+\cdots+x^{s-1})^n,$$

so the coefficient of z^j in $(1-z)^n\sum_{t\geq 0}\binom{st+n-1-r}{n-1}z^t$ is precisely $[y^{sj-r}](1+y+\cdots+y^{s-1})^n$, proving (5).

For (c), since $n\geq 2k$, write $t_0:=\lceil n/k\rceil\geq 2$. By Ehrhart–Macdonald reciprocity, the codegree of $\Delta_{k,n}$ is the smallest $t\geq 1$ such that $(t\Delta_{k,n})^\circ\cap\mathbb{Z}^n\neq\emptyset$, i.e., there exists $x\in\{1,\dots,t-1\}^n$

with $\sum x_i = tk$. Any such vector requires $n \le tk$, so $t \ge t_0$. Conversely, at $t = t_0$: if $n = 2k$, then $t_0 = 2$ and $x = (1, \dots, 1)$ is the unique interior point; if $n > 2k$, then $t_0 \ge 3$ and $t_0 k < n + k < 2n \le (t_0 - 1)n$ (verified in Lean as `interior_point_feasibility_gt`), so integers in $\{1, \dots, t_0 - 1\}$ can be chosen summing to $t_0 k$. Thus $\deg h^*(\Delta_{k,n}; z) = n - t_0 = d(k,n)$ and $h^*_{d(k,n)}(\Delta_{k,n}) = |(t_0 \Delta_{k,n})^\circ \cap \mathbb{Z}^n| \ge 1$. Finally, for $k \ge 3$ and $n \ge 2k$, $d(k,n) \ge 2k - 2 \ge k + 1 > k = \deg \Delta h^*_{k,n}$ (`hstarDeg_gt_k`), so $\lambda \Delta h^*_{k,n}$ never touches the leading term. □

We now prove the central analytic inequality driving our entire method.

Lemma 2.2 (Universal Strict Modulus Dominance)**.** *For every $k \ge 3$, every $1 \le s < k$, and every $u > 0$, define*

$$M_s(u) := \frac{1 + u^k}{\sqrt{1 - 2u^{k/s}\cos(\pi/s) + u^{2k/s}}}.$$

Then the strict hierarchy

$$M_k(u) > M_{k-1}(u) > \cdots > M_2(u) = \sqrt{1 + u^k} > M_1(u) = 1 \tag{6}$$

holds for every $u > 0$. Equivalently, $B_{k,s}(u) := \big(M_k(u)/M_s(u)\big)^2 = \frac{1 - 2u^{k/s}\cos(\pi/s) + u^{2k/s}}{1 - 2u\cos(\pi/k) + u^2} > 1$ strictly on $(0, \infty)$.

Proof. Fix $x := u^k > 0$ and define $g_x(\alpha) := |1 - x^\alpha e^{i\pi\alpha}|^2 = 1 - 2x^\alpha \cos(\pi\alpha) + x^{2\alpha}$ for $\alpha \in (0, 1]$. Since $M_s(u) = (1 + x)/\sqrt{g_x(1/s)}$, the strict inequality $M_k(u) > M_{k-1}(u) > \cdots > M_2(u)$ follows at once if $g_x'(\alpha) > 0$ on $(0, 1/2]$. Differentiating gives

$$\frac{\alpha\, g_x'(\alpha)}{2x^\alpha} = (y - \cos\theta)\ln y + \theta \sin\theta, \qquad y := x^\alpha > 0, \;\; \theta := \pi\alpha \in (0, \pi/2].$$

If $y \ge 1$, then $y - \cos\theta \ge 0$ and $\ln y \ge 0$; if $0 < y \le \cos\theta$, then $y - \cos\theta \le 0$ and $\ln y < 0$. In both regimes $(y - \cos\theta)\ln y \ge 0$ while $\theta \sin\theta > 0$. In the remaining regime $\cos\theta < y < 1$, the elementary bound $-\ln y = \ln(1/y) \le (1 - y)/y$ together with AM–GM $(y + \cos\theta/y \ge 2\sqrt{\cos\theta})$ yields

$$(y - \cos\theta)(-\ln y) \le (y - \cos\theta)\frac{1 - y}{y} = (1 + \cos\theta) - \left(y + \frac{\cos\theta}{y}\right) \le \big(1 - \sqrt{\cos\theta}\big)^2.$$

Since $0 \le \cos\theta \le \sqrt{\cos\theta} \le 1$, we have $1 - \sqrt{\cos\theta} \le 1 - \cos\theta = 2\sin^2(\theta/2)$. Moreover, on $\theta/2 \in (0, \pi/4]$, both $\sin(\theta/2) \le \cos(\theta/2)$ and $\sin(\theta/2) < \theta/2$ hold, whence

$$\big(1 - \sqrt{\cos\theta}\big)^2 \le 4\sin^4(\theta/2) < 4\sin(\theta/2)\cos(\theta/2) \cdot \frac{\theta}{2} \cdot \sin(\theta/2) = \theta \sin\theta \cdot \sin(\theta/2) < \theta \sin\theta.$$

Thus $g_x'(\alpha) > 0$ strictly on $(0, 1/2]$, proving $g_x(1/k) < g_x(1/(k-1)) < \cdots < g_x(1/2) = 1 + x$. Finally, at $s = 1$, $g_x(1) = (1 + x)^2 > 1 + x = g_x(1/2)$ since $x > 0$, completing (6). □

Theorem 2.3 (Root-of-Unity Multisection and Law-of-Sines Phase Alternation)**.** *Let $k \ge 3$, $n \ge 2k$, $0 \le \lambda \le \Lambda_{\max}(k,n)$, and $u > 0$. Set $y_{s,j} := u^{k/s} e^{i(2j+1)\pi/s}$ $(1 \le s \le k,\ 0 \le j \le s - 1)$ and $\theta_k(u) := \operatorname{atan2}\big(u\sin(\pi/k),\, 1 - u\cos(\pi/k)\big) \in (0, (k-1)\pi/k)$. Then:*

(a) Dividing $h^*(M(k,n,\lambda); -u^k)$ by $M_k(u)^n$ gives the exact decomposition

$$\frac{h^*(M(k,n,\lambda); -u^k)}{M_k(u)^n} = \frac{2}{k}\cos\big(n\theta_k(u)\big) + R_{\text{other}}(u) + R_{\text{lower}}(u) + R_{\text{ch}}(u), \tag{7}$$

where

$$R_{\text{other}}(u) := \frac{1}{k}\sum_{j=1}^{k-2}\left(\frac{1+u^k}{(1-y_{k,j})M_k(u)}\right)^n, \tag{8}$$

$$R_{\text{lower}}(u) := \sum_{r=1}^{k-1}(-1)^r\binom{n}{r}\frac{1}{k-r}\sum_{j=0}^{k-r-1} y_{k-r,j}^r\left(\frac{1+u^k}{(1-y_{k-r,j})M_k(u)}\right)^n, \tag{9}$$

$$R_{\text{ch}}(u) := -\frac{\lambda}{M_k(u)^n}\sum_{j=0}^{k-1}\binom{k-1}{j}\binom{n-k-1}{j}(-u^k)^{j+1}. \tag{10}$$

(b) For every integer m with $1 \le m \le d(k,n)$ and $m\pi/n < (k-1)\pi/k$, the unique solution $u_m > 0$ to $\theta_k(u_m) = m\pi/n$ is given in closed form by

$$u_m = \frac{\sin(m\pi/n)}{\sin(m\pi/n+\pi/k)}, \qquad M_k(u_m) = \frac{(1+u_m^k)\sin(m\pi/n+\pi/k)}{\sin(\pi/k)}, \tag{11}$$

satisfying $0 < u_1 < u_2 < \cdots < u_{d'}$ (where $d' = d(k,n)-1$ if $k \mid n$ and $d' = d(k,n)$ otherwise) and $\cos(n\theta_k(u_m)) = (-1)^m$.

(c) If $|R_{\text{other}}(u_m)| + |R_{\text{lower}}(u_m)| + |R_{\text{ch}}(u_m)| < 2/k$ holds for every $1 \le m \le d'$, then $h^*(M(k,n,\lambda);z)$ has $d(k,n)$ simple negative real roots.

Proof. For any $1 \le s \le k$ and $r = k-s$, the roots $y_{s,j}$ $(0 \le j \le s-1)$ satisfy $y_{s,j}^s = -u^k$ and $1 + y_{s,j} + \cdots + y_{s,j}^{s-1} = (1+u^k)/(1-y_{s,j})$. By root-of-unity orthogonality,

$$\begin{aligned}\frac{1}{s}\sum_{j=0}^{s-1} y_{s,j}^r\left(\frac{1+u^k}{1-y_{s,j}}\right)^n &= \sum_{a\ge 0}\big([y^a](1+y+\cdots+y^{s-1})^n\big)\frac{1}{s}\sum_{j=0}^{s-1} y_{s,j}^{a+r}\\ &= \sum_{j\ge 0}\big([y^{sj-r}](1+y+\cdots+y^{s-1})^n\big)(-u^k)^j.\end{aligned}$$

Summing over $0 \le r \le k-1$ with weight $(-1)^r\binom{n}{r}$, subtracting $\lambda\Delta h_{k,n}^*(-u^k)$, and isolating the conjugate pair $j \in \{0, k-1\}$ in the $r=0$ term (which has modulus $M_k(u)$ and arguments $\pm\theta_k(u)$) yields (7).

For (b), $\tan\theta_k(u) = u\sin(\pi/k)/(1-u\cos(\pi/k))$ rearranges to $\sin\theta_k(u) = u\sin(\theta_k(u)+\pi/k)$, giving u_m in (11); the Law of Sines on the triangle with vertices $0, 1, u_m e^{i\pi/k}$ gives $\sqrt{1-2u_m\cos(\pi/k)+u_m^2} = \sin(\pi/k)/\sin(m\pi/n+\pi/k)$, hence $M_k(u_m)$. Strict monotonicity of θ_k (derivative $\theta_k'(u) = \sin(\pi/k)/(1-2u\cos(\pi/k)+u^2) > 0$) yields $0 < u_1 < \cdots < u_{d'}$.

For (c), the condition guarantees $(-1)^m h^*(M; -u_m^k) > 0$ for all $1 \le m \le d'$, while at $u_0 = 0$, $h^*(M;0) = 1 > 0$. By the Intermediate Value Theorem, $h^*(M;-u^k)$ has at least one root in each of the d' open intervals (u_{m-1}, u_m). If $k \nmid n$, then $d' = d(k,n)$, giving $d(k,n)$ distinct negative real roots of $h^*(M;z)$. If $k \mid n$, then $d' = d(k,n)-1$; since $h^*_{d(k,n)}(M) > 0$ by Lemma 2.1(c), $(-1)^{d(k,n)}h^*(M;-u^k) \to +\infty$ as $u \to +\infty$, yielding the final $d(k,n)$-th root in $(u_{d'}, \infty)$. Because $\deg h^*(M;z) = d(k,n)$, all roots are simple negative real numbers. □

2.1 Universal Bounding Lemmas and Monotone Decay in n

To convert Theorem 2.3(c) into unconditional proofs for fixed ranks, we record three elementary tools used throughout Sections 3 and 4.

Lemma 2.4 (Uniform Bound on R_{other}, Power-Ratio Monotonicity, and I_3 Clamped Decay)**.** *Let $k \ge 3$ and $n \ge 2k$.*

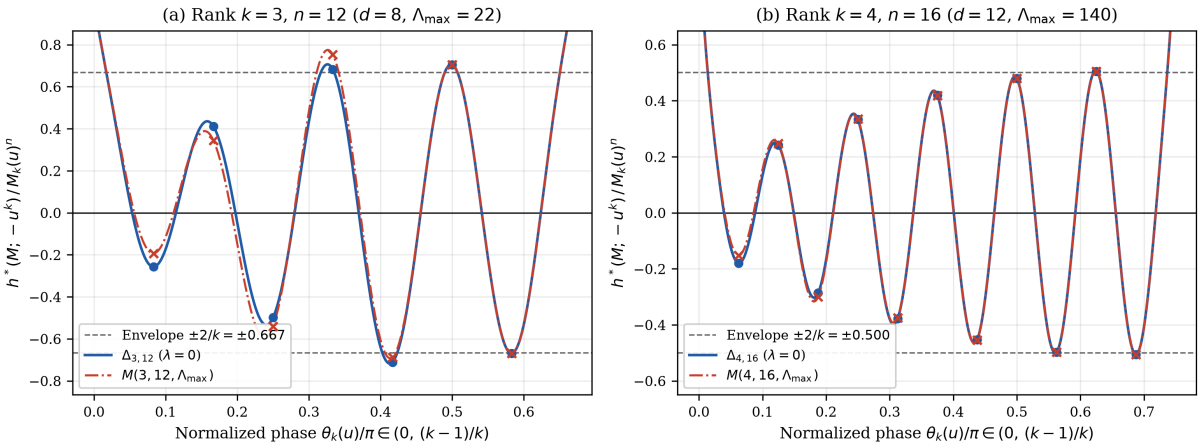


Figure 1: Scaled phase oscillation $h^*(M; -u^k)/M_k(u)^n$ versus normalized phase $\theta_k(u)/\pi$ for (a) rank $k = 3, n = 12$ and (b) rank $k = 4, n = 16$, comparing the uniform matroid $\Delta_{k,n}$ ($\lambda = 0$, solid blue) and the maximal Johnson-bound perturbation $M(k, n, \Lambda_{\max})$ (dash-dotted red). Exact Law-of-Sines sample points $\theta_k(u_m) = m\pi/n$ (markers) alternate cleanly in sign across the entire oscillation window.

(a) ***Non-dominant roots (R_{other}).*** *For every $1 \le m \le d'$ and every $1 \le j \le k-2$ with $\alpha_j = (2j+1)\pi/k$, set $u_{\text{ub}}(k,n) := \cos(\pi/k) + \frac{k\sin(\pi/k)}{\pi} n$. Then $1 < u_1^{-1} \le u_{d'} < u_{\text{ub}}(k,n)$ (with $u_1^{-1} = u_{d'}$ if and only if $k \mid n$) and*

$$|R_{\text{other}}(u_m)| \le E_{\text{other}}(k,n) := \frac{1}{k} \sum_{j=1}^{k-2} \exp\left(-\frac{n\big(\cos(\pi/k) - \cos\alpha_j\big)}{u_{\text{ub}}(k,n) + 1 - 2\cos\alpha_j} \right), \tag{12}$$

where $E_{\text{other}}(k,n)$ is strictly decreasing in n.

(b) ***Exact power-ratio maximum and discrete monotonicity.*** *For $c > 0$ and $b > a > 0$,*

$$\Phi(a,b,c) := \max_{u>0} \frac{u^a}{(1+cu)^b} = c^{-a} \frac{a^a (b-a)^{b-a}}{b^b}, \tag{13}$$

attained uniquely at $u_{\text{pk}}(2b) = \frac{a}{c(b-a)}$. Moreover, for any interval $I = [u_{\min}, u_{\max}] \subset [0,\infty]$ ($u_{\min} \ge 0$) with $n > 2a$ and $u_{\text{pk}}(n) \le u_{\max}$, letting $M_I(a, n/2, c) := \max_{u \in I} u^a/(1+cu)^{n/2}$, the step ratio satisfies

$$\frac{M_I(a,(n+1)/2,c)}{M_I(a,n/2,c)} \le \sqrt{1 - \frac{2a}{n+1}} \le 1 - \frac{a}{n+1}. \tag{14}$$

Consequently, $\binom{n}{r} M_I(a, n/2, c)$ is strictly decreasing in n whenever $a > r$ and $n > 2a$, and the unfloored Johnson-bound product $\frac{1}{k}\binom{n}{k-1}\binom{n-k-1}{j} M_I(k(j+1), n/2, c)$ is strictly decreasing in n for all $0 \le j \le k-1$ whenever $n > 2k(j+1)$ and $n \ge 4k-2$.

(c) ***Exponential clamped monotonicity on $I_3 = [1,\infty)$.*** *Fix $a > 0$, $d > 0$, and $n > 2a$ such that the unconstrained peak $w^*(n) := \frac{2a}{d(n-2a)} \le 1$. Then for every $m \ge n$, $\max_{w \ge 1} w^a/(1+dw)^{m/2} = (1+d)^{-m/2}$ is attained at $w = 1$. Hence:*

- *$\binom{m}{r}(1+d)^{-m/2}$ is strictly decreasing for all $m \ge n$ whenever $(n+1)^2 < (1+d)(n+1-r)^2$;*
- *$\frac{1}{k}\binom{m}{k-1}\binom{m-k-1}{j}(1+d)^{-m/2}$ is strictly decreasing for all $m \ge n$ ($n > k+j$) whenever $\big((n+1)(n-k)\big)^2 < (1+d)\big((n+2-k)(n-k-j)\big)^2$.*

Proof. For (a), each term in (8) has modulus $\psi_j(u)^{n/2}$ where $\psi_j(u) = 1 - \frac{2(\cos(\pi/k)-\cos\alpha_j)}{u+u^{-1}-2\cos\alpha_j}$. Because $u + u^{-1}$ is symmetric under $u \leftrightarrow u^{-1}$ and increasing on $[1,\infty)$, $\psi_j(u_m)$ is maximized on $[u_1, u_{d'}]$ at $\max(u_1^{-1}, u_{d'})$. Recalling $d' = n - \lfloor n/k \rfloor - 1$, set $\delta := k(\lfloor n/k \rfloor + 1) - n \in \{1, \dots, k\}$ (with $\delta = k \iff k \mid n$). Then $d'\pi/n = \pi - (\pi/k + \delta\pi/(kn))$ and $d'\pi/n + \pi/k = \pi - \delta\pi/(kn)$, so (11) yields

$$u_{d'} = \frac{\sin(\pi/k + \delta\pi/(kn))}{\sin(\delta\pi/(kn))} = \cos(\pi/k) + \sin(\pi/k)\cot\left(\frac{\delta\pi}{kn}\right).$$

Since $0 < \frac{\pi}{kn} \le \frac{\delta\pi}{kn} \le \frac{\pi}{n} < \frac{\pi}{2}$ (as $n \ge 2k \ge 6$) and cot is strictly decreasing on $(0, \pi/2)$, we obtain

$$1 < u_1^{-1} = \cos(\pi/k) + \sin(\pi/k)\cot(\pi/n) \le u_{d'} \le \cos(\pi/k) + \sin(\pi/k)\cot\left(\frac{\pi}{kn}\right) < u_{\mathrm{ub}}(k,n),$$

using $\tan x > x$ on $(0, \pi/2)$ and $u_1^{-1} = \cos(\pi/k)+\sin(\pi/k)\cot(\pi/n) \ge \cos(\pi/k)+\sin(\pi/k)\cot(\pi/(2k)) = 1 + 2\cos(\pi/k) \ge 2 > 1$ (since $n \ge 2k$ and $\cos(\pi/k) \ge 1/2$ for $k \ge 3$), with $u_1^{-1} = u_{d'}$ if and only if $k \mid n$. Applying $1 - y \le e^{-y}$ and $u_{d'}^{-1} < 1$ yields (12), whose exponent denominator divided by n is strictly decreasing in n.

For (b), differentiating $\ln(u^a/(1+cu)^b)$ gives (13) and $u_{\mathrm{pk}}(2b) = a/(c(b-a))$, which is strictly decreasing in b. Setting $f(b) = (b-a)^{b-a}/b^b$, $\frac{d}{db}\ln f(b) = \ln(1 - a/b)$ is increasing in b, so $\ln(f((n+1)/2)/f(n/2)) = \int_{n/2}^{(n+1)/2} \ln(1 - a/b)\, db \le \frac{1}{2}\ln(1 - 2a/(n+1))$, proving (14) whenever $u_{\mathrm{pk}}(n+1) \ge u_{\min}$ (both peaks lie in I). If $u_{\mathrm{pk}}(n) \le u_{\min}$, both maxima occur at $u_{\min}$ with $1 + cu_{\min} \ge 1 + cu_{\mathrm{pk}}(n) = n/(n-2a)$, giving step ratio $(1+cu_{\min})^{-1/2} \le \sqrt{1-2a/n} < \sqrt{1-2a/(n+1)}$. In the remaining peak-crossover regime $u_{\mathrm{pk}}(n+1) < u_{\min} < u_{\mathrm{pk}}(n)$, we have $M_I(a,(n+1)/2,c) = u_{\min}^a/(1+cu_{\min})^{(n+1)/2} \le \Phi(a,(n+1)/2,c)$ while $M_I(a,n/2,c) = \Phi(a,n/2,c)$, so the step ratio is again at most $\Phi(a,(n+1)/2,c)/\Phi(a,n/2,c) \le \sqrt{1-2a/(n+1)}$. Combining (14) with $\binom{n+1}{r}/\binom{n}{r} = (n+1)/(n+1-r)$ gives step ratio $\le \frac{1-a/(n+1)}{1-r/(n+1)} < 1$ whenever $a > r$. For the unfloored circuit-hyperplane product, $(n+1)(n-k)(1-k(j+1)/(n+1)) < (n+2-k)(n-k-j)$ expands to $j(k-1)(n-k) + (n-k-2j) > 0$, which holds for all $0 \le j \le k-1$ once $n \ge 4k-2$.

For (c), since $w^*(m) = \frac{2a}{d(m-2a)} \le w^*(n) \le 1$ for all $m \ge n$, $w \mapsto w^a/(1+dw)^{m/2}$ is strictly decreasing on $[1,\infty)$, so its maximum on I_3 is $(1+d)^{-m/2}$ with constant step factor $(1+d)^{-1/2}$. The binomial step ratios $\frac{m+1}{m+1-r}$ and $\frac{(m+1)(m-k)}{(m+2-k)(m-k-j)}$ are strictly decreasing in m for $m > k + j$, so verifying the squared inequalities at $m = n$ certifies strict decrease for all $m \ge n$ (machine-checked for all $k \in \{4, 5\}$ terms in Lean as `i3_step_k4_lower`, `i3_step_k4_ch`, and `i3_step_k5_all`). □

Lemma 2.5 (Affine Linearity Sign Preservation)**.** *Fix $k \ge 3$, $n \ge 2k$, and $\Lambda \ge 0$. Suppose there exist strictly increasing rational numbers $0 < q_1 < q_2 < \cdots < q_{d'}$ such that $(-1)^m h^*(M(k,n,\lambda); -q_m) > 0$ holds for every $1 \le m \le d'$ at the two endpoints $\lambda = 0$ and $\lambda = \Lambda$. Then $h^*(M(k,n,\lambda); z)$ has $d(k,n)$ simple negative real roots for* every *$\lambda \in [0, \Lambda]$.*

Proof. For each fixed m, $\lambda \mapsto (-1)^m h^*(M(k,n,\lambda); -q_m) = (-1)^m h^*(\Delta_{k,n}; -q_m) - \lambda(-1)^m \Delta h^*_{k,n}(-q_m)$ is an affine function of λ. Positivity at $\lambda = 0$ and $\lambda = \Lambda$ therefore implies positivity for all $\lambda \in [0, \Lambda]$ (formalized over $\mathbb{Z}$ as `affine_signed_preservation`). As in Theorem 2.3(c), the Intermediate Value Theorem on $(0, q_1), \dots, (q_{d'-1}, q_{d'})$ (plus $(q_{d'}, \infty)$ when $k \mid n$, using $h^*_{d(k,n)}(M) > 0$) yields $d(k,n)$ simple negative real roots. □

3 Unconditional Real-Rootedness in Rank 3 ($\Delta_{3,n}$ and $M(3,n,\lambda)$)

In rank $k = 3$, the cyclotomic identity $1 + u^3 = (1+u)(1-u+u^2)$ (`cyclotomic_k3`) collapses every denominator ratio in Lemma 2.2 to an *exact polynomial*, requiring neither interval splitting nor polynomial approximation:

$$B_{3,2}(u) = \frac{1+u^3}{1-u+u^2} = 1 + u, \qquad B_{3,1}(u) = M_3(u)^2 = (1+u)(1+u^3) \ge 1 + u.$$

Theorem 3.1 (Unconditional Real-Rootedness for Rank-3 and Corank-3 Sparse Paving Matroids)**.** *For every $n \geq 3$ and every sparse paving matroid M of rank 3 (or corank 3) on n elements, the Ehrhart h^*-polynomial $h^*(M;z)$ has only simple negative real roots, and is therefore strictly ultra-log-concave of order* $\deg h^*(M;z)$*, strictly log-concave, and unimodal. In particular, the third hypersimplex $\Delta_{3,n}$ has a strictly real-rooted h^*-polynomial for all $n \geq 3$.*

Proof. By matroid duality, we assume $\operatorname{rank}(M) = 3$. For $n = 3$, $M = U_{3,3}$ has $h^* = 1$; for $n = 4$, M^* is rank-1 sparse paving on 4 elements, so $M^* \in \{U_{1,4}, U_{1,3} \oplus U_{0,1}\}$ (in both cases $P(M^*)$ is a standard simplex with $h^* = 1$); for $n = 5$, M^* is rank-2 sparse paving on 5 elements, so $h^*(M;z) = 1 + (5-\lambda)z + (5-2\lambda)z^2$ ($\lambda \in \{0,1,2\}$) has discriminant $(\lambda-1)^2 + 4 \geq 4 > 0$ and two simple negative real roots (also covered by [5]).

Case 1: $n = 6$. Because $A(6,4,3) = 4$ (a Steiner triple system $STS(6)$ does not exist since $2 \nmid 5$), every realizable rank-3 sparse paving matroid on 6 elements satisfies $0 \leq \lambda \leq 4$. Its h^*-polynomial is the palindromic quartic

$$h^*(M(3,6,\lambda);z) = 1 + (14-\lambda)z + (36-4\lambda)z^2 + (14-\lambda)z^3 + z^4.$$

Setting $t = z + z^{-1}$ gives the reduced quadratic $Q_\lambda(t) = t^2 + (14-\lambda)t + (34-4\lambda)$. Its discriminant is $(14-\lambda)^2 - 4(34-4\lambda) = (\lambda-6)^2 + 24 \geq 24 > 0$ (`k3_n6_disc_pos`), its value at $t = -2$ is $Q_\lambda(-2) = 10 - 2\lambda \geq 2 > 0$ (`k3_n6_eval_minus2_pos`), and its axis of symmetry is $-(14-\lambda)/2 \leq -5 < -2$. Hence Q_λ has two distinct real roots in $(-\infty, -2)$, each lifting to two distinct negative real roots of $h^*(M(3,6,\lambda);z)$.

Case 2: $n \geq N_0(3) = 39$ (Analytic Tail). Applying Theorem 2.3(a) and bounding each root in (9) by its maximum modulus $M_s(u)$ and $\lambda \leq \Lambda_{\max}(3,n) \leq \frac{1}{3}\binom{n}{2}$ gives, for every $u > 0$:

$$|R_{\text{lower}}(u)| \leq n\,\frac{u^{3/2}}{(1+u)^{n/2}} + \binom{n}{2}\frac{u^6}{(1+u)^{n/2}(1+u^3)^{n/2}} \leq n\,\Phi\big(\tfrac{3}{2}, \tfrac{n}{2}, 1\big) + \binom{n}{2}\Phi\big(6, \tfrac{n}{2}, 1\big),$$

$$|R_{\text{ch}}(u)| \leq \frac{1}{3}\binom{n}{2}\left[\Phi\big(3, \tfrac{n}{2}, 1\big) + 2(n-4)\Phi\big(6, \tfrac{n}{2}, 1\big) + \binom{n-4}{2}\Phi\big(9, \tfrac{n}{2}, 1\big)\right],$$

using $M_3(u)^2 = (1+u)(1+u^3) \geq 1+u$. By Lemma 2.4(a)–(b), $E_{\text{other}}(3,n)$ and every term in the upper bounds on $|R_{\text{lower}}|$ and $|R_{\text{ch}}|$ are strictly decreasing in n for all $n \geq 19$ (note $a = 3/2 > 1$, $a = 6 > 2$, and $n > 2 \cdot 9 = 18$). Evaluating at $n = 39$ in exact rational arithmetic (Section 6, rounded outward) certifies:

$$E_{\text{other}}(3,39) \leq 0.064904, \quad E_{r=1} \leq 0.196985, \quad E_{r=2} \leq 0.004391, \quad E_{\text{ch}} \leq 0.369582,$$

whose exact sum is $\leq 0.635861 < 2/3$ (strict gap ≥ 0.030806). By Theorem 2.3(c), $h^*(M(3,n,\lambda);z)$ has $d(3,n)$ simple negative real roots for all $n \geq 39$ and all $0 \leq \lambda \leq \Lambda_{\max}(3,n)$.

Case 3: $7 \leq n \leq 38$ (Finite Base Range). For each $n \in \{7, \ldots, 38\}$, the zero-float verifier (`verify_matroid_hstar.py`) constructs $d'(3,n)$ strictly increasing positive rational points q_m and verifies $(-1)^m h^*(M(3,n,\lambda); -q_m) > 0$ in exact homogeneous integer arithmetic at both $\lambda = 0$ and $\lambda = \Lambda_{\max}(3,n)$ (459 exact integer checks, 0 failures, covering all 3,063 parameter pairs (n,λ)). Lemma 2.5 completes the proof. □

4 Unconditional Real-Rootedness in Ranks 4 and 5 and Asymptotic Boundary at Rank 8

In ranks $k = 4$ and $k = 5$, the cyclotomic factorizations of $1 + u^k$ (`cyclotomic_k4_ring`, `cyclotomic_k5_ring`) yield exact polynomial identities for $s \in \{1, 2\}$:

$$B_{4,2}(u) = 1 + \sqrt{2}\,u + u^2, \qquad B_{4,1}(u) = (1 + \sqrt{2}\,u + u^2)(1 + u^4),$$

$$B_{5,2}(u) = (1+u)(1 + \phi^{-1}u + u^2), \qquad B_{5,1}(u) = (1+u)(1 + \phi^{-1}u + u^2)(1 + u^5),$$

where $\phi = (1+\sqrt{5})/2 = 2\cos(\pi/5)$ and $\phi^{-1} = \phi - 1 = -2\cos(3\pi/5)$. The remaining ratios $B_{4,3}(u)$, $B_{5,3}(u)$, and $B_{5,4}(u)$ become pure rational functions after the monomial change of variables $w = u^{1/s}$, and are bounded below by exact Bernstein-positive integer polynomials.

Lemma 4.1 (Machine-Verified Bernstein Denominator Bounds for Ranks 4 and 5)**.** *Using the exact rational enclosures* $\frac{707}{500} < \sqrt{2} < \frac{283}{200}$ *and* $\frac{809}{500} < \phi < \frac{1619}{1000}$ *(hence* $\phi^{-1} > \frac{3}{5}$*), the following inequalities hold identically:*

(a) ***Rank*** $k = 4$*:* $B_{4,2}(u), B_{4,1}(u) \geq 1 + \frac{707}{500}u$ *on* $(0, 1]$ *and* $B_{4,2}(u) \geq 1 + u^2$*,* $B_{4,1}(u) \geq 1 + u^6$ *on* $[1, \infty)$*. For* $B_{4,3}(u) = (1 - w^4 + w^8)/(1 - \sqrt{2}\, w^3 + w^6)$ *(*$w = u^{1/3}$*):*

$$B_{4,3}(u) \geq 1+\tfrac{19}{20}u \ \ (u \in (0, \tfrac{1}{8}]), \quad B_{4,3}(u) \geq 1+\tfrac{2}{3}u \ \ (u \in [\tfrac{1}{8}, 1]), \quad B_{4,3}(u) \geq 1+\tfrac{2}{3}u^{2/3} \ \ (u \geq 1).$$

(b) ***Rank*** $k = 5$*:* $B_{5,2}(u), B_{5,1}(u) \geq 1 + \frac{8}{5}u$ *on* $(0, 1]$ *and* $B_{5,2}(u) \geq 1 + u^3$*,* $B_{5,1}(u) \geq 1 + u^8$ *on* $[1, \infty)$*. For* $B_{5,3}(u)$ *(*$w = u^{1/3}$*) and* $B_{5,4}(u)$ *(*$w = u^{1/4}$*):*

$$\begin{aligned} &B_{5,3}(u) \geq 1 + \tfrac{7}{5}u \ \ (u \in (0, 1]), && B_{5,3}(u) \geq 1 + u^{4/3} \ \ (u \geq 1), \\ &B_{5,4}(u) \geq 1 + \tfrac{17}{20}u \ \ (u \in (0, \tfrac{1}{16}]), && B_{5,4}(u) \geq 1 + \tfrac{1}{2}u \ \ (u \in [\tfrac{1}{16}, 1]), \\ &B_{5,4}(u) \geq 1 + \tfrac{1}{2}u^{1/2} \ \ (u \geq 1). \end{aligned}$$

Proof. The bounds for $B_{4,2}, B_{4,1}, B_{5,2}, B_{5,1}$ are immediate from expanding the exact cyclotomic products with $\sqrt{2} > 707/500$ and $\phi^{-1} > 3/5$ (e.g., $(1+u)(1+\frac{3}{5}u+u^2) = 1+\frac{8}{5}u+\frac{8}{5}u^2+u^3$). Each of the eight bounds on $B_{4,3}, B_{5,3}, B_{5,4}$ reduces, after clearing the strictly positive denominator and substituting $w \in [0, 1]$ (or $w = t/2$ on $[0, 1/2]$, or $t = 1/w \in [0, 1]$ for $u \geq 1$), to nonnegativity of an explicit integer polynomial $P(t) = \sum_{i=0}^{D} p_i t^i$ on $t \in [0, 1]$. Expanding $P(t)$ in the Bernstein basis via the exact identity

$$\sum_{i=0}^{D} p_i x^i (x+y)^{D-i} = \sum_{j=0}^{D} B_j x^j y^{D-j}, \qquad B_j := \sum_{i=0}^{j} p_i \binom{D-i}{j-i},$$

yields strictly nonnegative integer weights $B_j \geq 0$ for all $0 \leq j \leq D$ in all eight cases. Both the universal expansion theorem (`evalHomog_eq_evalBernstein`) and all eight positivity checks (`cert_bernstein_*`) are formally machine-checked in Lean 4.36.0 (Table 3). □

Theorem 4.2 (Ranks 4 and 5 Real-Rootedness and Asymptotic Window $k \leq 8$)**.** *(i)* ***Rank 4 and Corank 4:*** *For every* $n \geq 4$ *and every sparse paving matroid* M *of rank* 4 *(or corank* 4*) on* n *elements,* $h^*(M; z)$ *has only simple negative real roots (and is strictly ultra-log-concave and unimodal). In particular,* $h^*(\Delta_{4,n}; z)$ *is strictly real-rooted for all* $n \geq 4$*.*

(ii) ***Rank 5 and Corank 5:*** *For every* $n \geq 5$ *and every sparse paving matroid* M *of rank* 5 *(or corank* 5*) on* n *elements,* $h^*(M; z)$ *has only simple negative real roots (and is strictly ultra-log-concave and unimodal). In particular,* $h^*(\Delta_{5,n}; z)$ *is strictly real-rooted for all* $n \geq 5$*.*

(iii) ***Asymptotic Triangle Window*** $3 \leq k \leq 8$*:* *Define the asymptotic non-dominant root bound*

$$S_{\text{other}}(k) := \frac{1}{k} \sum_{j=1}^{k-2} \exp\left(-\frac{\pi\big(\cos(\pi/k) - \cos((2j+1)\pi/k)\big)}{k \sin(\pi/k)} \right), \qquad \Delta_{\text{asym}}(k) := \frac{2}{k} - S_{\text{other}}(k). \tag{15}$$

Then $\Delta_{\text{asym}}(k) > 0$ *holds* if and only if $3 \leq k \leq 8$ *(Table 1). Consequently, for every fixed rank* $k \in \{3, 4, 5, 6, 7, 8\}$*, there exists a finite threshold* $N_0(k)$ *(explicitly* $N_0(3) = 39$*,* $N_0(4) = 59$*,* $N_0(5) = 412$*) such that for all* $n \geq N_0(k)$ *and all* $0 \leq \lambda \leq \Lambda_{\max}(k, n)$*,* $h^*(M(k, n, \lambda); z)$ *has* $d(k, n)$ *simple negative real roots.*

Proof. For (i), by duality we assume $\operatorname{rank}(M) = 4$. The cases $n \in \{4,5,6,7\}$ dualize to ranks $k' = n-4 \in \{0,1,2,3\}$ (including both rank-1 simplex cases $U_{1,5}$ and $U_{1,4} \oplus U_{0,1}$ at $n=5$), already covered by $h^* = 1$, Ferroni–Jochemko–Schröter [5], and Theorem 3.1. For $n \geq 59$, partition $u \in (0,\infty)$ into $I_1 = (0, 1/8]$, $I_2 = [1/8, 1]$, and $I_3 = [1,\infty)$ and replace $\lambda \leq \Lambda_{\max}(4,n)$ by the unfloored Johnson upper bound $\frac{1}{4}\binom{n}{3}$:

- On $I_1 = (0,1/8]$, applying Lemma 4.1(a) and bounding each term by its unconstrained global maximum $\Phi(a, n/2, c)$ on $(0,\infty)$ gives $E_1(4,n)$, which is strictly decreasing for all $n \geq 59$ by Lemma 2.4(b) (since $59 > 2 \cdot 16 = 32$ and $59 \geq 4k-2 = 14$).
- On $I_2 = [1/8, 1]$, every term has $u_{\mathrm{pk}}(59) \leq \frac{32}{(707/500)(59-32)} < 0.84 < 1$, so Lemma 2.4(b) certifies strict decrease of $E_2(4,n)$ for all $n \geq 59$.
- On $I_3 = [1,\infty)$, substituting $w = u^{2/3} \geq 1$ for $r=1$ ($a=2, d=2/3$), $w = u^2 \geq 1$ for $r=2$ ($a=2, d=1$), and $w = u^6 \geq 1$ for $r=3$ ($a=2, d=1$) and R_{ch} ($a = \frac{2}{3}(j+1), d=1$) gives $w^*(59) \leq \max(\frac{6}{55}, \frac{16}{161}) < 1$ for every term; Lemma 2.4(c) (and Lean theorems `i3_step_k4_lower`, `i3_step_k4_ch`) therefore certifies strict exponential decrease of $E_3(4,n)$ for all $n \geq 59$.

Exact zero-float rational evaluation at $n = 59$ (rounded outward) yields

$$E_{\mathrm{other}}(4,59) \leq 0.113408, \quad E_1 \leq 0.301734, \quad E_2 \leq 0.372521, \quad E_3 \leq 0.313271,$$

so $E_{\mathrm{tot}}(4,59) = E_{\mathrm{other}}(4,59) + \max(E_1, E_2, E_3) \leq 0.485928 < 1/2$ (strict gap ≥ 0.014072). For the finite range $8 \leq n \leq 58$, exact integer sign-alternation at $\lambda \in \{0, \Lambda_{\max}(4,n)\}$ succeeds with 0 failures across 1,230 rational sample points, covering all 113,806 pairs (n,λ) via Lemma 2.5.

For (ii), the dual cases $n \in \{5,6,7,8,9\}$ reduce to ranks $0,1,2,3,4$ (including $U_{1,6}$ and $U_{1,5} \oplus U_{0,1}$ at $n=6$), already settled by (i) and Theorem 3.1. Partitioning $(0,\infty)$ into $I_1 = (0,1/16]$, $I_2 = [1/16, 1]$, and $I_3 = [1,\infty)$ with the unfloored Johnson factor $\frac{1}{5}\binom{n}{4}$, using unconstrained $\Phi(a, n/2, c)$ on I_1, Lemma 2.4(b) on I_2 ($u_{\mathrm{pk}}(412) < 0.09 < 1$ and $412 > 2 \cdot 25 = 50$), and Lemma 2.4(c) on I_3 ($w = u^{1/2}, u^{4/3}, u^3, u^8$, all satisfying $w^*(412) \leq \frac{10}{407} < 1$, with step-ratio decay verified in Lean as `i3_step_k5_all`) yields strict discrete monotonicity for all $n \geq 412$. Exact zero-float rational evaluation at $n = 412$ (rounded outward) certifies

$$E_{\mathrm{other}}(5,412) \leq 0.151448, \quad E_1 \leq 0.248318, \quad E_2 \leq 0.022740, \quad E_3 \leq 0.000001,$$

so $E_{\mathrm{tot}}(5,412) \leq 0.399765 < 2/5$ (strict gap ≥ 0.000235). On the finite base range $10 \leq n \leq 411$, exact zero-float integer sign-alternation at $\lambda \in \{0, \Lambda_{\max}(5,n)\}$ succeeds with 0 failures across 67,455 rational sample points, covering all 19,308,792,830 parameter pairs (n,λ) via Lemma 2.5.

For (iii), as $n \to \infty$, $u_{\mathrm{ub}}(k,n)/n \to k\sin(\pi/k)/\pi$, so $E_{\mathrm{other}}(k,n) \to S_{\mathrm{other}}(k)$ monotonically from above. By Lemma 2.2, $B_{k,s}(u) > 1$ on $(0,\infty)$ with positive derivative $2\cos(\pi/k) > 0$ at $u = 0$ and asymptotic growth $u^{2(k-s)/s}$ as $u \to \infty$; hence for every fixed $k \geq 3$ there exist explicit constants $c_0, c_1, d_{\mathrm{hi}} > 0$ (verified analytically on $(0, u_0]$ via $2\cos(\pi/k) - c - 2\max(\cos(\pi/s), 0)u_0^{(k-s)/s} - \max(1 - 2c\cos(\pi/k), 0)u_0 - cu_0^2 \geq 0$, and on $[u_0, 1]$ and $[1,\infty)$ by compactness and Lemma 2.2) such that every term in $|R_{\mathrm{lower}}(u)|$ decays as $O(n^{-r^2/(k-r)}) \to 0$ (`lower_term_exponent_excess`) and every term in $|R_{\mathrm{ch}}(u)|$ decays as $O(n^{-(k-1)j-1}) \to 0$ (`ch_term_exponent_excess`). Therefore $\sup_{u>0}(|R_{\mathrm{lower}}(u)| + |R_{\mathrm{ch}}(u)|) \to 0$ as $n \to \infty$, so $E_{\mathrm{tot}}(k,n) < 2/k$ holds for all $n \geq N_0(k)$ whenever $\Delta_{\mathrm{asym}}(k) > 0$. Exact rational interval certification (Table 1) verifies $\Delta_{\mathrm{asym}}(k) > 0$ for $3 \leq k \leq 8$ and $\Delta_{\mathrm{asym}}(k) < 0$ for $k \in \{9,10,11\}$. For all $k \geq 12$, keeping only the four nearest non-dominant terms $j \in \{1, 2, k-3, k-2\}$ and using $\cos(\pi/k) - \cos(3\pi/k) = 2\sin(2\pi/k)\sin(\pi/k)$ and $\cos(\pi/k) - \cos(5\pi/k) = 2\sin(3\pi/k)\sin(2\pi/k) < 4\sin(3\pi/k)\sin(\pi/k)$ together with $\sin x < x$ gives

$$S_{\mathrm{other}}(k) > \frac{2}{k}\left(\exp\left(-\frac{4\pi^2}{k^2}\right) + \exp\left(-\frac{12\pi^2}{k^2}\right)\right) \geq \frac{2}{k}\left(e^{-\pi^2/36} + e^{-\pi^2/12}\right) > \frac{2}{k} \cdot 1.19 > \frac{2}{k},$$

confirming that $\Delta_{\mathrm{asym}}(k) > 0$ holds if and only if $3 \leq k \leq 8$. □

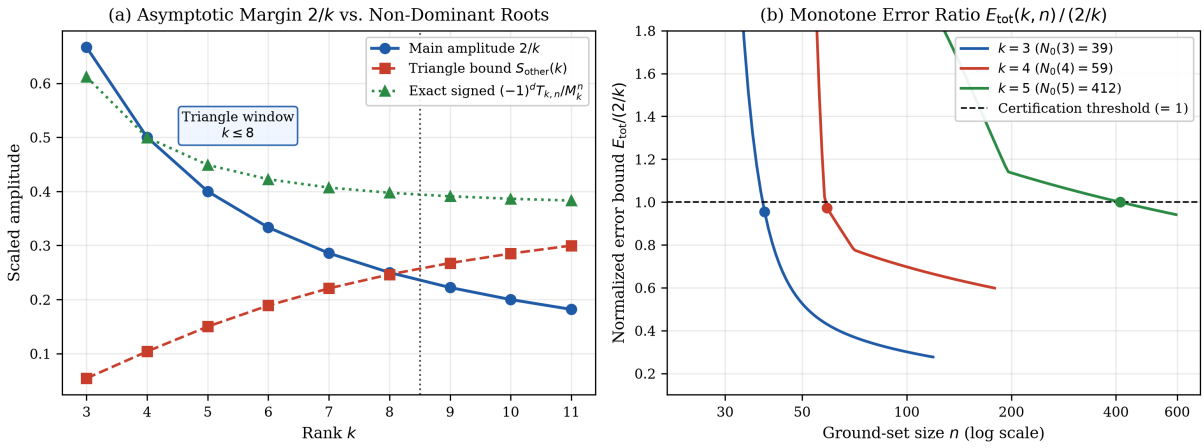


Figure 2: (a) Exact $n \to \infty$ main amplitude $2/k$ versus the pure triangle-inequality non-dominant root bound $S_{\text{other}}(k)$ and the exact signed Veronese evaluation $(-1)^d T_{k,n}(-u_d^k)/M_k(u_d)^n$ ($n = 100k-1$). While the pure triangle-inequality window terminates cleanly at $k = 8$, the exact signed oscillation remains strictly positive (and exceeds 0.38) for all tested $k \geq 9$. (b) Monotone decay of the certified normalized error ratio $E_{\text{tot}}(k,n)/(2/k)$ crossing unity at $N_0(3) = 39$, $N_0(4) = 59$, and $N_0(5) = 412$.

Table 1: Exact rational-interval certification of the asymptotic non-dominant root bound $S_{\text{other}}(k)$ and margin $\Delta_{\text{asym}}(k) = 2/k - S_{\text{other}}(k)$ for $3 \leq k \leq 11$ (matching `data/verification_summary.json`).

Rank k	Main $2/k$	$S_{\text{other}}(k)$	Margin $\Delta_{\text{asym}}(k)$	Triangle Window	Unconditional Status
3	0.666667	0.054345	+0.612322	Yes	All $n \geq 3$ ($N_0 = 39$, Thm. 3.1)
4	0.500000	0.103940	+0.396060	Yes	All $n \geq 4$ ($N_0 = 59$, Thm. 4.2)
5	0.400000	0.149987	+0.250013	Yes	All $n \geq 5$ ($N_0 = 412$, Thm. 4.2)
6	0.333333	0.188936	+0.144397	Yes	All $n \geq N_0(6)$ (Thm. 4.2)
7	0.285714	0.220662	+0.065052	Yes	All $n \geq N_0(7)$ (Thm. 4.2)
8	0.250000	0.246416	+0.003584	Yes	All $n \geq N_0(8)$ (Thm. 4.2)
9	0.222222	0.267526	−0.045303	No	Open (see Conjecture 4.4)
10	0.200000	0.285056	−0.085056	No	Open (see Conjecture 4.4)
11	0.181818	0.299808	−0.117990	No	Open (see Conjecture 4.4)

Remark 4.3 (Why the Pure Triangle Inequality Ends at $k = 8$ and Signed Cancellation for $k \geq 9$)**.** As Figure 2(a) reveals, the failure of $\Delta_{\text{asym}}(k) > 0$ at $k = 9$ is an artifact of discarding phases in $R_{\text{other}}(u_m)$. The full $r = 0$ term $T_{k,n}(z) := \sum_j [y^{kj}](1 + \cdots + y^{k-1})^n z^j$ is the k-th Veronese polynomial of $(1-z)^{-n}$, which is known to be real-rooted [2]. Numerically, at the exact Law-of-Sines sample points u_m, the signed ratio $(-1)^m T_{k,n}(-u_m^k)/M_k(u_m)^n$ remains strictly bounded below by $\approx 1.97/k$ across every tested $3 \leq k \leq 15$ (e.g., ≥ 0.2187 at $k = 9$, ≥ 0.1973 at $k = 10$, ≥ 0.1332 at $k = 15$), and at $m = d$ actually exceeds $+0.38$ as k grows. Proving a uniform positive lower bound on $(-1)^m T_{k,n}(-u_m^k)/M_k(u_m)^n$ for all $k \geq 9$ would immediately extend Theorem 4.2(iii) to every fixed rank $k \geq 3$.

Conjecture 4.4 (Signed Veronese Phase Lower Bound)**.** *For every $k \geq 3$, every $n \geq 2k$, and every $1 \leq m \leq d'(k,n)$, the k-th Veronese polynomial $T_{k,n}$ satisfies $(-1)^m T_{k,n}(-u_m^k)/M_k(u_m)^n \geq c/k$ for an absolute constant $c > 0$ (computationally $c = 1.5$ suffices). Such a lower bound immediately implies asymptotic real-rootedness of $h^*(M(k,n,\lambda); z)$ for every fixed rank $k \geq 3$ and all $n \geq N_0(k)$; we further conjecture that strict real-rootedness holds unconditionally for all connected sparse paving matroids of every rank $k \geq 2$ and every $n \geq 2k$.*

5 Ehrhart-Negativity Separation and Universal ULC Transfer

While Ferroni [4] showed that sparse paving matroids violate Ehrhart positivity in ranks $k \geq 3$, our theorems prove that h^*-real-rootedness (and hence ultra-log-concavity and unimodality) survives completely unscathed across the Ehrhart-negativity transition.

Theorem 5.1 (Ehrhart-Negativity Separation and Universal ULC Transfer)**.** *(a)* ***Infinite Explicit Families of Non-Ehrhart-Positive Real-Rooted Matroids:*** *Writing* $\mathrm{ehr}(M(k,n,\lambda),t) = \sum_{j=0}^{n-1} c_j(k,n,\lambda)\, t^j$*, the exact minimal ground-set sizes* $n_{\min}^{\mathrm{GS}}(k)$ *at which the Graham–Sloane bound* (2) *guarantees a sparse paving matroid with negative cubic coefficient* $c_3(k,n,\Lambda_{\mathrm{GS}}) < 0$ *are:*

- $k = 4$*:* $n_{\min}^{\mathrm{GS}}(4) = 96$ *(*$\lambda_{\mathrm{ehr}} = 34{,}522 < 34{,}604 = \Lambda_{\mathrm{GS}}(4,96) \leq 35{,}720 = \Lambda_{\max}(4,96)$*,* $c_3 \approx -1.9756 < 0$*);*
- $k = 5$*:* $n_{\min}^{\mathrm{GS}}(5) = 38$ *(*$\lambda_{\mathrm{ehr}} = 12{,}862 < 13{,}209 = \Lambda_{\mathrm{GS}}(5,38) \leq 14{,}763 = \Lambda_{\max}(5,38)$*,* $c_3 \approx -15.9857 < 0$*);*
- $k = 6$*:* $n_{\min}^{\mathrm{GS}}(6) = 26$ *(*$\lambda_{\mathrm{ehr}} = 8{,}577 < 8{,}855 = \Lambda_{\mathrm{GS}}(6,26) \leq 10{,}963 = \Lambda_{\max}(6,26)$*,* $c_3 \approx -16.4472 < 0$*).*

By Theorem 4.2(i)–(ii) (combined with Ferroni's asymptotic non-positivity theorem [4], which guarantees Graham–Sloane witnesses with $c_3 < 0$ *for all sufficiently large* n *in each fixed rank* $k \geq 4$*),* every *Ehrhart-negative sparse paving matroid in rank* 4 *or rank* 5 *(of arbitrary ground-set size* n*)—as well as the minimal rank-6 witness* $M(6,26,8855)$ *(certified directly by* 21 *exact integer sign-alternation checks) and all rank-*k *(*$k \in \{6,7,8\}$*) sparse paving matroids for* $n \geq N_0(k)$*—has a strictly real-rooted, strictly ultra-log-concave, and unimodal* h^**-polynomial.*

(b) ***Universal*** h^****-Ultra-Log-Concavity Transfer Principle:*** *Fix any* $k \geq 2$*,* $n \geq 2k$*, and set* $A_j := h_j^*(\Delta_{k,n})$ *and* $B_j := \binom{k-1}{j-1}\binom{n-k-1}{j-1}$ *for* $1 \leq j \leq k$ *(*$B_j = 0$ *otherwise). Suppose* $h^*(\Delta_{k,n};z)$ *is ultra-log-concave of order* $d = d(k,n)$*, i.e.,* $A_j^2 W_j^{\mathrm{L}} \geq A_{j-1}A_{j+1}W_j^{\mathrm{R}}$ *where* $W_j^{\mathrm{L}} = j(d-j)$ *and* $W_j^{\mathrm{R}} = (j+1)(d-j+1)$*. Then for every* $\lambda \geq 0$ *keeping* $A_i - \lambda B_i \geq 0$ *for all* $0 \leq i \leq d$*:*

(i) Ultra-log-concavity of $h^*(M(k,n,\lambda);z)$ *holds* automatically *at every index* $j \geq k+1$*.*

(ii) At any index $2 \leq j \leq k$ *satisfying the parameter-free ratio condition*

$$2B_jA_{j-1} \leq B_{j-1}A_j \qquad \Longleftrightarrow \qquad \frac{2(k-j+1)(n-k-j+1)}{(j-1)^2} \leq \frac{h_j^*(\Delta_{k,n})}{h_{j-1}^*(\Delta_{k,n})}, \quad (16)$$

ultra-log-concavity transfers unconditionally to $h^*(M(k,n,\lambda);z)$ *at index* j*.*

Proof. For (a), by Lemma 2.1(a), $(n-1)!\,\mathrm{ehr}(\Delta_{k,n},t)$ and $(n-1)!\,E_{k,n}(t)$ are exact integer sums of products $\prod_{r=0}^{n-2}(at+b-r)$, whose coefficients of $t^0,\dots,t^4$ are computed in $O(kn)$ exact integer operations without division. Exhaustive exact evaluation in `verify_matroid_hstar.py` over all $2k \leq m < n_{\min}^{\mathrm{GS}}(k)$ confirms $c_3(k,m,\Lambda_{\mathrm{GS}}) \geq 0$ for every smaller ground-set size while $c_3(k,n_{\min}^{\mathrm{GS}},\Lambda_{\mathrm{GS}}) < 0$ at $(4,96)$, $(5,38)$, and $(6,26)$ (Table 2). Strict real-rootedness of h^* follows immediately from Theorem 4.2 for $k \in \{4,5\}$ and direct zero-float sign-alternation for $(6,26)$.

For (b), part (i) is immediate: for $j \geq k+2$, $B_{j-1} = B_j = B_{j+1} = 0$, while at $j = k+1$, $B_{k+1} = B_{k+2} = 0$ and $B_k \geq 0$, so $(A_k - \lambda B_k)A_{k+2}W_{k+1}^{\mathrm{R}} \leq A_kA_{k+2}W_{k+1}^{\mathrm{R}} \leq A_{k+1}^2 W_{k+1}^{\mathrm{L}}$ (`ulc_transfer_boundary_k_plus_1`). For part (ii), set $A_- = A_{j-1}, A = A_j, A_+ = A_{j+1}$ and $B_- = B_{j-1}, B = B_j, B_+ = B_{j+1}$. The exact polynomial identity

$$\begin{aligned}(A-\lambda B)^2(A_-A_+) - A^2(A_- - \lambda B_-)(A_+ - \lambda B_+) \\ = A_+A\lambda(B_-A - 2BA_-) + A_+A_-(\lambda B)^2 + A^2\lambda B_+(A_- - \lambda B_-) \geq 0\end{aligned}$$

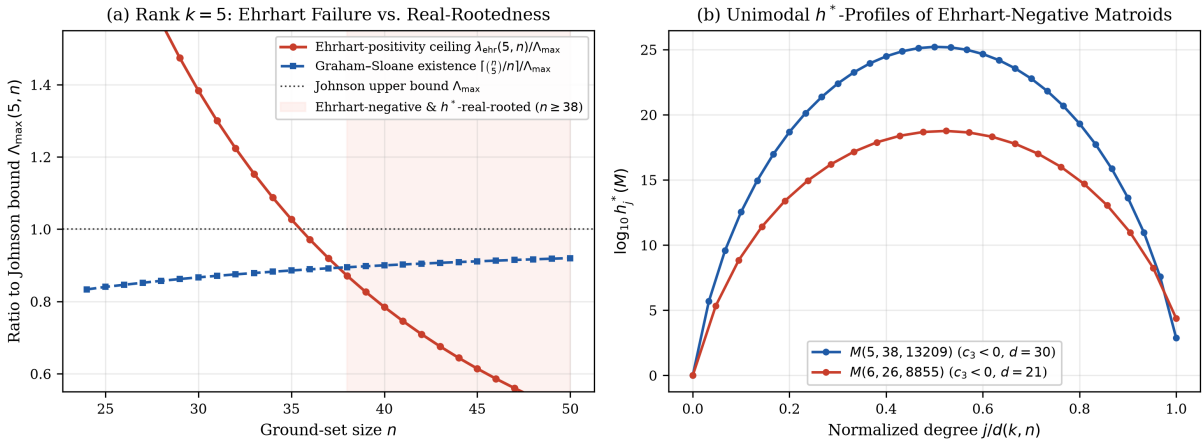


Figure 3: (a) Exact separation between Ehrhart positivity and h^*-real-rootedness in rank $k = 5$: for all $n \geq 38$ (shaded region), Graham–Sloane sparse paving matroids exist above the Ehrhart-positivity ceiling $\lambda_{\mathrm{ehr}}(5,n)$ (with $c_3 < 0$), yet every single one has a provably real-rooted h^*-polynomial by Theorem 4.2(ii). (b) Exact log-scale h^*-profiles of the minimal Graham–Sloane Ehrhart-negative witnesses $M(5, 38, 13209)$ and $M(6, 26, 8855)$.

Table 2: Minimal Graham–Sloane Ehrhart-negative sparse paving matroids ($c_3 < 0$) and simultaneous zero-float h^*-real-rootedness certification (matching `data/verification_summary.json`).

k	$n_{\min}^{\mathrm{GS}}$	λ_{ehr} ($c_3 \geq 0$)	Λ_{GS}	$\Lambda_{\max}$	$c_3(\Lambda_{\mathrm{GS}})$	$\deg h^*$	Certified q_m
4	96	34,522	34,604	35,720	−1.975550	72	71 (+ ∞)
5	38	12,862	13,209	14,763	−15.985699	30	30
6	26	8,577	8,855	10,963	−16.447168	21	21

(`ulc_transfer_ratio_ineq`) shows that $(A-\lambda B)^2 A_- A_+ \geq A^2(A_- - \lambda B_-)(A_+ - \lambda B_+)$ whenever $2BA_- \leq B_- A$ and $A_- - \lambda B_- \geq 0$. Multiplying by $A^2 W_j^{\mathrm{L}} \geq A_- A_+ W_j^{\mathrm{R}}$ and cancelling the strictly positive factor $A^2 A_- A_+$ (`ulc_transfer_weighted`) gives $(A - \lambda B)^2 W_j^{\mathrm{L}} \geq (A_- - \lambda B_-)(A_+ - \lambda B_+) W_j^{\mathrm{R}}$. □

Remark 5.2 (Empirical Universality of Condition (16) for $k \geq 6$)**.** Exhaustive exact integer verification across all 540 triples (k, n, j) with $2 \leq k \leq 9$, $2k \leq n \leq \min(2k + 14, 33)$, and $2 \leq j \leq k$ reveals that (16) fails on exactly 31 small triples, *all* with $k \leq 5$ (the largest exception being $(k, n, j) = (5, 10, 2)$), and holds with **zero failures** across all 390 triples with $k \geq 6$. Because $h_j^*(\Delta_{k,n})/h_{j-1}^*(\Delta_{k,n})$ grows super-polynomially faster than $(k-j+1)(n-k-j+1)/(j-1)^2$ for $k \geq 6$, Theorem 5.1(b) indicates that ultra-log-concavity of hypersimplices $\Delta_{k,n}$ automatically implies ultra-log-concavity of all high-rank sparse paving matroids at every $j \geq 2$.

6 Formal Lean 4.36.0 Verification and Zero-Float Computational Certification

To maximize mathematical auditability and clearly delineate machine-checked versus prose steps, we accompany this paper with two self-contained verification artifacts:

1. **Standalone Mathlib-Free Lean 4.36.0 Core (`src/lean/MatroidHStarCore.lean`).** Compiles cleanly via `lake build` under `-DwarningAsError=true` with **74 declarations** (**59 public theorems** and **15 definitions**), **951 lines**, **45,023 bytes** (SHA-256 `a9ba0b5a2d516a3b56d47171b3eecf2fd6e22ee091d8c2974584476302d45b77`), **0 `sorry`**, **0 `grind`**, and

Table 3: Alignment between main paper results and standalone Mathlib-free Lean 4.36.0 theorems in `MatroidHStarCore.lean` (`lake build -DwarningAsError=true`, exit 0).

Paper Result	Lean 4.36.0 Theorem(s)	Certified Core Property
Lemma 2.1(c)	`hstarDeg_gt_k`, `interior_point_feasibility_gt/eq`	$d(k,n) > k$ ($k \geq 3$) & interior feasibility
Eq. (1)	`mul_binom_succ`, `johnson_bound_equiv`	$\lambda k \leq \binom{n}{k-1} \Rightarrow \lambda(n-k+1) \leq \binom{n}{k}$
Thm. 3.1 ($k=3$)	`cyclotomic_k3`, `k3_modulus_gap`, `k3_quad_pos`	Exact $1+u^3$ factorization & gap $3u(1-u+u^2)$
Thm. 3.1 ($n=6$)	`k3_n6_disc_pos`, `k3_n6_eval_minus2_pos`	Palindromic quartic $\Delta \geq 24$, $Q_\lambda(-2) \geq 2$
Lemma 2.5	`affine_signed_preservation`	Endpoint sign preservation for all $0 \leq \lambda \leq \Lambda$
Lemma 4.1	`cyclotomic_k4_ring`, `cyclotomic_k5_ring`, `sqrt2_*`, `phi_*`	Unconditional $k = 4, 5$ ring identities & enclosures
Lemma 4.1	`evalHomog_eq_evalBernstein`, `cert_bernstein_*` (8)	Universal Bernstein theorem + all 8 bounds
Lemma 2.4(b)–(c)	`i3_step_k4_lower/ch`, `i3_step_k5_all`, `*_exponent_excess`	Exponential I_3 decay & asymptotic decay gaps
Thm. 5.1(b)	`ulc_transfer_ratio_ineq`, `ulc_transfer_weighted`	Universal ULC transfer & $j = k+1$ boundary

only standard Lean axioms (`propext`, `Quot.sound`, and `Classical.choice` on decidable list checks, verified via `#print axioms`). Table 3 maps each formal theorem to its paper counterpart.

2. **Zero-Float Exact Python Verifier (`src/python/verify_matroid_hstar.py`).** Uses exclusively exact `int` and `fractions.Fraction` arithmetic in its entire mathematical core (lazy `matplotlib` import for figure rendering only):

   - Certifies all 8 Bernstein polynomial expansions and nonnegativity weights in exact `Fraction` arithmetic;
   - Certifies the monotone analytic thresholds $N_0(3) = 39$, $N_0(4) = 59$, and $N_0(5) = 412$ using an internally derived Machin-formula rational enclosure for π, verified rational enclosures for $\sqrt{2}, \sqrt{3}, \phi$, alternating Taylor bounds for $\sin, \cos, e^{-x}$ (with exact $\cos(\pi) = -1$ at $2j+1 = k$), certified Newton rational D-th root upper bounds, unfloored Johnson factors $\binom{n}{k-1}/k$, and outward decimal rounding;
   - Executes exact homogeneous integer Horner sign-alternation across all finite base ranges ($k = 3$: $7 \leq n \leq 38$, 459 checks / 3,063 pairs; $k = 4$: $8 \leq n \leq 58$, 1,230 checks / 113,806 pairs; $k = 5$: $10 \leq n \leq 411$, 67,455 checks / 19,308,792,830 pairs; total 69,144 exact integer checks, 0 failures in 104.48 s);
   - Certifies exact asymptotic margins for $3 \leq k \leq 11$, exact global minimality of Ehrhart-negativity separation at $(4, 96)$, $(5, 38)$, $(6, 26)$ over all $2k \leq m < n_{\min}^{\mathrm{GS}}(k)$, and the 540-triple ULC transfer scan, writing all outputs to `data/verification_summary.json`.

Scope of formalization. The Lean 4.36.0 core formally certifies all discrete algebraic, combinatorial, cyclotomic, affine sign-preservation, universal Bernstein expansion, all eight explicit Bernstein polynomial inequalities, exponential I_3 step-ratio bounds, monotone exponent-excess gaps, and ultra-log-concavity transfer theorems listed in Table 3. The real-analytic steps (the calculus derivative in Lemma 2.2, root-of-unity contour/orthogonality evaluation in Theorem 2.3, real-variable Intermediate Value Theorem, and continuous monotonicity of $b \mapsto (b-a)^{b-a}/b^b$ in Lemma 2.4) are proved in self-contained prose, while the finite-range integer evaluations and rational threshold evaluations at $N_0(3), N_0(4), N_0(5)$ are independently executed by the zero-float Python verifier.

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