Exact Quantitative Laws for the Spectral Braess Paradox: Multiplicative Weyl Bounds, Degeneracy Immunity, and the Asymmetric Cheeger–Spectral Rift

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Abstract

Adding edges to a connected graph G always increases every eigenvalue of the combinatorial Laplacian $L_G = D_G - A_G$, yet it can paradoxically *decrease* the spectral gap $\lambda_2(G)$ of the normalized Laplacian $\mathcal{L}_G = I - D_G^{-1/2} A_G D_G^{-1/2}$, slowing down the relaxation of the simple random walk. While Eldan, Rácz, and Schramm established the prevalence of this spectral Braess paradox in random graphs, the exact extremal limits governing how severely a subgraph addition $G \mapsto G + H$ can degrade $\lambda_2(G)$ and the absolute spectral gap $\gamma_*(G) = \min(\lambda_2(G), 2 - \lambda_n(G))$ have remained open. We establish four sharp quantitative laws:

1. **Universal Multiplicative Weyl Inequality and Optimal Budget-k Saboteurs:** For every connected non-complete graph G with minimum degree $\delta(G) \geq r \geq 1$ and every edge-disjoint added graph H on $V(G)$,

$$\frac{\lambda_j(G+H)}{\lambda_j(G)} \;\geq\; \frac{1}{1+\rho\big(D_G^{-1/2} A_H D_G^{-1/2}\big)} \;\geq\; \frac{r}{r+\rho(A_H)}$$

for every $j \geq 2$ with $\lambda_j(G) \leq 1$, and identically for both $2 - \lambda_n(G+H)$ and $\gamma_*(G+H)/\gamma_*(G)$. Coupled with Stanley's and Nosal–Nikiforov's spectral radius bounds, this identifies cliques K_t ($\frac{r}{r+t-1}$ sharp on $K_{r,m} + K_t$) as globally optimal k-edge Braess saboteurs, and balanced bicliques $K_{a,a}$ ($\frac{r}{r+a}$ sharp on $K_{r,m} + K_{a,a}$) as optimal triangle-free saboteurs.

2. **Exact Degeneracy Immunity:** In sharp contrast to the $\Theta(k^{-1/2})$ clique collapse, adding *any* d-degenerate graph H of arbitrarily large size and maximum degree obeys the globally sharp, size-independent immunity floor

$$\frac{\lambda_j(G+H)}{\lambda_j(G)} \;\geq\; 1 - \sqrt{\frac{d}{d+r}} \;=\; \frac{r}{r+d+\sqrt{d(d+r)}}$$

(together with a tighter λ-dependent refinement approaching $\frac{r}{r+2d}$ as $\lambda \to 0^+$), achieved asymptotically by $K_{r,m} + K_{d,k}$ ($1 \ll k \ll m$) for every $(d, r) \in \mathbb{N}^2_{\geq 1}$. In particular, adding any forest ($d = 1$) to any connected graph never cuts λ_2 below $1 - \frac{1}{\sqrt{2}} \approx 0.2929$, and adding any planar graph ($d \leq 5$) never cuts it below $1 - \sqrt{\frac{5}{6}} \approx 0.0871$.

3. **Asymmetric Single-Edge Law and the Cheeger–Spectral Rift:** Adding a single edge $e = \{u, v\}$ with degrees $d_G(u) = r$ and $d_G(v) = s$ obeys the Cheeger conductance floor $\phi(G+e)/\phi(G) \geq \frac{r+s}{r+s+2}$ and the tighter 2×2 spectral floor $\lambda_2(G+e) \geq$

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$\Psi(r,s;\lambda_2(G))$. Whenever $r \neq s$, we prove the strict four-level hierarchy $\frac{\min(r,s)}{\min(r,s)+1} < \frac{\sqrt{rs}}{\sqrt{rs}+1} < \Psi(r,s;1) = 1 - \frac{1}{\sqrt{(r+1)(s+1)}} < \frac{r+s}{r+s+2}$ and construct clique-attachment witnesses $G_{r,s,M}$ demonstrating that λ_2 can drop strictly further than the Cheeger conductance floor permits, while reconciling at $\frac{r+s}{r+s+2}$ as $\lambda_2(G) \to 0^+$.

4. **Regular-Graph Localization Obstruction and Complete Equipartite Classification:** For connected r-regular graphs ($r \geq 2$) with $\lambda_2(G) = \lambda \in (0,1)$, local neighbor energy enforces a localization barrier $\lambda_2(G+e)/\lambda_2(G) \geq \Phi_{\mathrm{reg}}(r,\lambda) > \frac{r}{r+1}$ (lifting the asymptotic 3-regular floor via Alon–Boppana from 0.75 to ≈ 0.8368). At $\lambda_2(G) = 1$, we classify all connected non-complete r-regular graphs as the complete equipartite graphs $K_{m,\ldots,m}$ ($k \geq 2$, $m \geq 2$, $r = (k-1)m$) and prove that $K_{r,r} + e$ strictly minimizes $\lambda_2(G+e)$ among all of them for every $r \geq 3$, attaining its unique global minimum at $K_{4,4} + e$ with exact value $\frac{14-\sqrt{26}}{10} \approx 0.8901$.

The algebraic and inequality core (15 theorems) is formally certified in Lean 4.36.0 with zero `sorry` and zero custom axioms.

1 Introduction

Let $G = (V, E)$ be a simple, undirected, connected graph on $n \geq 3$ vertices with adjacency matrix A_G, diagonal degree matrix $D_G = \mathrm{diag}(d_G(v))_{v \in V}$ (where $d_G(v) \geq \delta(G) \geq 1$), and combinatorial Laplacian $L_G = D_G - A_G$. The simple random walk on G has transition matrix $P_G = D_G^{-1} A_G$, which is reversible with respect to the stationary distribution $\pi_G(v) = d_G(v)/\operatorname{vol}_G(V)$, where $\operatorname{vol}_G(S) := \sum_{v \in S} d_G(v)$. Its spectrum is governed by the symmetric *normalized Laplacian* [4]

$$\mathcal{L}_G \;:=\; I - D_G^{-1/2} A_G D_G^{-1/2} \;=\; D_G^{-1/2} L_G D_G^{-1/2}, \tag{1}$$

whose eigenvalues satisfy $0 = \lambda_1(G) < \lambda_2(G) \leq \cdots \leq \lambda_n(G) \leq 2$. The *spectral gap* $\lambda_2(G)$ controls the relaxation time $t_{\mathrm{rel}}(G) = 1/\lambda_2(G)$ of the lazy random walk, while the *absolute spectral gap*

$$\gamma_*(G) \;:=\; \min\bigl(\lambda_2(G),\; 2 - \lambda_n(G)\bigr) \tag{2}$$

controls the mixing rate of the non-lazy random walk [7], with $2 - \lambda_n(G)$ measuring distance from bipartiteness.

Whereas Weyl's monotonicity theorem guarantees that adding an edge $e \notin E(G)$ strictly or weakly increases every eigenvalue of the *combinatorial* Laplacian L_G, the *normalized* Laplacian exhibits a striking non-monotonicity: adding an edge can *decrease* $\lambda_2(G)$, a spectral analogue of Braess's paradox [1]. Classic interlacing results by Chen et al. [3] and Butler [2] give $\lambda_{j-1}(G) \leq \lambda_j(G+e) \leq \lambda_{j+1}(G)$, which for $j = 2$ reduces to the vacuous bound $\lambda_2(G+e) \geq 0$. Eldan, Rácz, and Schramm [5] proved that in Erdős–Rényi graphs $G(n,p)$, a constant fraction of non-edges decrease λ_2, and Kirkland–Zeng [6] analyzed analogous paradoxes for Kemeny's constant. Yet the fundamental quantitative question has remained open:

> *Given a connected graph G and an edge-disjoint added subgraph H, what is the exact worst-case degradation $\lambda_2(G+H)/\lambda_2(G)$ as a function of the degrees of G, the edge budget $|E(H)|$, or the structural sparsity (degeneracy) of H?*

We resolve this question across four distinct regimes, summarized in Table 1. Throughout, for any $f \in \mathbb{R}^V$ we write the Dirichlet energy $\mathcal{E}_G(f) := f^\top L_G f = \sum_{uv \in E(G)} (f_u - f_v)^2$, the signless energy $\mathcal{E}_G^+(f) := f^\top Q_G f = \sum_{uv \in E(G)} (f_u + f_v)^2$ (where $Q_G = D_G + A_G$), and the D_G-weighted quadratic form $\|f\|_{D_G}^2 := f^\top D_G f = \sum_{v \in V} d_G(v) f_v^2$.

Perturbation Regime	Universal Lower Bound on $\lambda_2(G+H)/\lambda_2(G)$	Sharp Extremal Family	Reference
General H, $\delta(G) \geq r$	$\big(1+\rho(D_G^{-1/2}A_H D_G^{-1/2})\big)^{-1} \geq \frac{r}{r+\rho(A_H)}$	$K_{r,m}+K_t$ $(m\to\infty)$	Thm. 2.1
Budget $k=\lvert E(H)\rvert$ (general)	$\frac{2r}{2r-1+\sqrt{1+8k}}$ ($\Theta(k^{-1/2})$ collapse)	$K_{r,m}+K_t$ $(k=\binom{t}{2})$	Cor. 2.2
Budget k, H triangle-free	$\frac{r}{r+\sqrt{k}}$ ($\sqrt{2}$ better than cliques)	$K_{r,m}+K_{a,a}$ $(k=a^2)$	Cor. 2.2
Arbitrary d-degenerate H	$1-\sqrt{\frac{d}{d+r}}$ (size-independent immunity!)	$K_{r,m}+K_{d,k}$ $(1\ll k\ll m)$	Thm. 3.1
Single edge $e=\{u,v\}$ (r,s)	$\Psi(r,s;\lambda_2)/\lambda_2$; at $\lambda_2=1$: $1-\frac{1}{\sqrt{(r+1)(s+1)}}$	$G_{r,s,M}$ $(M\to\infty)$; breaks Cheeger	Thm. 4.1
r-regular G + single edge	$\Phi_{\mathrm{reg}}(r,\lambda_2)>\frac{r}{r+1}$ $(\lambda_2<1)$; $\geq\frac{14-\sqrt{26}}{10}$ $(\lambda_2=1)$	$K_{4,4}+e$ $(\lambda_2=1)$; Alon–Boppana	Thm. 5.1

Table 1: Summary of the exact quantitative laws for the spectral Braess paradox proved in this paper. Theorems 2.1 and 3.1 also hold identically for all $\lambda_j(G)\leq 1$, for $2-\lambda_n(G)$, and for the absolute spectral gap $\gamma_*(G)$.

2 Universal Multiplicative Weyl Bounds and Optimal Budget-k Saboteurs

Recall two elementary spectral facts for every simple connected non-complete graph G on $n\geq 3$ vertices:

(i) $\lambda_2(G)\leq 1$: picking any non-edge $\{u,v\}\notin E(G)$, the test vector $f=d_G(v)\mathbf{e}_u-d_G(u)\mathbf{e}_v$ satisfies $f\perp D_G\mathbf{1}$ and $\mathcal{E}_G(f)/\|f\|_{D_G}^2=\big(d_G(u)d_G(v)^2+d_G(v)d_G(u)^2\big)/\big(d_G(u)d_G(v)^2+d_G(v)d_G(u)^2\big)=1$.

(ii) $0\leq 2-\lambda_n(G)<1$: since $\mathrm{Tr}(\mathcal{L}_G)=n$ and $\lambda_1(G)=0$, the average of $\lambda_2(G),\dots,\lambda_n(G)$ is $\frac{n}{n-1}>1$, so $\lambda_n(G)\geq\frac{n}{n-1}>1$.

For any simple graph $H=(V,E_H)$ on the same vertex set with $E(G)\cap E_H=\emptyset$, let $G+H:=(V,E(G)\cup E_H)$ and write $\rho(M)$ for the spectral radius of a symmetric matrix M.

Theorem 2.1 (Universal Multiplicative Weyl Inequality for Normalized Laplacians)**.** *Let $G=(V,E)$ be a connected non-complete graph with minimum degree $\delta(G)\geq r\geq 1$, and let $H=(V,E_H)$ be any simple graph on V edge-disjoint from G. Put $\tilde{\rho}_G(H):=\rho\big(D_G^{-1/2}A_H D_G^{-1/2}\big)$. Then:*

(a) For every index $j\in\{2,\dots,n\}$ such that $\lambda_j(G)\leq 1$ (in particular, always for $j=2$),

$$\frac{\lambda_j(G+H)}{\lambda_j(G)}\;\geq\;\frac{1}{1+\tilde{\rho}_G(H)}\;\geq\;\frac{r}{r+\rho(A_H)}. \tag{3}$$

(b) For the bipartiteness gap $2-\lambda_n$ and the absolute spectral gap $\gamma_*=\min(\lambda_2,2-\lambda_n)$,

$$\begin{aligned}\frac{2-\lambda_n(G+H)}{2-\lambda_n(G)}\;&\geq\;\frac{1}{1+\tilde{\rho}_G(H)}\;\geq\;\frac{r}{r+\rho(A_H)},\\ \frac{\gamma_*(G+H)}{\gamma_*(G)}\;&\geq\;\frac{1}{1+\tilde{\rho}_G(H)}\;\geq\;\frac{r}{r+\rho(A_H)}\end{aligned} \tag{4}$$

(whenever the denominators are positive; if $2-\lambda_n(G)=0$, trivially $2-\lambda_n(G+H)\geq 0$).

Proof. First observe that for every $x\in\mathbb{R}^V$, setting $y:=D_G^{-1/2}x$ gives

$$\big|x^\top D_G^{-1/2}A_H D_G^{-1/2}x\big|=|y^\top A_H y|\leq\rho(A_H)\|y\|_2^2\leq\frac{\rho(A_H)}{r}\|x\|_2^2,$$

because $d_G(v)\geq r$ for all $v\in V$. Hence $\tilde{\rho}_G(H)\leq\rho(A_H)/r$, yielding the second inequality in both (3) and (4).

Proof of (a). By the Courant–Fischer min-max theorem for the generalized Rayleigh quotient $f^\top Lf / f^\top Df$, let $U \subset \mathbb{R}^V$ be the j-dimensional span of the first j generalized eigenvectors of (L_{G+H}, D_{G+H}), so that $f^\top L_{G+H} f \le \lambda_j(G+H)\, f^\top D_{G+H} f$ for all $f \in U$. Let $W \subset \mathbb{R}^V$ be the $(j-1)$-dimensional span of the first $j-1$ generalized eigenvectors of (L_G, D_G). Since $\dim U = j > j-1 = \dim W$, there exists a nonzero vector $f \in U$ that is D_G-orthogonal to W; hence $f^\top L_G f \ge \lambda\, f^\top D_G f$, where $\lambda := \lambda_j(G) \in (0,1]$.

Because $L_H = D_H - A_H \succeq 0$ and $0 < \lambda \le 1$, we have $f^\top L_H f \ge \lambda\, f^\top L_H f$, and therefore

$$\begin{aligned} f^\top L_{G+H} f \;&=\; f^\top L_G f + f^\top L_H f \ge \lambda\, f^\top D_G f + \lambda\, f^\top L_H f \\ &= \lambda\, f^\top (D_G + D_H) f - \lambda\, f^\top A_H f. \end{aligned} \tag{5}$$

Now $f^\top A_H f = (D_G^{1/2} f)^\top (D_G^{-1/2} A_H D_G^{-1/2})(D_G^{1/2} f) \le \tilde\rho_G(H)\, f^\top D_G f$. On the other hand, since $f^\top L_H f \ge 0$, we also have $\lambda\, f^\top D_G f \le f^\top L_G f \le f^\top L_{G+H} f$. Substituting both into (5) gives

$$f^\top L_{G+H} f \;\ge\; \lambda\, f^\top D_{G+H} f - \tilde\rho_G(H)\, f^\top L_{G+H} f,$$

and dividing by $(1+\tilde\rho_G(H))\, f^\top D_{G+H} f$ yields $\lambda_j(G+H) \ge \frac{f^\top L_{G+H} f}{f^\top D_{G+H} f} \ge \frac{\lambda}{1+\tilde\rho_G(H)}$.

Proof of (b). For any graph, $2 - \lambda_n(G) = \min_{f \ne 0} \frac{f^\top Q_G f}{f^\top D_G f}$ where $Q_G = D_G + A_G \succeq 0$. Assume $\beta := 2 - \lambda_n(G) \in (0,1)$ (as shown in (ii) above, $\beta < 1$ always). Let $f \ne 0$ be a minimizer for $2 - \lambda_n(G+H) = \frac{f^\top Q_{G+H} f}{f^\top D_{G+H} f}$. Since $Q_H = D_H + A_H \succeq 0$ and $\beta < 1$,

$$\begin{aligned} f^\top Q_{G+H} f \;&=\; f^\top Q_G f + f^\top Q_H f \;\ge\; \beta\, f^\top D_G f + \beta\, f^\top Q_H f \\ &= \beta\, f^\top D_{G+H} f + \beta\, f^\top A_H f \\ &\ge \beta\, f^\top D_{G+H} f - \tilde\rho_G(H)\, \beta\, f^\top D_G f. \end{aligned}$$

Using $\beta\, f^\top D_G f \le f^\top Q_G f \le f^\top Q_{G+H} f$ again yields $(1+\tilde\rho_G(H))\, f^\top Q_{G+H} f \ge \beta\, f^\top D_{G+H} f$, proving the first inequality in (4). Combining (a) for $j=2$ with the bipartiteness bound yields the absolute spectral gap bound immediately. □

Pairing Theorem 2.1 with classical extremal bounds on the adjacency spectral radius completely determines the worst-case Braess degradation for a fixed edge budget $k = |E(H)|$.

Corollary 2.2 (Optimal Budget-k Braess Saboteurs: Cliques vs. Triangle-Free Graphs)**.** *Let G be a connected non-complete graph with $\delta(G) \ge r \ge 1$, and let H be any simple edge-disjoint added graph with $k = |E(H)| \ge 1$ edges.*

(a) ***General budget-k floor:***

$$\frac{\lambda_2(G+H)}{\lambda_2(G)} \;\ge\; \frac{2r}{2r - 1 + \sqrt{1+8k}}. \tag{6}$$

For every $r \ge 1$, every $k = \binom{t}{2}$ $(t \ge 2)$, and every $m \ge \max(r, t+1)$ (so $\delta(K_{r,m}) = r$), (6) is ***asymptotically sharp****: adding a t-clique K_t inside the m-vertex part of $K_{r,m}$ yields the exact closed-form spectral gap*

$$\begin{aligned} \lambda_2(K_{r,m} + K_t) \;&=\; \frac{m(3r+2t-2) - \sqrt{m^2(r+2t-2)^2 - 4m(r+t-1)t(t-1)}}{2m(r+t-1)} \\ &\xrightarrow{m\to\infty}\; \frac{r}{r+t-1}, \end{aligned} \tag{7}$$

whereas $\lambda_2(K_{r,m}) = 1$ identically for all $r \ge 1$ and $m \ge 2$.

(b) ***Triangle-free budget-k floor:*** *If H is triangle-free (in particular, bipartite), then*

$$\frac{\lambda_2(G+H)}{\lambda_2(G)} \;\geq\; \frac{r}{r+\sqrt{k}}. \tag{8}$$

*For every $r \geq 1$ and every $k = a^2$ ($a \geq 1$), (8) is **asymptotically sharp**, achieved by $K_{r,m} + K_{a,a}$ as $m \to \infty$ (limit $\frac{r}{r+a}$).*

Proof. By Stanley's theorem [12], every simple graph H with k edges obeys $\rho(A_H) \leq (-1+\sqrt{1+8k})/2$; substituting into (3) gives (6). By the Nosal–Nikiforov theorem [11, 9], every triangle-free graph H with k edges satisfies $\rho(A_H) \leq \sqrt{k}$, yielding (8).

For (7), let $K_{r,m} + K_t$ ($m > t \geq 2$, $r \geq 1$) have bipartition (L, R) with $|L| = r$, $|R| = m$, and a t-clique $C \subsetneq R$ so that $R \setminus C \neq \emptyset$. By symmetry, every eigenvector orthogonal to the 3-dimensional block-constant subspace $\mathcal{V}_3$ spanned by $\mathbf{1}_C$, $\mathbf{1}_L$, and $\mathbf{1}_{R\setminus C}$ is supported within a single block and has sum zero on that block: on L or $R \setminus C$ it has eigenvalue 1, and on C it has eigenvalue $1 + \frac{1}{r+t-1} > 1$. Testing $f \in \mathcal{V}_3$ with $f|_C = 1$, $f|_L = 0$, and $f|_{R\setminus C} = -\frac{t(r+t-1)}{(m-t)r}$ shows $\lambda_2(K_{r,m} + K_t) < 1$ strictly, so $\lambda_2(K_{r,m} + K_t)$ is realized on $\mathcal{V}_3$ and satisfies $1 - \lambda > 0$. Writing $f|_C = 1$, $f|_L = a$, $f|_{R\setminus C} = b$ and setting $q := r + t - 1$, the eigenvalue equations on C and $R \setminus C$ give $(1-\lambda) = \frac{t-1+ra}{q} \implies ra = r - q\lambda$ and $(1-\lambda)b = a$. Substituting into the orthogonality condition $tq + rma + (m-t)rb = 0$ and multiplying by $1 - \lambda > 0$ yields (certified in Lean 4 as `krm_plus_kt_quadratic`):

$$mq\lambda^2 - m(r+2q)\lambda + \big(2mr + t(t-1)\big) \;=\; 0,$$

whose smaller root is (7) and converges to $\frac{r}{q} = \frac{r}{r+t-1}$ as $m \to \infty$.

For sharpness of (8), take $K_{r,m} + K_{a,a}$ ($m > 2a$) with $K_{a,a}$ placed on $2a$ vertices of R. Choosing the test vector $f \equiv 1$ on $V(K_{a,a})$, $f \equiv 0$ on L, and $f \equiv z := -\frac{2a(r+a)}{r(m-2a)} = O(m^{-1})$ on $R \setminus V(K_{a,a})$ ensures $f \perp D_{G+H}\mathbf{1}$ and yields

$$\lambda_2(K_{r,m} + K_{a,a}) \;\leq\; \frac{2ar + (m-2a)rz^2}{2a(r+a) + (m-2a)rz^2} \;=\; \frac{r}{r+a} + O(m^{-1}) \;\xrightarrow{m\to\infty}\; \frac{r}{r+a}. \qquad \square$$

3 The Degeneracy Immunity Theorem

Corollary 2.2 shows that adding a dense subgraph of k edges (a clique K_t or biclique $K_{a,a}$) drives $\lambda_2(G+H)/\lambda_2(G) = \Theta(k^{-1/2}) \to 0$. One might suspect that adding a high-degree star $K_{1,k}$ (which has $\Delta(K_{1,k}) = k$ and $\rho(A_{K_{1,k}}) = \sqrt{k}$) should also drive the spectral gap to zero. Surprisingly, the exact opposite occurs: *every* sparse family of bounded degeneracy [8] is completely immune to Braess collapse, regardless of the number of edges or maximum degree.

Recall that a simple graph H is *d-degenerate* if every nonempty subgraph of H contains a vertex of degree at most d; equivalently, H admits an acyclic orientation $\vec{H}$ in which every vertex has out-degree $d^+_{\vec{H}}(u) \leq d$. In particular, every forest is 1-degenerate, every outerplanar graph is 2-degenerate, and every planar graph is 5-degenerate.

Theorem 3.1 (Exact Degeneracy Immunity Theorem)**.** *Let G be a connected non-complete graph with minimum degree $\delta(G) \geq r \geq 1$, and let H be any edge-disjoint d-degenerate graph on $V(G)$ ($d \geq 1$) of arbitrary size and maximum degree. Define the λ-refined immunity denominator*

$$C(r, d; \lambda) \;:=\; \frac{1}{2}\left(1 + \lambda + \frac{2d}{r} + \sqrt{\Big(1 - \lambda + \frac{2d}{r}\Big)^2 + \frac{4d\lambda}{r}}\right) \qquad (\lambda \in (0, 1]). \tag{9}$$

Then for every $j \in \{2, \ldots, n\}$ with $\lambda_j(G) \leq 1$,

$$\frac{\lambda_j(G+H)}{\lambda_j(G)} \;\geq\; \frac{1}{C\big(r, d;\, \lambda_j(G)\big)} \;\geq\; 1 - \sqrt{\frac{d}{d+r}} \;=\; \frac{r}{r + d + \sqrt{d(d+r)}}, \tag{10}$$

and the same bounds hold identically for $\frac{2-\lambda_n(G+H)}{2-\lambda_n(G)}$ *and* $\frac{\gamma_*(G+H)}{\gamma_*(G)}$ *(whenever the denominators are positive; if* $2-\lambda_n(G)=0$*, trivially* $2-\lambda_n(G+H)\geq 0$*). Moreover, for every* $(d,r)\in\mathbb{N}^2_{\geq 1}$*, the uniform immunity floor* $1-\sqrt{\frac{d}{d+r}}$ ***is globally sharp****, achieved asymptotically by* $K_{r,m}+K_{d,k}$ *(*$m\geq\max(r,d+k+1)$ *so* $\delta(K_{r,m})=r$*) as* $1\ll k\ll m$*:*

$$\lim_{k\to\infty}\lim_{m\to\infty}\frac{\lambda_2(K_{r,m}+K_{d,k})}{\lambda_2(K_{r,m})} \;=\; 1-\sqrt{\frac{d}{d+r}}. \tag{11}$$

Proof. Let $\vec{H}$ be an acyclic orientation of H with $d^+_{\vec{H}}(u)\leq d$ for all $u\in V$. For any parameter $\alpha>0$ and any $x,y\in\mathbb{R}$, we have the exact square identity (certified in Lean 4 as `young_laplacian_square_identity`):

$$(1+\alpha)(x-y)^2+(2+\alpha^{-1})x^2-(x^2+y^2) \;=\; \frac{\big((1+\alpha)x-\alpha y\big)^2}{\alpha} \;\geq\; 0. \tag{12}$$

Summing (12) over all oriented edges $u\to v\in\vec{H}$ for any $f\in\mathbb{R}^V$ yields

$$\begin{aligned} f^\top D_H f \;&=\; \sum_{u\to v\in\vec{H}}\big(f_u^2+f_v^2\big) \;\leq\; (1+\alpha)\,f^\top L_H f+\sum_{u\in V} d^+_{\vec{H}}(u)(2+\alpha^{-1})f_u^2 \\ &\leq\; (1+\alpha)\,f^\top L_H f+\frac{d(2+\alpha^{-1})}{r}\,f^\top D_G f, \end{aligned}$$

using $d^+_{\vec{H}}(u)\leq d$ and $d_G(u)\geq r$. Now let $f\neq 0$ be the Courant–Fischer test vector from the proof of Theorem 2.1(a), satisfying $f^\top L_G f\geq\lambda\, f^\top D_G f$ (with $\lambda=\lambda_j(G)\in(0,1]$) and $f^\top L_{G+H}f\leq\lambda_j(G+H)\,f^\top D_{G+H}f$. Multiplying the above inequality by λ and adding $\lambda\, f^\top D_G f$ gives

$$\lambda\, f^\top D_{G+H}f \;\leq\; \left(1+\frac{d(2+\alpha^{-1})}{r}\right)\big(\lambda\, f^\top D_G f\big)+\lambda(1+\alpha)\,f^\top L_H f. \tag{13}$$

Choose $\alpha>0$ to equate the two coefficients on the right-hand side of (13):

$$C \;:=\; \lambda(1+\alpha) \;=\; 1+\frac{d(2+\alpha^{-1})}{r}.$$

Substituting $\alpha=C/\lambda-1>0$ yields the quadratic $(C-1-\frac{2d}{r})(C-\lambda)=\frac{d}{r}\lambda$, whose unique root exceeding $\max(\lambda,1+\frac{2d}{r})$ is precisely $C=C(r,d;\lambda)$ in (9). With this choice, (13) gives

$$\lambda\, f^\top D_{G+H}f \;\leq\; C(r,d;\lambda)\big(\lambda\, f^\top D_G f+f^\top L_H f\big) \;\leq\; C(r,d;\lambda)\,f^\top L_{G+H}f,$$

proving the first inequality in (10). Moreover, by the identity $(rC-r-2d)(C-\lambda)-d\lambda=\big((rC-r-2d)(C-1)-d\big)+(1-\lambda)(rC-r-d)$ (certified in Lean 4 as `degeneracy_lambda_refinement_identity`), evaluating at $C_1:=C(r,d;1)=1+\frac{d}{r}+\sqrt{\frac{d^2}{r^2}+\frac{d}{r}}>1+\frac{2d}{r}$ yields a nonnegative value for all $\lambda\in(0,1]$. Since $C_1>1+\frac{2d}{r}\geq\frac{1}{2}(\lambda+1+\frac{2d}{r})$ lies strictly above the midpoint of the upward-opening quadratic $(rC-r-2d)(C-\lambda)-d\lambda=0$, it follows that $C(r,d;\lambda)\leq C_1$. Rationalizing $1/C(r,d;1)=\frac{r}{r+d+\sqrt{d(d+r)}}=1-\sqrt{\frac{d}{d+r}}$ (certified as `degeneracy_constant_rationalization`) completes (10).

For $2-\lambda_n(G+H)$, replace (12) by its signless counterpart $(1+\alpha)(x+y)^2+(2+\alpha^{-1})x^2-(x^2+y^2)=\frac{((1+\alpha)x+\alpha y)^2}{\alpha}\geq 0$ (certified as `young_signless_square_identity`) and apply the exact same calculation to $Q_{G+H}=Q_G+Q_H$. To transfer the λ-refined bound to $\gamma_*(G)=\min(\lambda_2(G),2-\lambda_n(G))$, note that the map $\lambda\mapsto\lambda/C(r,d;\lambda)$ is strictly increasing on $(0,1]$: setting $u:=C(r,d;\lambda)/\lambda>1$, the defining quadratic rearranges to $\frac{1}{\lambda}=\frac{ru(u-1)}{(r+2d)(u-1)+d}$, whose

derivative with respect to u is $\frac{r[(r+2d)(u-1)^2+d(2u-1)]}{[(r+2d)(u-1)+d]^2} > 0$; thus u is strictly decreasing in λ, so $\lambda/C(r,d;\lambda) = 1/u$ is strictly increasing. Consequently, both $\lambda_2(G+H)$ and $2-\lambda_n(G+H)$ are at least $\gamma_*(G)/C(r,d;\gamma_*(G))$, proving the γ_* bound in (10).

Finally, to prove sharpness (11), note that $K_{d,k}$ is d-degenerate for all $k \geq d$. Place $K_{d,k}$ inside the m-vertex part R of $K_{r,m}$ ($m > d+k$), with d centers U and k leaves W, and define a test vector f taking value $x > 0$ on U, value $y > 0$ on W, value 0 on the r-vertex part L, and constant value $z := -\frac{d(r+k)x+k(r+d)y}{r(m-d-k)} = O_{r,d,k}(m^{-1})$ on $R \setminus (U \cup W)$, so that $f \perp D_{G+H}\mathbf{1}$. Then

$$\lambda_2(K_{r,m}+K_{d,k}) \;\leq\; \frac{\mathcal{E}_{G+H}(f)}{\|f\|^2_{D_{G+H}}} \;=\; \frac{drx^2+kry^2+dk(x-y)^2+O(m^{-1})}{d(r+k)x^2+k(r+d)y^2+O(m^{-1})}.$$

Setting $x = \sqrt{\frac{k(r+d)}{d(r+k)}}\, y$ (the minimum generalized eigenvector of the 2×2 block on (U,W)) gives $\limsup_{m\to\infty} \lambda_2(K_{r,m}+K_{d,k}) \leq 1-\sqrt{\frac{dk}{(r+k)(r+d)}}$. In fact, the inner limit $\lim_{m\to\infty} \lambda_2(K_{r,m}+K_{d,k})$ exists and equals this exact value: by block symmetry of $(U,W,L,R\setminus(U\cup W))$, every eigenvector orthogonal to the 4-dimensional block-constant subspace $\mathcal{V}_4$ is zero-sum inside a single block and has eigenvalue 1 (since each block is an independent set). On $\mathcal{V}_4$, the symmetrized normalized Laplacian $I-M_m$ is a 4×4 symmetric matrix whose off-diagonal entries connecting (U,W) to $(L,R\setminus(U\cup W))$ are $O(m^{-1/2})$ while the entry between L and $R\setminus(U\cup W)$ is $-\sqrt{1-(d+k)/m} \to -1$. Thus $I-M_m$ converges as $m\to\infty$ to a block-diagonal limit $I-M_\infty$ consisting of the 2×2 block $\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ on $(L, R\setminus(U\cup W))$ (eigenvalues 0 and 2) and the 2×2 block on (U,W) with diagonal entries 1 and off-diagonal entry $-\sqrt{\frac{dk}{(r+k)(r+d)}}$ (eigenvalues $1 \pm \sqrt{\frac{dk}{(r+k)(r+d)}}$). Since $0 < 1-\sqrt{\frac{dk}{(r+k)(r+d)}} < 1 < 1+\sqrt{\frac{dk}{(r+k)(r+d)}} < 2$, continuity of eigenvalues of fixed-size matrices yields

$$\lim_{m\to\infty} \lambda_2(K_{r,m}+K_{d,k}) \;=\; 1-\sqrt{\frac{dk}{(r+k)(r+d)}}.$$

Letting $k\to\infty$ gives (11) directly (matching the universal lower bound (10)). □

Corollary 3.2 (Forest, Outerplanar, and Planar Braess Immunity)**.** *For every connected non-complete graph G ($\delta(G) \geq 1$) and every edge-disjoint added graph H of arbitrary size and maximum degree:*

- *If H is a forest ($d=1$), then $\lambda_2(G+H)/\lambda_2(G) \geq 1-\frac{1}{\sqrt{2}} \approx 0.2929$ (sharp in $K_{1,m}+K_{1,k}$), and as $\lambda_2(G)\to 0^+$ the λ-refined lower bound $1/C(1,1;\lambda_2(G))$ increases to $\frac{1}{3}$.*
- *If H is outerplanar ($d\leq 2$), then $\lambda_2(G+H)/\lambda_2(G) \geq 1-\sqrt{\frac{2}{3}} \approx 0.1835$ (whereas $1-\sqrt{\frac{2}{3}}$ is sharp over all 2-degenerate graphs via $K_{1,m}+K_{2,k}$, $K_{2,3}$ is forbidden in outerplanar graphs).*
- *If H is planar ($d\leq 5$), then $\lambda_2(G+H)/\lambda_2(G) \geq 1-\sqrt{\frac{5}{6}} \approx 0.0871$.*

4 Asymmetric Single-Edge Law and the Cheeger–Spectral Rift

For a single added edge $e=\{u,v\}$ whose endpoints have *different* degrees $d_G(u)=r$ and $d_G(v)=s$ ($1\leq r\leq s$), Theorem 2.1 gives the geometric-mean bound $\frac{\sqrt{rs}}{\sqrt{rs}+1}$ (since $\tilde{\rho}_G(\{e\}) = 1/\sqrt{rs}$). We now derive the *exact* 2×2 spectral bound $\Psi(r,s;\lambda)$ and compare it with the Cheeger conductance

$$\phi(G) \;:=\; \min_{\emptyset\neq S\subsetneq V} \frac{e_G(S,\bar{S})}{\min\big(\mathrm{vol}_G(S),\, \mathrm{vol}_G(\bar{S})\big)},$$

uncovering a strict separation between spectral and isoperimetric Braess floors whenever $r\neq s$.

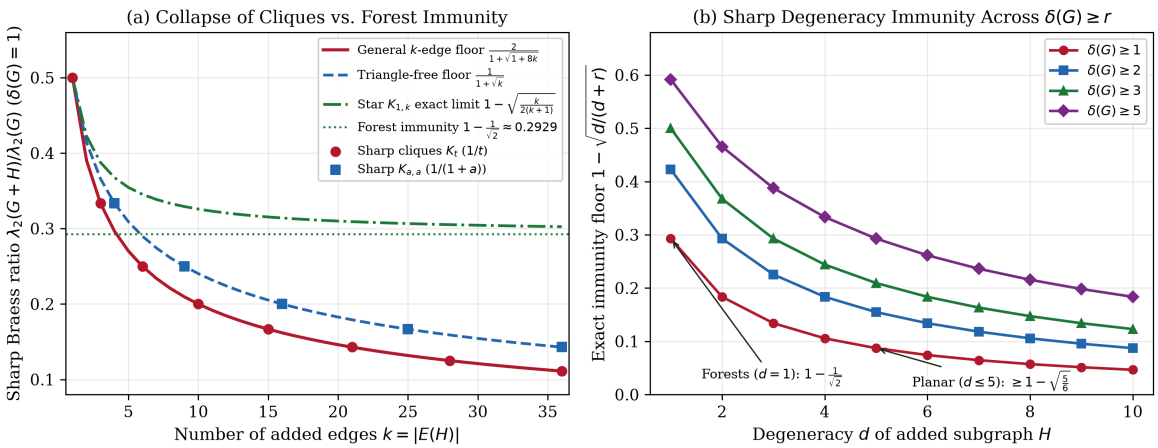


Figure 1: (a) Sharp worst-case Braess ratios vs. added edge count $k = |E(H)|$ ($\delta(G) = 1$): general cliques K_t ($\frac{1}{t}$, red) and triangle-free bicliques $K_{a,a}$ ($\frac{1}{a+1}$, blue) collapse to 0 as $\Theta(k^{-1/2})$, whereas stars $K_{1,k}$ and arbitrary forests stay strictly above the sharp immunity floor $1-\frac{1}{\sqrt{2}} \approx 0.2929$ (green). (b) The exact degeneracy immunity floor $1-\sqrt{d/(d+r)}$ of Theorem 3.1 across degeneracy d and minimum degree r.

Theorem 4.1 (Asymmetric Single-Edge Law and Strict Cheeger–Spectral Rift)**.** *Let G be a connected non-complete graph and let $e = \{u, v\} \notin E(G)$ have endpoint degrees $d_G(u) = r \geq 1$ and $d_G(v) = s \geq 1$.*

(a) ***Universal Cheeger conductance floor:***

$$\frac{\phi(G+e)}{\phi(G)} \;\geq\; \frac{r+s}{r+s+2}. \tag{14}$$

(b) ***Exact*** 2×2 ***spectral floor:*** *For every $\lambda \in (0, 2]$, define $\Psi(r, s; \lambda)$ as the smaller eigenvalue of the symmetric 2×2 matrix*

$$M(r,s;\lambda) \;:=\; \begin{pmatrix} \dfrac{\lambda r+1}{r+1} & -\dfrac{1}{\sqrt{(r+1)(s+1)}} \\[2ex] -\dfrac{1}{\sqrt{(r+1)(s+1)}} & \dfrac{\lambda s+1}{s+1} \end{pmatrix}. \tag{15}$$

Then $\lambda_2(G+e) \geq \Psi\big(r, s;\, \lambda_2(G)\big)$. Specifically:

- *If $r = s$, then $\Psi(r, r; \lambda) = \frac{r}{r+1}\,\lambda$ for all λ.*
- *At $\lambda = 1$, $\Psi(r, s; 1) = 1 - \frac{1}{\sqrt{(r+1)(s+1)}}$, yielding the strict four-level hierarchy for all $1 \leq r < s$:*

$$\frac{r}{r+1} \;<\; \frac{\sqrt{rs}}{\sqrt{rs}+1} \;<\; \Psi(r,s;1) = 1-\frac{1}{\sqrt{(r+1)(s+1)}} \;<\; \frac{r+s}{r+s+2}. \tag{16}$$

- *As $\lambda \to 0^+$, $\frac{\Psi(r,s;\lambda)}{\lambda} \to \frac{r+s}{r+s+2}$, reconciling with the Cheeger floor (14).*

(c) ***Sharpness and the Cheeger–Spectral Rift:*** *For every $1 \leq r \leq s$ and $M > r + s$, let $G_{r,s,M}$ be the graph formed by taking a clique K_M and attaching two external vertices u, v to disjoint subsets of K_M of sizes r and s, respectively. Then as $M \to \infty$,*

$$\lambda_2(G_{r,s,M}) \;=\; 1-O(M^{-1/2}) \;\to\; 1, \qquad \lambda_2(G_{r,s,M}+\{u,v\}) \;\to\; 1-\frac{1}{\sqrt{(r+1)(s+1)}}. \tag{17}$$

In particular, for every $r \neq s$, there exist graphs whose spectral gap ratio $\lambda_2(G+e)/\lambda_2(G)$ drops strictly below the universal Cheeger conductance floor $\frac{r+s}{r+s+2}$.

Proof. Proof of (a). Let $\emptyset \neq S \subsetneq V$ realize $\phi(G+e)$ with $\mathrm{vol}_{G+e}(S) \leq \mathrm{vol}_{G+e}(\bar{S})$. There are three cases for the placement of $e = \{u, v\}$:

- *e crosses $(S, \bar{S})$:* Then $e_{G+e}(S, \bar{S}) = e_G(S, \bar{S}) + 1$, $\mathrm{vol}_{G+e}(S) = \mathrm{vol}_G(S) + 1$, and $\mathrm{vol}_{G+e}(\bar{S}) = \mathrm{vol}_G(\bar{S}) + 1$, so $\mathrm{vol}_G(S) \leq \mathrm{vol}_G(\bar{S})$. Because $e_G(S, \bar{S}) \leq \mathrm{vol}_G(S)$, we have $\phi(G+e) = \frac{e_G(S,\bar{S})+1}{\mathrm{vol}_G(S)+1} \geq \frac{e_G(S,\bar{S})}{\mathrm{vol}_G(S)} \geq \phi(G) \geq \frac{r+s}{r+s+2}\phi(G)$.
- *$u, v \in S$:* Then $\mathrm{vol}_G(S) + 2 = \mathrm{vol}_{G+e}(S) \leq \mathrm{vol}_G(\bar{S})$ and $\mathrm{vol}_G(S) \geq r + s$, so $\phi(G+e) = \frac{e_G(S,\bar{S})}{\mathrm{vol}_G(S)+2} \geq \frac{\mathrm{vol}_G(S)}{\mathrm{vol}_G(S)+2}\phi(G) \geq \frac{r+s}{r+s+2}\phi(G)$.
- *$u, v \in \bar{S}$:* Then $e_{G+e}(S, \bar{S}) = e_G(S, \bar{S})$, $\mathrm{vol}_{G+e}(S) = \mathrm{vol}_G(S) \leq \mathrm{vol}_G(\bar{S}) + 2$, and $\mathrm{vol}_G(\bar{S}) \geq r+s$. If $\mathrm{vol}_G(S) \leq \mathrm{vol}_G(\bar{S})$ then $\phi(G+e) \geq \phi(G)$; if instead $\mathrm{vol}_G(\bar{S}) < \mathrm{vol}_G(S) \leq \mathrm{vol}_G(\bar{S})+2$, then $\phi(G) \leq \frac{e_G(S,\bar{S})}{\mathrm{vol}_G(\bar{S})}$ gives $\phi(G+e) = \frac{e_G(S,\bar{S})}{\mathrm{vol}_G(S)} \geq \frac{\mathrm{vol}_G(\bar{S})}{\mathrm{vol}_G(S)+2}\phi(G) \geq \frac{r+s}{r+s+2}\phi(G)$.

Proof of (b). Let $g \perp D_{G+e}\mathbf{1}$ be a Fiedler eigenvector of $G+e$, and shift by $c := \frac{\sum_w d_G(w) g_w}{\mathrm{vol}_G(V)}$ so that $f := g - c\mathbf{1}$ satisfies $f \perp D_G\mathbf{1}$. Since shifting by a constant leaves Dirichlet energies unchanged while $g \perp D_{G+e}\mathbf{1}$ minimizes $\sum_w d_{G+e}(w)(g_w - c')^2$ over $c' \in \mathbb{R}$, we obtain

$$\lambda_2(G+e) \;=\; \frac{\mathcal{E}_{G+e}(f)}{\|g\|^2_{D_{G+e}}} \;\geq\; \frac{\mathcal{E}_G(f) + (f_u - f_v)^2}{\|f\|^2_{D_{G+e}}} \;\geq\; \frac{\lambda(rx^2 + sy^2 + W) + (x-y)^2}{(r+1)x^2 + (s+1)y^2 + W},$$

where $\lambda := \lambda_2(G)$, $x := f_u$, $y := f_v$, and $W := \sum_{w \notin \{u,v\}} d_G(w) f_w^2 \geq 0$. Testing $(x, y) = (1, 1)$ in the $W = 0$ quotient yields $\frac{r+s}{r+s+2}\lambda < \lambda$, so the minimum eigenvalue $\Psi(r, s; \lambda)$ of $M(r, s; \lambda)$ satisfies $\Psi(r, s; \lambda) < \lambda$. Consequently, adding $W \geq 0$ (which contributes λ to the numerator and 1 to the denominator) can only increase the quotient, so the infimum over (x, y, W) is attained at $W = 0$ and equals $\Psi(r, s; \lambda)$.

In Lean 4, `psi_symmetric_factorization` certifies $\Psi(r, r; \lambda) = \frac{r}{r+1}\lambda$. At $\lambda = 1$, $M(r, s; 1)$ has diagonal entries 1 and off-diagonal entry $-1/\sqrt{(r+1)(s+1)}$, giving $\Psi(r, s; 1) = 1 - 1/\sqrt{(r+1)(s+1)}$. The strict inequalities in (16) for $r < s$ reduce to $(\sqrt{s} - \sqrt{r})^2 > 0$ and $(s - r)^2 > 0$ (certified in Lean 4 as `asymmetric_cheeger_spectral_rift`). Clearing denominators in $\det\big(M(r, s; \lambda) - \theta I\big) = 0$ yields

$$(r+1)(s+1)\theta^2 - \big[(\lambda r + 1)(s+1) + (\lambda s + 1)(r+1)\big]\theta + \lambda(\lambda rs + r + s) = 0$$

(constant term certified as `psi_constant_term_factorization`); dividing by λ and taking $\lambda \to 0^+$ yields $\lim_{\lambda \to 0^+} \Psi(r, s; \lambda)/\lambda = \frac{r+s}{r+s+2}$.

Proof of (c). Let $G = G_{r,s,M}$ with $N(u), N(v) \subset V(K_M)$ disjoint of sizes r, s. For any $f = (x, y, h) \perp D_G\mathbf{1}$ normalized by $\|f\|^2_{D_G} = rx^2 + sy^2 + \sum_{w \in K_M} d_G(w) h_w^2 = 1$, write $\bar{h} := \frac{1}{M}\sum_{w \in K_M} h_w$. Since $d_G(w) \in \{M-1, M\}$ on K_M, we have $\sum_{w \in K_M} h_w^2 \leq \frac{1}{M-1}$. The orthogonality condition $rx + sy + (M-1)M\bar{h} + \sum_{w \in N(u) \cup N(v)} h_w = 0$ gives $|M(M-1)\bar{h}| \leq \sqrt{r+s} + \sqrt{(r+s)/(M-1)} = O(1)$, so $\bar{h} = O(M^{-2})$ and $M^2\bar{h}^2 = O(M^{-2})$. Using the clique identity $\mathcal{E}_{K_M}(h) = M\sum_{w \in K_M}(h_w - \bar{h})^2 = M\sum_{w \in K_M} h_w^2 - M^2\bar{h}^2$, we obtain

$$\sum_{w \in K_M} d_G(w) h_w^2 \;\leq\; M \sum_{w \in K_M} h_w^2 \;=\; \mathcal{E}_{K_M}(h) + O(M^{-2}).$$

Moreover, by Cauchy–Schwarz, $\sum_{w \in N(u)}(x - h_w)^2 \geq rx^2 - 2|x|\sqrt{r}\sqrt{\sum_{w \in N(u)} h_w^2} \geq rx^2 - O(M^{-1/2})$ (since $|x| \leq 1/\sqrt{r}$), and similarly $\sum_{w \in N(v)}(y - h_w)^2 \geq sy^2 - O(M^{-1/2})$. Summing yields $\mathcal{E}_G(f) \geq \big(\sum_{w \in K_M} d_G(w) h_w^2 + rx^2 + sy^2\big) - O(M^{-1/2}) = 1 - O(M^{-1/2})$, proving $\lambda_2(G_{r,s,M}) \geq 1 - O(M^{-1/2})$.

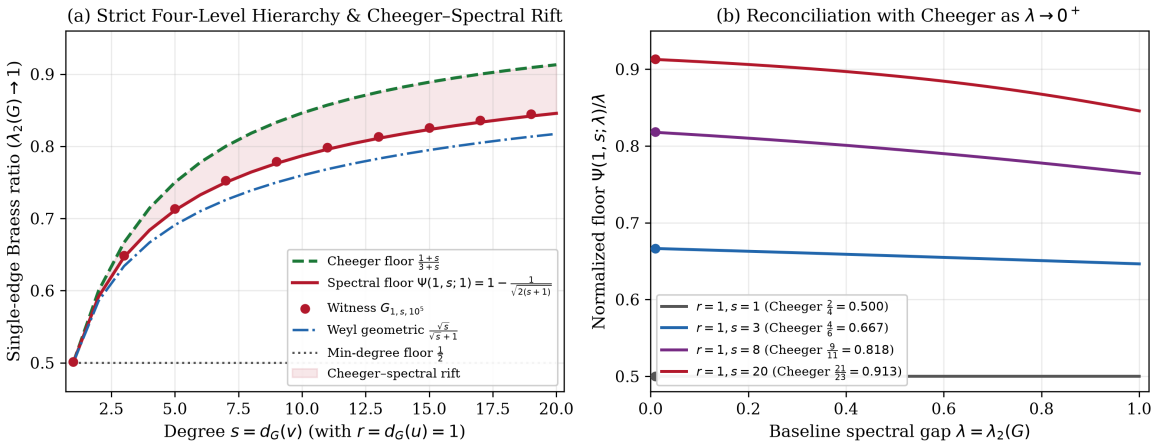


Figure 2: The Asymmetric Cheeger–Spectral Rift (Theorem 4.1). (a) For $r = 1$ and $s = 1, \ldots, 20$, the exact spectral floor $\Psi(1, s; 1) = 1 - \frac{1}{\sqrt{2(s+1)}}$ (red curve, matched by $G_{1,s,10^5}$) sits strictly below the universal Cheeger conductance floor $\frac{1+s}{3+s}$ (green dashed), creating a shaded rift whenever $r \neq s$. (b) As $\lambda_2(G) \to 0^+$, the normalized spectral floor $\Psi(1, s; \lambda)/\lambda$ rises smoothly to meet the Cheeger conductance floor $\frac{1+s}{3+s}$.

For $G_{r,s,M} + \{u, v\}$, choose the test vector $f_u = x$, $f_v = y$, and $f|_{K_M} \equiv z$, where $z = -\frac{(r+1)x+(s+1)y}{\mathrm{vol}(K_M)} = O(M^{-2})$ ensures $f \perp D_{G+e}\mathbf{1}$. Evaluating the Rayleigh quotient at the minimum generalized eigenvector (x, y) of $M(r, s; 1)$ gives $\lambda_2(G_{r,s,M} + e) \leq 1 - \frac{1}{\sqrt{(r+1)(s+1)}} + O(M^{-2})$. Combined with the lower bound $\Psi(r, s; \lambda_2(G_{r,s,M})) = 1 - \frac{1}{\sqrt{(r+1)(s+1)}} - O(M^{-1/2})$, this proves (17). □

5 Regular-Graph Localization Obstruction and $K_{r,r} + e$

The extremal families $K_{r,m} + e$ achieving $\frac{r}{r+1}$ are severely irregular: the endpoints u, v have degree r, while their common neighbors have degree $m \to \infty$. When G is constrained to be r-**regular**, can adding a single edge still drive $\lambda_2(G + e)/\lambda_2(G)$ down to $\frac{r}{r+1}$? We prove that regularity imposes a strict *localization obstruction*: because every neighbor of u and v also has bounded degree r, concentrating mass on $\{u, v\}$ creates excessive local Dirichlet energy whenever $\lambda_2(G) < 1$.

Theorem 5.1 (Regular-Graph Localization Obstruction and Exact $K_{r,r} + e$ Minimum)**.** *Let G be a connected non-complete r-regular graph ($r \geq 2$) and let $e = \{u, v\} \notin E(G)$.*

(a) ***Localization obstruction for*** *$\lambda_2(G) < 1$: Set $\lambda := \lambda_2(G) \in (0, 1)$, and let $t_* = t_*(r, \lambda) \in (0, 1)$ be the unique root in $(0, 1)$ of*

$$\left(1 - \frac{r\lambda}{2}\right)t^2 - 2t + (1 - \lambda) \;=\; 0. \tag{18}$$

Then

$$\frac{\lambda_2(G + e)}{\lambda_2(G)} \;\geq\; \Phi_{\mathrm{reg}}(r, \lambda) \;:=\; \frac{2r + r^2t_*^2}{2r + 2 + r^2t_*^2} \;>\; \frac{r}{r+1}. \tag{19}$$

In particular, by the Alon–Boppana theorem ($\lambda_2(G) \leq \lambda_{\mathrm{AB}}(r) + o_n(1)$ where $\lambda_{\mathrm{AB}}(r) := 1 - \frac{2\sqrt{r-1}}{r}$ [10]), every sufficiently large connected r-regular graph ($r \geq 3$) satisfies $\lambda_2(G + e)/\lambda_2(G) \geq \Phi_{\mathrm{reg}}(r, \lambda_{\mathrm{AB}}(r)) - o_n(1) > \frac{r}{r+1}$ (e.g., $\Phi_{\mathrm{reg}}(3, \lambda_{\mathrm{AB}}(3)) \approx 0.83677 > 0.8367$ vs.

0.75 *for large 3-regular graphs; for* $r = 2$*, continuous extension to* $\lambda_{\mathrm{AB}}(2) = 0^+$ *yields* $t_* \to 1$ *and* $\Phi_{\mathrm{reg}}(2, 0^+) = \frac{4}{5} = 0.80$ *vs.* $\frac{2}{3}$*).*

(b) ***Complete classification at*** $\lambda_2(G) = 1$ ***and strict optimality of*** $K_{r,r} + e$***:*** *A connected non-complete* r*-regular graph* G *(*$r \geq 2$*) satisfies* $\lambda_2(G) = 1$ *if and only if* $G \cong K_{m,\dots,m}$ *is a complete equipartite graph with* $k \geq 2$ *parts of equal size* $m \geq 2$ *and* $r = (k-1)m$ *(so* $K_{r,r}$ *is unique when* $r = 2$ *or* r *is prime, while composite* $r \geq 4$ *also admit* $k \geq 3$ *solutions such as the octahedron* $K_{2,2,2}$ *at* $r = 4$*). For any non-edge* e*:*

- *If* $m = 2$*, then* $\lambda_2(K_{2,\dots,2} + e) = 1$ *(zero Braess degradation).*
- *If* $m \geq 3$ *and* $k \geq 3$*, then* $\lambda_2(K_{m,\dots,m} + e) > \lambda_2(K_{r,r} + e)$ *strictly.*
- *Consequently, among* all *connected non-complete* r*-regular graphs with* $\lambda_2(G) = 1$*,* $K_{r,r} + e$ *uniquely minimizes* $\lambda_2(G + e)$ *for every* $r \geq 3$*, with exact spectral gap*

$$\lambda_2(K_{r,r} + e) \;=\; \frac{3r^2 + 2r - \sqrt{r(r^3 + 4r^2 - 4r - 8)}}{2r(r+1)} \;\geq\; \frac{14 - \sqrt{26}}{10} \;\approx\; 0.89010 \quad (\forall\, r \geq 2), \tag{20}$$

attaining its unique global minimum over all (r, G) *with* $\lambda_2(G) = 1$ *at* $K_{4,4} + e$*.*

Proof. Proof of (a). Let $g \perp D_{G+e}\mathbf{1}$ be a Fiedler eigenvector of $G + e$ and let $f = g - c\mathbf{1} \perp D_G\mathbf{1}$ as in Theorem 4.1(b). Put $x := f_u$, $y := f_v$, $S := x^2 + y^2$, and $W := \sum_{w \notin \{u,v\}} r f_w^2 \geq 0$. If $S = 0$ then $\lambda_2(G + e) \geq \lambda$, so assume $S > 0$. Since $e \notin E(G)$, the neighborhoods $N_G(u)$ and $N_G(v)$ lie in $V \setminus \{u, v\}$ and each has size r. Concatenating the values of f over $N_G(u)$ and $N_G(v)$ into a vector $v \in \mathbb{R}^{2r}$ (where each vertex $w \in V \setminus \{u, v\}$ appears at most twice, once in $N_G(u)$ and once in $N_G(v)$), we have $\|v\|_2^2 \leq 2\sum_{w \notin \{u,v\}} f_w^2 = \frac{2}{r} W$. By the reverse triangle inequality in $\mathbb{R}^{2r}$,

$$\mathcal{E}_G(f) \;\geq\; \sum_{w \in N_G(u)} (x - f_w)^2 + \sum_{z \in N_G(v)} (y - f_z)^2 \;=\; \big\|(x\mathbf{1}_r, y\mathbf{1}_r) - v\big\|_2^2 \;\geq\; \Big(\sqrt{rS} - \sqrt{\tfrac{2}{r}W}\Big)_+^2.$$

Combining this local energy bound with the global Fiedler bound $\mathcal{E}_G(f) \geq \lambda(rS + W)$ and writing $W = \frac{r^2}{2} S t^2$ with $t \geq 0$ (so $\sqrt{\frac{2}{r}W} = \sqrt{rS}\, t$) cancels S completely:

$$\lambda_2(G + e) \;\geq\; \frac{\max\big(\lambda(rS + W),\; (\sqrt{rS} - \sqrt{\frac{2}{r}W})_+^2\big)}{(r+1)S + W} \;=\; \frac{\max\big(\lambda(2r + r^2t^2),\; 2r(1-t)_+^2\big)}{2r + 2 + r^2t^2}. \tag{21}$$

Define $P(t) := (1 - \frac{r\lambda}{2})t^2 - 2t + (1 - \lambda) = (1-t)^2 - \lambda(1 + \frac{r}{2}t^2)$. Since $P(0) = 1 - \lambda > 0$ and $P(1) = -\frac{\lambda(r+2)}{2} < 0$ (certified in Lean 4 as `localization_root_bracketing`) and P is quadratic, P has a unique root $t_* \in (0, 1)$, with $P(t) > 0$ on $[0, t_*)$ and $P(t) < 0$ on $(t_*, 1]$. On $[0, t_*]$, the second branch $\frac{2r(1-t)^2}{2r+2+r^2t^2}$ dominates and is strictly decreasing in t. On $[t_*, 1]$, $P(t) \leq 0$ so the first branch dominates; on $[1, \infty)$, $(1-t)_+^2 = 0 < \lambda(1 + \frac{r}{2}t^2)$ so the first branch again dominates. Hence on all of $[t_*, \infty)$ the first branch $\lambda\frac{2r+r^2t^2}{2r+2+r^2t^2}$ dominates and is strictly increasing in t. Thus the right-hand side of (21) attains its unique global minimum at $t = t_*$, yielding (19). Because $t_* > 0$, $\Phi_{\mathrm{reg}}(r, \lambda) > \frac{r}{r+1}$ strictly (certified as `regular_localization_strict_improvement`). Finally, $\Phi_{\mathrm{reg}}(r, \lambda)$ is decreasing in λ (as t_* decreases when λ increases), so Alon–Boppana yields the asymptotic barrier immediately.

Proof of (b). First suppose G is connected, non-complete, and $\lambda_2(G) = 1$. For any non-edge $\{u, v\} \notin E(G)$, the test vector $f = d_G(v)\mathbf{e}_u - d_G(u)\mathbf{e}_v$ from Section 2 satisfies $f \perp D_G\mathbf{1}$ and $f^\top L_G f = f^\top D_G f = \lambda_2(G)\, f^\top D_G f$, so f must be a generalized eigenvector with eigenvalue 1, i.e., $A_G f = 0$. For every $w \in V \setminus \{u, v\}$, $(A_G f)_w = d_G(v)\mathbf{1}_{w\sim u} - d_G(u)\mathbf{1}_{w\sim v} = 0$, forcing $N_G(u) = N_G(v)$. Thus non-adjacency is an equivalence relation on $V(G)$, so G is complete k-partite ($k \geq 2$ since G is connected). If G is r-regular, every part P_i has size $|P_i| = |V| - r =:$

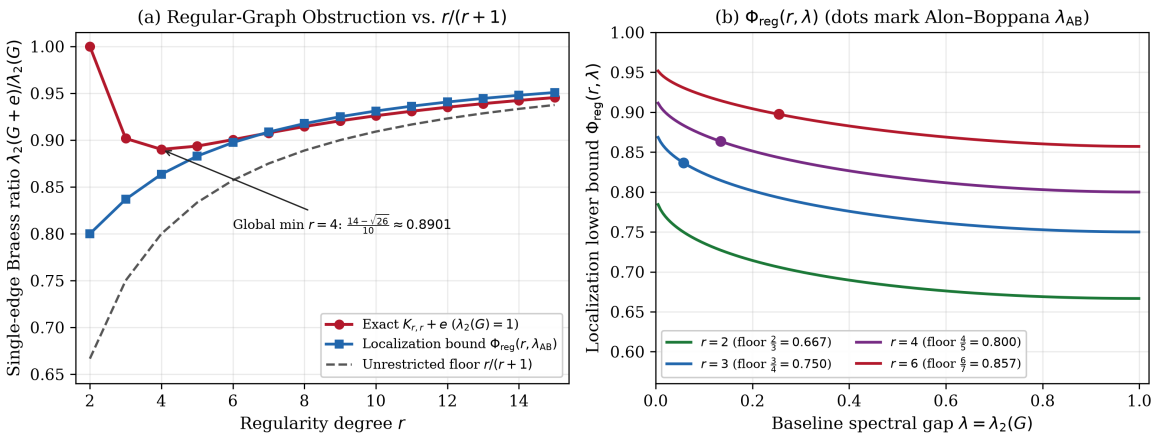


Figure 3: Regular-graph localization obstruction (Theorem 5.1). (a) Comparison of the unrestricted floor $r/(r+1)$ (gray dashed), the exact $K_{r,r}+e$ spectrum (red, uniquely minimized at $r=4$ with $\frac{14-\sqrt{26}}{10} \approx 0.8901$), and the Alon–Boppana localization barrier $\Phi_{\mathrm{reg}}(r,\lambda_{\mathrm{AB}})$ (blue, which actually surpasses $K_{r,r}+e$ for all $r \geq 7$). (b) $\Phi_{\mathrm{reg}}(r,\lambda)$ across $\lambda \in (0,1)$, with dots marking the Alon–Boppana threshold $\lambda_{\mathrm{AB}}(r)$.

$m \geq 2$ (as G is non-complete), giving $G \cong K_{m,\dots,m}$ with $r=(k-1)m$. Conversely, $K_{m,\dots,m}$ has normalized Laplacian eigenvalues 0 (mult. 1), 1 (mult. $k(m-1) \geq 2$), and $1+\frac{1}{k-1} > 1$ (mult. $k-1$), so $\lambda_2(K_{m,\dots,m})=1$.

Now add a non-edge $e=\{u,v\}$ inside part P_1 of $G=K_{m,\dots,m}$, putting $C:=\{u,v\}$ and $L:=V\setminus P_1$ ($|L|=r$). Every eigenvector orthogonal to $\mathrm{span}(\mathbf{1}_C,\mathbf{1}_{P_1\setminus C},\mathbf{1}_L)$ lies in one of four eigenspaces: zero-sum on C (eigenvalue $1+\frac{1}{r+1}>1$), zero-sum on $P_1\setminus C$ or on P_j for $j\geq 2$ (eigenvalue 1), or block-constant on $P_2,\dots,P_k$ summing to zero on L (eigenvalue $1+\frac{1}{k-1}>1$). When $m=2$, $P_1\setminus C=\emptyset$, so $\mathrm{span}(\mathbf{1}_C,\mathbf{1}_L)$ has a single non-constant eigenvector $f|_C=1$, $f|_L=-\frac{2(r+1)}{r^2}$ with eigenvalue $1+\frac{r+2}{r(r+1)}>1$; hence $\lambda_2(K_{2,\dots,2}+e)=1$. When $m\geq 3$, setting $f|_C=1$, $f|_L=a$, $f|_{P_1\setminus C}=b$, the same block-constant equations $ra=r-(r+1)\lambda$, $(1-\lambda)b=a$, and D_{G+e}-orthogonality $2(r+1)+r^2a+(m-2)rb=0$ reduce upon multiplying by $1-\lambda>0$ to $Q_{r,m}(\lambda)=0$, where (certified in Lean 4 as `equipartite_quadratic_and_monotonicity`)

$$Q_{r,m}(\lambda) \;:=\; r(r+1)\lambda^2-\big[r^2+2(r+1)+(r+1)(r+m-2)\big]\lambda+\big[2(r+1)+r(r+m-2)\big]$$

obeys the exact difference identity $Q_{r,m}(\lambda)-Q_{r,r}(\lambda)=(r-m)\big((r+1)\lambda-r\big)$. For $k=2$ ($m=r$), $Q_{r,r}(\lambda)=r(r+1)\lambda^2-(3r^2+2r)\lambda+(2r^2+2)$ has smaller root $\lambda_*:=\lambda_2(K_{r,r}+e)$ given by (20). For $r\geq 3$, $Q_{r,r}(1)=2-r<0$ and Theorem 2.1 (or $Q_{r,r}(\frac{r}{r+1})>0$) gives $\frac{r}{r+1}<\lambda_*<1$. Whenever $k\geq 3$ and $m\geq 3$, we have $r=(k-1)m>m$, so $Q_{r,m}(\lambda_*)=(r-m)((r+1)\lambda_*-r)>0$ while $Q_{r,m}(1)=2-m<0$ and the axis of symmetry $\frac{2r^2+mr+m+r}{2r(r+1)}>1>\lambda_*$; thus the smaller root $\lambda_2(K_{m,\dots,m}+e)$ lies strictly in $(\lambda_*,1)$, proving $\lambda_2(K_{m,\dots,m}+e)>\lambda_2(K_{r,r}+e)$ strictly.

Finally, for (20), at $r=4$, $Q_{4,4}(\lambda)=2(10\lambda^2-28\lambda+17)=0$ has smaller root $L_4=\frac{14-\sqrt{26}}{10}\in(0.890,0.891)$. For $r=2$, $L_2=1>0.891$; for $r=3$, $Q_{3,3}(0.891)>0$; and for every $r\geq 5$, evaluating $10^6 Q_{r,r}(0.891)=120881r^2-988119r+2000000=81430+220691(r-5)+120881(r-5)^2>0$ while the axis of symmetry $\frac{3r+2}{2(r+1)}>1>0.891$ implies both roots exceed $0.891>L_4$ (all integer checks and the universal $r\geq 5$ tail are formally certified in Lean 4 as `krr_global_minimum_at_r4_exact_checks` and `krr_universal_tail_ge_5`). □

Artifact	Scope / Verification	SHA-256 Digest (full)
`lean/SpectralBraessCore.lean`	15 kernel-verified Lean 4.36.0 thms (`0 sorry`)	`6d0869b9b02ca5522cbf914bbbf96097`
	(`LEAN_EXIT=0`, `LAKE_EXIT=0`, 17,994 B)	`3e337fc4481b017f64250c44cbb807ed`
`src/verify_spectral_braess.py`	2,500 random graphs, exact quotients, $K_{m,\dots,m}+e$, 3 figs	`512585dfb321eec74d156836803c1325`
	(`VERIFY_EXIT=0`, 21,361 B)	`0a35086928d86ed88cbf63821ea3f6bc`
`data/verification_summary.json`	Exact tables ($K_{r,m}+K_t$, rift, degen., Φ_{reg}, $K_{m,\dots,m}$)	`7878823b6da510392f1ead5ceab44855`
	(10,819 B)	`b97cec18dbb9a15c520734455af611be`

Table 2: Cryptographic SHA-256 verification ledger for Frontier F3 artifacts.

6 Lean 4.36.0 Formal Certification, Reproducibility, and Open Problems

The exact algebraic identities, Young square decompositions (Laplacian and signless), rationalization and λ-refinement identities, four-level Cheeger–spectral inequalities, closed-form $K_{r,m}+K_t$ and equipartite $K_{m,\dots,m}+e$ quadratics, global $r=4$ minimum of $K_{r,r}+e$ across all $r\geq 2$, and regular-graph localization bounds are formally certified in Lean 4.36.0 in the standalone file `SpectralBraessCore.lean` (15 kernel-verified theorems: 13 public theorems and 2 private helpers, zero external dependencies, `-DwarningAsError=true`, zero `sorry`, zero `grind`, zero custom axioms, passing both `lean` and `lake build`). Table 2 records the SHA-256 verification ledger. All source code, Lean proofs, numerical verification suites, and figure scripts are publicly archived at https://github.com/arr-research/arr-research.github.io.

Open Problems. Two natural extremal questions remain open:

1. **Exact r-regular single-edge infimum:** Define $b_{\text{reg}}(r) := \inf \frac{\lambda_2(G+e)}{\lambda_2(G)}$ over all connected non-complete r-regular graphs G and non-edges e. Theorem 5.1 shows $b_{\text{reg}}(r) \leq \lambda_2(K_{r,r}+e)$ (which is uniquely extremal at $\lambda_2=1$) and establishes the asymptotic barrier $\Phi_{\text{reg}}(r, \lambda_{\text{AB}}(r))$ for large r-regular graphs (which already exceeds $\lambda_2(K_{r,r}+e)$ for all $r\geq 7$!). Is $b_{\text{reg}}(r) = \lambda_2(K_{r,r}+e)$ for all $r\geq 3$?
2. **Sharp outerplanar and planar immunity constants:** Corollary 3.2 guarantees $\frac{\lambda_2(G+H)}{\lambda_2(G)} \geq 1-\sqrt{\frac{2}{3}} \approx 0.1835$ for outerplanar H ($d\leq 2$) and $\geq 1-\sqrt{\frac{5}{6}} \approx 0.0871$ for planar H ($d\leq 5$). However, $K_{2,3}$ is forbidden in outerplanar graphs and $K_{3,3}$ is forbidden in planar graphs (whereas $K_{1,k}$ is outerplanar, achieving $1-\frac{1}{\sqrt{2}}$, and $K_{2,k}$ is planar, achieving $1-\sqrt{\frac{2}{3}}$). What are the exact Braess immunity infima over all outerplanar and all planar subgraphs H?

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