An exact two-mean law for singleton–avoidance probabilities

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Abstract

We determine the infimum of $\mathbb{P}(X = 1, Z = 0)$ over finitely supported nonnegative integer-valued pairs with real-stable probability generating polynomial and prescribed means (α, b). In the positive region $0 < \alpha < 2$, $0 \le b < \min(1, 2 - \alpha)$, the answer is the smaller of a degree-two value and an explicit categorical–Poisson limit. For $\alpha \le 1$ the limiting categorical trial has no X mass; for $\alpha > 1$ and $b > 0$ it necessarily has positive X mass. A determinantal pinching preserves the coefficient and both means exactly. Three perturbations of a hypothetical finite global minimizer then exclude all intermediate Bernoulli block sizes, using an analytic estimate for large blocks and three elementary polynomial factorizations for the remaining sizes. The bound is sharp even for finite determinantal measures with Hermitian positive-contraction kernels, counted on two disjoint blocks. The formula concerns unrestricted finite degree and one singleton target; no application to a TSP approximation constant is asserted.

Draft status. The preceding exact draft received a single-model assessment recommending minor revision. This corrected draft awaits a check of those changes and is not published. Exact algebraic checks supplement the proofs below.

1 The exact envelope

A real polynomial is *real stable* if it is nonzero whenever all variables have positive imaginary parts. Let $\mathcal{S}(\alpha, b)$ consist of real-stable polynomials with nonnegative coefficients, $P(1, 1) = 1$, and $\nabla P(1, 1) = (\alpha, b)$. Degrees are finite but have no common bound. Define

$$C(\alpha, b) = \inf_{P \in \mathcal{S}(\alpha, b)} [xz^0]P.$$

These classes are nonempty for all nonnegative means: independent monomials times Bernoulli factors realize every such pair. For the positive region put

$$\varepsilon = 1 - b, \qquad U = \min(\alpha, \varepsilon), \tag{1}$$

$$a_1 = \min\{U, \max\{0, (2\alpha - b)/4\}\}, \qquad H = \varepsilon\alpha - (2\alpha - b)a_1 + 2a_1^2. \tag{2}$$

Define $a_\infty = 0$ when $\alpha \le 1$. For $1 < \alpha < 2$, set

$$a_\infty = \frac{2b(\alpha - 1)}{2 - \alpha + b + \sqrt{(2 - \alpha + b)^2 + 4b(\alpha - 1)}}. \tag{3}$$

This includes $b = 0$. Finally put

$$J = e^{a_\infty - \alpha}\{\varepsilon(\alpha - a_\infty) + a_\infty(1 - \alpha + a_\infty)\}. \tag{4}$$

Theorem 1. *For $\alpha, b \geq 0$, the envelope is zero if $\alpha = 0$, $b \geq 1$, $\alpha \geq 2$, or $\alpha + b \geq 2$. At every remaining point,*

$$C(\alpha, b) = \min\{H, J\} > 0. \tag{5}$$

The value H is attained by a polynomial of degree at most two; J is approached by normalized categorical–binomial products. The same infimum holds for disjoint-block counts of strongly Rayleigh measures, and even when restricted to finite determinantal measures with Hermitian positive-contraction kernels $0 \preceq K \preceq I$.

The zero cases in the theorem overlap harmlessly. The positive domain is precisely $0 < \alpha < 2$, $0 \leq b < \min(1, 2 - \alpha)$. When $\alpha > 1$, the quadratic vertex satisfies $(2\alpha - b)/4 - (\alpha - 1) = (4 - 2\alpha - b)/4 > 0$ in this domain, so the definition of a_1 also respects the lower feasibility bound $a \geq \alpha - 1$ for the degree-two realizer. This domain was already covered by refined positivity estimates in Gurvits–Klein–Leake [5, Theorem 5.11 and Corollary 5.13]. Our question is the exact value, including the competing finite and limiting mechanisms. We do not claim a new positivity region.

For $\alpha \leq 1$, the limiting branch simplifies to $J = \alpha(1-b)e^{-\alpha}$. For $\alpha > 1$, $b > 0$, $a_\infty \in (0, \alpha - 1)$: using a pure Poisson block and a disjoint avoidance trial misses the smaller mixed limit. An exact finite example illustrating that failure is given in Section 7.

The deep input is Borcea–Brändén's bivariate determinantal representation [1, Corollary 6.7]. The Bernoulli equalization argument is classical Hoeffding theory [3]; we give the needed proof. Capacity productization in [4, 5] does not by itself assert exact preservation of our boundary coefficient. These comparisons identify dependencies and scope, not worldwide originality. Higher targets and several simultaneous positive target counts remain outside the theorem.

2 Exact coefficient preservation

Proposition 2. *For $P \in \mathcal{S}(\alpha, b)$ of exact degree $D \geq 1$, there is a product $Q = \prod_{i=1}^{D}(c_i + a_i x + b_i z)$ with nonnegative entries, $a_i + b_i + c_i = 1$, and $a_i + b_i > 0$, having the same means and coefficient $[xz^0]$ as P.*

Proof. The representation in [1] gives $P = \pm \det(xA + zB + C)$ with real symmetric D by D matrices, $A, B \succeq 0$. The coefficient of t^D in $P(t, t)$ is positive and is $\pm \det(A + B)$. Thus $G = A + B \succ 0$ and the sign is positive. Since

$$P(t, t) = \det G \det(tI + G^{-1/2} C G^{-1/2}) > 0 \quad (t > 0),$$

the symmetric matrix $G^{-1/2} C G^{-1/2}$ has no negative eigenvalue. Hence $C \succeq 0$. Now $A + B + C \succ 0$ has determinant one; congruence by its inverse square root preserves P and normalizes the sum to I. Jacobi's polynomial identity yields

$$\partial_x P(1, 1) = \operatorname{tr} A, \quad \partial_z P(1, 1) = \operatorname{tr} B, \quad [xz^0]P = \operatorname{tr}(\operatorname{adj}(C)A).$$

The last identity also holds for singular C. Diagonalize C and replace A, B by their diagonals in that basis. This preserves both traces and $\operatorname{tr}(\operatorname{adj}(C)A)$, since $\operatorname{adj}(C)$ is diagonal. The diagonal entries are the required a_i, b_i, c_i. Their sums are one, and $A + B \succ 0$ implies $a_i + b_i > 0$. The determinant becomes Q. □

No inverse of C is used. The proposition preserves a coefficient and means, not the full distribution or higher coefficients. The converse inclusion holds because the displayed products themselves are stable.

3 Reduction to one categorical trial and an equal block

Throughout this section $0 < b < \min(1, 2-\alpha)$ and $0 < \alpha < 2$. For $k \geq 2$, $0 \leq a \leq U$, put

$$G_k(a) = a(1-u)^k + (\varepsilon - a)(\alpha - a)(1-u)^{k-1}, \qquad u = (\alpha - a)/k. \tag{6}$$

For $k = 1$, use the same polynomial formula on $\max(0, \alpha - 1) \leq a \leq U$. These values are realized by

$$(ax + bz + \varepsilon - a)(1 - u + ux)^k. \tag{7}$$

A factor with $u = 0$ can be dropped.

Lemma 3. *Fix a finite number of Bernoulli slots, their total mean $m < 2$, and nonnegative weights A, C. A minimum of $A\mathbb{P}(S=0) + C\mathbb{P}(S=1)$ can be chosen with all fractional Bernoulli parameters equal and at most one parameter equal to one.*

Proof. The fixed-slot feasible set is compact. Among its minimizers choose one with fewest fractional parameters. For two such parameters u, v with fixed sum, the objective is affine in uv. A positive coefficient forces an endpoint, a zero coefficient permits an endpoint with fewer fractional entries, and a negative coefficient forces $u = v$. Thus all fractional entries are equal. There is at most one deterministic one because the total mean is less than two. Zero entries may be removed. □

The fixed number of slots is chosen *before* applying compactness; only afterwards do we take the infimum over all finite slot counts.

Proposition 4. *$C(\alpha, b)$ equals the infimum of all the values in (6), including the stated $k = 1$ domain.*

Proof. Use Proposition 2. At fixed constants c_i, the target coefficient $\sum_i a_i \prod_{j \neq i} c_j$ is linear on $0 \leq a_i \leq 1 - c_i$, $\sum_i a_i = \alpha$. A vertex has at most one coordinate strictly between its bounds. Therefore a minimizing replacement has at most one factor containing both x and z. This reasoning remains valid when some constants vanish.

If there is a mixed factor, write a for its x mass and t for the sum of pure z masses. Merging the latter into $1 - t + tz$ decreases the coefficient, since their constant product is at least $1 - t$. The mixed factor becomes $ax + (b-t)z + \varepsilon - a + t$. For the remaining pure x block write P_0, P_1 for its zero and one probabilities; its mean is $\alpha - a < 2$. The objective is

$$(1-t)\{aP_0 + (\varepsilon - a + t)P_1\}, \qquad \max(0, a - \varepsilon) \leq t \leq b.$$

This is concave in t, so an endpoint is no larger. At $t = 0$, $a \leq U$, and Lemma 3 gives an equal block, except possibly for a deterministic one. In that case $\alpha \geq 1$, $a \leq \alpha - 1$, and the objective is at least

$$(\varepsilon - a)(2 - \alpha + a) \geq \delta, \qquad \delta = 2 - \alpha - b.$$

Indeed the left side is concave on $[0, \alpha - 1]$ and its endpoint values are $\varepsilon(2-\alpha)$ and δ; their difference is $b(\alpha - 1) \geq 0$. The value $\delta = G_1(\alpha - 1)$ is feasible. The empty block has $a = \alpha \leq \varepsilon$ and is included with $u = 0$.

At $t = b$, or with no mixed factor to start with, the objective is ε times a pure Bernoulli one-probability of mean α. Equal blocks give $G_k(0)$; the deterministic alternative is at least $\varepsilon(2-\alpha) \geq \delta$ when it exists. At the other possible endpoint $t = a - \varepsilon > 0$ one has $a > \varepsilon > \alpha - 1$, so $\alpha - a < 1$. Then

$$\begin{aligned} a(1 - a + \varepsilon)P_0 &\geq a(1 - a + \varepsilon)(1 - \alpha + a) \\ &\geq \varepsilon(1 - \alpha + \varepsilon) = G_1(\varepsilon). \end{aligned}$$

For the second inequality subtract the right side and factor:

$$(a-\varepsilon)\{(1-a)(1-\alpha+a)+\varepsilon\}\geq 0.$$

Here $a\leq 1$, and $\varepsilon<a\leq\alpha$ makes $G_1(\varepsilon)$ feasible. All reductions are thus dominated by the scalar union. Conversely, (7) realizes every scalar value. □

4 The limit and the two boundaries

Uniformly on $a\in[0,U]$, $G_k(a)$ tends to

$$L(a)=e^{a-\alpha}\{\varepsilon(\alpha-a)+a(1-\alpha+a)\}. \tag{8}$$

For example, for $k\geq 4$ and $0\leq\mu=\alpha-a<2$,

$$|k\log(1-\mu/k)+\mu|\leq\frac{\mu^2}{2k(1-\mu/k)}\leq 4/k.$$

Together with (6) this proves uniform convergence, including when a varies with k. The sign of $L'(a)$ is the sign of

$$a^2+(2-\alpha+b)a-b(\alpha-1).$$

It is increasing on $a\geq 0$. Its minimum occurs at zero for $\alpha\leq 1$, and otherwise at (3), strictly between zero and $\alpha-1$. Thus $\min L=J$. Also $\min G_1=H$.

We will exclude any global scalar minimum below both H and J. Uniform convergence ensures that such an infimum is attained in a finite collection of compact scalar families: the large-k tail is separated from it by a positive gap. At $a=U=\alpha$ the value is $\alpha=G_1(\alpha)\geq H$. At $a=U=\varepsilon$, $\mu=\alpha-\varepsilon=1-\delta\in[0,1)$, and

$$G_k(\varepsilon)=\varepsilon(1-\mu/k)^k\geq\varepsilon(1-\mu)=\varepsilon\delta=G_1(\varepsilon)\geq H.$$

To handle $a=0$, we record the classical univariate boundary explicitly.

Lemma 5. *For a finite sum of independent Bernoulli variables of mean α, $0<\alpha<2$, the infimum of its one-probability is*

$$M(\alpha)=\begin{cases}\alpha e^{-\alpha}, & 0<\alpha\leq 1,\\ \min\{\alpha e^{-\alpha},\alpha(1-\alpha/2)\}, & 1<\alpha<2.\end{cases}$$

Proof. Lemma 3 reduces the objective to equal blocks or one deterministic success. The latter gives at least $2-\alpha$, which exceeds $\alpha(1-\alpha/2)$ by $(2-\alpha)^2/2$. Equal blocks have $q(k)=\alpha(1-\alpha/k)^{k-1}$. For $\alpha\leq 1$, $q(k)>\alpha e^{-\alpha}$ for every finite feasible k, by $\log(1-t)>-t/(1-t)$; the deterministic $k=1$ endpoint is separate. For $\alpha>1$, extend k to real $k\geq 2$ and put $t=\alpha/k$. Direct differentiation gives

$$(1-t)\frac{d\log q}{dk}=t^2\{R(t)-1/\alpha\},\qquad R(t)=\frac{t+(1-t)\log(1-t)}{t^2}=\sum_{j\geq 0}\frac{t^j}{(j+1)(j+2)}.$$

R increases with t, so the derivative sign decreases as k increases. A critical point can only be a maximum. The infimum is therefore at $k=2$ or $k\to\infty$, as stated. □

At $a=0$, Lemma 5 excludes a value below $\min(H,J)$. For $\alpha\leq 1$ its lower bound is J. For $\alpha>1$ the Poisson bound is $L(0)\geq J$, while the quadratic bound $\varepsilon\alpha(1-\alpha/2)$ is at least H. To check the latter, put $d=2-\alpha$, $\delta=d-b$. The unconstrained quadratic branch is

$$Q=\delta-(b+2\delta)^2/8=\alpha\delta/2-b^2/8;$$

$H=Q$ if $b\leq 2\delta$, and $H=E:=\varepsilon\delta$ otherwise. The differences from these two branches are respectively $b\alpha(\alpha-1)/2+b^2/8>0$ and $\varepsilon(b-d^2/2)>0$; in the clipped branch $b\geq 2d/3>d^2/2$.

5 Three variations exclude intermediate finite blocks

Assume a scalar value lies below $\min(H, J)$, and select an attained global minimum V as above. It has $k \geq 2$, $0 < a < U$, and $p = (\alpha - a)/k \in (0, 1)$. Lemma 3 and the deterministic bound in Proposition 4 show that every finite admissible product with the fixed means (α, b) has value at least V. This applies also after changing a or appending a zero Bernoulli slot; both types of perturbation below are therefore covered. Write $\mu = kp$, $q = \varepsilon - a$, $y = (k+1)p$, and $B = 2 - y$.

First change a to $a+t$ and one pure probability p to $p-t$, keeping b fixed. This preserves the two means and is feasible for small t of either sign. Dividing its linear and quadratic coefficients by the positive factor $(1-p)^{k-2}$ gives

$$S = B(a - p) - b(\mu - 1) = 0, \qquad B \geq 0.$$

If $B = 0$, then $\mu = 2 - p > 1$ and $S < 0$, impossible. Set $\beta = b/B > 0$. Stationarity gives

$$\begin{aligned} a &= p + \beta(\mu - 1), & q &= (1-p)(1-\beta), \\ \alpha &= y + \beta(\mu - 1), & \delta &= B - \beta(1-p), \\ G_k &= (1-p)^k(y - \beta), & b &= \beta B. \end{aligned} \tag{9}$$

Second perturb two pure probabilities to $p + t, p - t$. The coefficient of their product is a positive factor times $a(1-p) + q(\mu - 2) = (1-p)(\beta - B)$. Since their product decreases by t^2, minimality requires $\beta \leq B$. Third split one probability into $p - t, t$, $0 < t < p$, using a new slot. The exact objective change is

$$t(p - t)(1-p)^{k-2}\{a(1-p) + q(y-2)\}.$$

The bracket equals $(1-p)(p-\delta)$ by stationarity, hence $\delta \leq p$. Combining both necessary conditions gives

$$B - p \leq \beta(1-p) \leq B(1-p), \qquad B \leq 1.$$

Equality $B = 1$ forces $\beta = 1$ and $b = 1$, excluded. Thus

$$0 < B < 1, \quad n = k + 1 \geq 3, \quad p = (2 - B)/n, \quad \max(0, \ell) \leq \beta \leq B, \quad \ell = \frac{(n+1)B - 2}{n - 2 + B}. \tag{10}$$

This deduction does not require $\alpha > 1$.

5.1 The quadratic branch remains feasible

Before comparing with H, its branch formula must be justified also when the original mean is below one. At a stationary point,

$$2\alpha - b = 4 - 2B + \beta(2 - 3B - 2p) > 0.$$

Indeed, if the coefficient of β is negative, use $\beta \leq B$ and $p \leq (2-B)/3$ to bound it below by $4 - 7B^2/3 - 4B/3 > 1/3$; the other sign is immediate. Thus $a_* = (2\alpha - b)/4$ is positive and at most α. Furthermore $a_* - (\alpha - 1) = (4 - 2\alpha - b)/4 > 0$. Only the constraint $a_* \leq \varepsilon$ may clip it. Consequently at these points

$$H = \begin{cases} Q, & b \leq 2\delta, \\ E, & b \geq 2\delta, \end{cases} \quad Q = \delta - (b + 2\delta)^2/8, \quad E = (1-b)\delta. \tag{11}$$

The identity $Q - E = -(b - 2\delta)^2/8 \leq 0$ also proves $H \leq E$ even if the formal endpoint $a = \varepsilon$ would not have been feasible before imposing the stationary conditions.

5.2 All $k \geq 5$

The ratio $(y-\beta)/(1-\beta B)$ decreases with β; its derivative is $-(y-1)^2/(1-\beta B)^2$. Hence

$$\frac{G_k}{\varepsilon} \geq A_n(y) := \frac{2(1-y/n)^{n-1}}{3-y}, \qquad 1 < y < 2.$$

For $n \geq 6$,

$$A_n(y) \geq \min(y/n, 2-y). \tag{12}$$

To prove this, put $y_* = 2n/(n+1)$. On $[1, y_*]$ the logarithmic derivative of $A_n(y)/(y/n)$ is $-(n-1)/(n-y) + 1/(3-y) - 1/y < 0$. On $[y_*, 2)$ that of $A_n(y)/(2-y)$ is $-(n-1)/(n-y) + 1/(3-y) + 1/(2-y) > 0$: here $y_* \geq 3/2$, $1/(2-y) \geq 2$, and $(n-1)/(n-y) \leq 2$. The minimum ratio is at y_* and equals

$$R_n = \frac{(n+1)^2}{n+3}\left(\frac{n-1}{n+1}\right)^{n-1}.$$

$R_6 = 3125/3087 > 1$, and for real $n \geq 6$,

$$\frac{d\log R_n}{dn} \geq \frac{4}{n+1} - \frac{1}{n+3} - \frac{2}{n-1} = \frac{n^2-17}{(n-1)(n+1)(n+3)} > 0,$$

using $\log(1-u) \geq -u/(1-u)$. This proves (12). Since $\delta \leq \min(p, B)$, it gives $G_k \geq \varepsilon\delta = E \geq H$, contradicting the violating minimum.

5.3 The three remaining sizes

Under (10), the exact factorizations in Appendix A prove $G_k \geq H$ for $n = 3, 4, 5$. Thus no finite global minimum can lie below both H and J. The uniform-convergence argument proves the lower bound everywhere. Conversely (7) with $k = 1, a = a_1$ attains H and with $a = a_\infty, k \to \infty$ approaches J. This proves (5) for $b > 0$.

6 Zero region, boundary and determinantal sharpness

At $b = 0$, nonnegative coefficients force P to be univariate in x. Its real nonpositive roots factor it into normalized Bernoulli factors; Lemma 5 proves exactly (5). At $\alpha = 0$ the target coefficient vanishes. If $b \geq 1$, a factor z ensures its vanishing, with remaining means realized independently. If $\alpha \geq 2$, use a factor x^2. For $1 < \alpha < 2$ and $b \geq 2 - \alpha$, use

$$\big((\alpha/2)x + (1-\alpha/2)z\big)^2$$

and an independent z-only count of mean $b - (2-\alpha)$. These cover the remaining zero cases. Positivity inside the stated region follows also from the formula: each feasible degree-two product has positive target value, and the limiting prefactor in (4) is positive.

Here a finite determinantal measure means one with a Hermitian kernel $0 \preceq K \preceq I$; such measures are strongly Rayleigh [2, Proposition 3.5]. If a strongly Rayleigh measure counts disjoint blocks A, B, identifying the variables in each block and setting the other variables to one produces a bivariate stable polynomial with the same means and event. Conversely every categorical factor $c + ax + bz$ is realized on a two-point block by the determinantal kernel

$$K = \begin{pmatrix} a & \sqrt{ab} \\ \sqrt{ab} & b \end{pmatrix}, \qquad 0 \preceq K \preceq I.$$

Its probabilities are $c, a, b, 0$ for neither, the first point, the second point and both. Independent blocks realize the products in (7); one-point Bernoulli blocks suffice for pure factors. The upper constructions therefore belong to the determinantal class, which itself lies in the strongly Rayleigh class [2]. This proves the sharpness assertion of Theorem 1. Additional avoided blocks can all be identified with z; only the sum of their expected counts is then prescribed.

7 A mixed limit and the boundary of the result

At $\alpha = 13/10, b = 3/10$, the pure-Poisson candidate is $(91/100)e^{-13/10}$ and the degree-two value is $199/800$. The valid finite product

$$\left(\frac{x}{10} + \frac{3z}{10} + \frac{3}{5}\right)\left(\frac{247}{250} + \frac{3x}{250}\right)^{100}$$

has coefficient

$$\left(\frac{247}{250}\right)^{99}\left(\frac{41}{50} - \frac{3}{2500}\right) = 0.2478080145\ldots,$$

strictly below both naive candidates. The exponential comparison can be certified by rational Taylor bounds; this finite example is not used as proof of the main theorem. The correct exceptional mass is $a_\infty = (\sqrt{34} - 5)/10$. A further tempting shortcut would compare $G_k(a)$ to $G_1(a)$ and $L(a)$ at the *same* a; that assertion is false. For $\alpha = 1, b = 13/20, a = 1/5, k = 3$, $G_3 = 484/3375$ is smaller than both $G_1 = 4/25$ and $L = (8/25)e^{-4/5}$. The global quadratic minimum, however, is $391/3200 < G_3$. The proof therefore requires a global minimizer and feasible perturbations, rather than a pointwise block-size comparison.

The theorem does not give a fixed-degree envelope, classify all equality cases, preserve higher coefficients under pinching, or handle several positive targets simultaneously. Its limiting branch denotes an infimum approached by finite polynomials, not a finite polynomial identified with a Poisson law. The univariate ingredients are classical; the determinantal representation is a cited deep theorem. Neither a new TSP approximation ratio nor exhaustive priority is asserted.

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A Three exact small-block comparisons

Here $n = 3, 4, 5$, $p = (2 - B)/n$, and the constraints are (10). Write $t = \beta$ and

$$G = (1-p)^{n-1}(2 - B - t), \quad \delta = B - t(1-p), \quad b = tB.$$

Let Q, E be as in (11). The branch threshold is

$$T = \frac{2B}{B + 2(1-p)}; \qquad H = Q \ (t \le T), \quad H = E \ (t \ge T).$$

The identity $G - Q = (2 - B)^2 P_n / D_n$ holds with

$$\begin{aligned}
P_3 &= t^2 + 4t + 4 - 8B, && D_3 = 72,\\
P_4 &= 2t^2 + (6 - B)t + 4 - 8B - B^2, && D_4 = 64,\\
P_5 &= 225t^2 + (588 - 128B - 8B^2)t && \\
&\quad + 324 - 656B - 112B^2 - 8B^3, && D_5 = 5000.
\end{aligned}$$

Each P_n increases with $t \ge 0$, $0 < B < 1$. At $t = 0$ and $B \le 2/(n+1)$ its nonconstant-in-B expression is nonnegative: for $n = 3$ it is $4 - 8B$; for $n = 4$ its decreasing value at $2/5$ is $16/25$;

for $n = 5$ its decreasing value at $1/3$ is $2500/27$. For $B \geq 2/(n+1)$, evaluate at $t = \ell$:

$$(G-Q)_{n=3} = \frac{B(2-B)^3(2B-1)}{18(B+1)^2},$$
$$(G-Q)_{n=4} = \frac{B(2-B)^3(B^2+19B-2)}{64(B+2)^2},$$
$$(G-Q)_{n=5} = \frac{(2-B)^3(B^4+28B^3+343B^2+60B+18)}{625(B+3)^2}.$$

These are nonnegative on the respective ranges. Thus $G \geq Q$ throughout the relaxed feasible set.

For the clipped branch, $G - E$ is concave in t, with second derivative $-2B(1-p) < 0$. Its feasible interval is $[\max(0, \ell, T), B]$. If nonempty, $T \leq B$ implies $B \geq B_c = 4/(n+2)$. At $t = B$ its values are

$$n = 3: \quad (1-B)(B+1)(3B^2-4B+2)/9,$$
$$n = 4: \quad (1-B)(9B^3-2B^2-4B+8)/32,$$
$$n = 5: \quad (1-B)(2B^4+149B^3-17B^2-34B+162)/625.$$

They are positive: complete the square in the first, and bound the negative terms by 6 and 51 respectively in the last two. At the lower endpoint T, $Q = E$ and the preceding comparison applies. If the lower endpoint is instead $\ell > T$, the exact values are

$$\frac{(1-B)R_n(B)}{d_n(B+n-2)}, \qquad (d_3, d_4, d_5) = (9, 64, 625),$$

where

$$R_3 = B^3+18B^2-12B-2,$$
$$R_4 = B^4+12B^3+128B^2-48B-16,$$
$$R_5 = B^5+20B^4+150B^3+1290B^2-180B-102.$$

Each increases for $B \geq B_c$: the positive linear term in its derivative already dominates the sole negative constant. Their values at B_c are $54/125$, $1024/81$, and $4143750/16807$, all positive. Concavity now gives $G \geq E$ on the clipped interval. This proves all three remaining comparisons. The displayed identities are rational polynomial identities; their checks require no numerical minimization or sampled sign inference.

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