The sharp thermal regime for entropy-budget minimax losses

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Abstract

We establish an extension of the entropy-budget minimax law for oscillator reset channels to a thermal attenuator in the transmitted branch. The necessary and sufficient regime for the full loss curves is environment occupation $n = o(E)$, where E is the input energy, with the channel fixed throughout each code. The full Shannon and fixed super-Shannon curves have the same bounded absolute loss as for an identity or pure-loss signal, relative to the thermal channel's own entanglement-assisted benchmark. The key argument combines a bounded Choi form, a largest-eigenvalue bound for near-capacity inputs, output pinching, and Audenaert's established telescopic relative-entropy inequality. It avoids a continuity error proportional to $\log E$. Explicit input/output truncation errors give the matching common-code bound without an energy constraint on the reset. A separate extension treats unknown fixed signal parameters in a known temperature envelope, with the same sharp sublinear criterion. Necessity follows from a strictly positive, dimension-independent gap below the binary flag entropy whenever the signal mutual information is bounded. The gap combines established purification/Gram, fidelity and relative-entropy inequalities; it is transferred to a common energy-constrained code for arbitrary fixed resets. The result does not determine the critical-regime minimax curve. Working-argument audits support these arguments; they are not a full assessment of this exact PDF, a priority certificate or editorial acceptance.

1 Model, scope and theorem

All logarithms in information quantities are base two; ln is natural. Let $H = \sum_{j\geq 0} j|j\rangle\langle j|$, $g(x) = (x+1)\log_2(x+1) - x\log_2 x$, and let h denote binary entropy. The thermal state of mean x is γ_x. A thermal attenuator $\Phi_{\kappa,n}$ mixes the input with an environment thermal state of mean n, through a beamsplitter of photon transmissivity $0 < \kappa < 1$. Write $\delta = 1-\kappa$, $q = 1-\lambda$, with $0 < \lambda < 1$, and

$$\mathcal{N}_{\kappa,n,\tau}(\rho) = \lambda\Phi_{\kappa,n}(\rho) + q\tau\,\mathrm{Tr}\,\rho.$$

The reset τ has no energy or orientation restriction. All parameters are fixed throughout a code block. The input constraint is expected physical block-average energy at most E, not peak energy. Entanglement assistance is unlimited. Classical error is average-message error. In a compound capacity one encoder and decoder serve the whole fixed-parameter family, with uniform error. The coding limit is taken at each fixed E, n, B, before the energy limit.

Put $\mathcal{T}_{\alpha,B} = \{\tau : S_\alpha(\tau) \leq B\}$, where $S_1 = S$ and $S_\alpha = (1-\alpha)^{-1}\log_2 \mathrm{Tr}\,\tau^\alpha$. The established constrained Gaussian assisted capacity [1, 2] is

$$C_{\kappa,n}(E) = g(E) + g(M) - g(u_l) - g(u_s), \qquad M = \kappa E + \delta n, \tag{C}$$
$$V = \delta E + (1+\kappa)n + 1, \qquad D = \sqrt{V^2 - 4\kappa n(n+1)},$$
$$u_l = \tfrac{1}{2}[D + \delta(E-n) - 1], \qquad u_s = \tfrac{1}{2}[D - \delta(E-n) - 1].$$

Here $g(u_l) + g(u_s)$ is the entropy of the *true two-mode* complement, including purification of the initial thermal environment. A one-mode weak complement cannot replace it.

Define the known-reset and unknown-reset deficits

$$D_{\mathrm{ind}}(E, n, B) = \lambda C_{\kappa,n}(E) - \inf_{\tau \in \mathcal{T}_{\alpha,B}} C_{\mathrm{EA}}(\mathcal{N}_{\kappa,n,\tau}, H, E),$$
$$D_{\mathrm{cmp}}(E, n, B) = \lambda C_{\kappa,n}(E) - C_{\mathrm{EA}}^{\mathrm{cmp}}(\{\mathcal{N}_{\kappa,n,\tau} : \tau \in \mathcal{T}_{\alpha,B}\}, H, E).$$

Thus $D_{\mathrm{ind}} \le D_{\mathrm{cmp}}$. Let

$$b(u, v) = (u + v)h\left(\frac{u}{u+v}\right), \quad f(a) = b(\lambda, qa),$$

$$t_\alpha(E, B) = \begin{cases} \min\{1, B/\log_2(E+1)\}, & \alpha = 1, \\ 1 - 2^{-((\alpha-1)/\alpha)B}, & 1 < \alpha < \infty. \end{cases}$$

Use continuous boundary values in b.

Theorem 1 (Growing-noise law)**.** *Fix $\kappa, \lambda \in (0, 1)$ and $\alpha = 1$ or a fixed finite $\alpha > 1$. For every prescribed path $n(E) > 0$ with $n(E)/E \to 0$,*

$$\sup_{0 \le B < \infty} \max_{Y \in \{\mathrm{ind}, \mathrm{cmp}\}} |D_Y(E, n(E), B) - f(t_\alpha(E, B))| \longrightarrow 0. \tag{P}$$

The argument below uses the scientific dependencies disclosed at the end. It claims neither a uniform blocklength nor uniformity over all asymptotic paths at a fixed energy without a common envelope.

This removes a sufficient $n = o(\sqrt{E})$ restriction of an earlier working proof, rather than establishing that restriction as a physical threshold. The proposed addition concerns operational quantifiers and uniform errors. Gaussian extremality, entropy mixing, Helstrom testing, telescopic relative entropy and compound coding are prior ingredients.

Theorem 2 (Sharp regime for the complete curves)**.** *Fix $\kappa, \lambda \in (0, 1)$, and either $\alpha = 1$ or a fixed finite $\alpha > 1$. For every finite nonnegative path $n(E)$, the complete uniform-in-budget law (P) holds if and only if $n(E) = o(E)$. For the unknown fixed signal family $\kappa \in [\kappa_0, 1]$, $n \in [0, N(E)]$, with fixed $0 < \kappa_0 < 1$ and finite $N(E) \ge 0$, its compound analogue relative to $\lambda C_{\kappa_0, N(E)}(E)$ holds if and only if $N(E) = o(E)$. There is no smoothness or monotonicity assumption on either path.*

Sufficiency is Theorem 1 and Proposition 4; zero-temperature subsequences use the previous pure-loss law [9]. Necessity, proved below, excludes the complete curve on a growing budget. It does not assert necessity for every restricted small-budget problem or give the actual critical-temperature loss curve.

2 A uniform Gaussian entropy estimate

The input/output energy bounds are $S(\rho) \le g(E)$, $S(\Phi_{\kappa,n}(\rho)) \le g(M)$. For input mean x, the true complementary entropy is bounded by $g(\delta x + \kappa n) + g(n)$. Let $\beta_x = \log_2(1 + 1/x)$ for $x > 0$. The complementary Gibbs logarithm pulled back to the input is a quadratic form $a(E; n)H + b(E; n)I$, with

$$a(E; n) = \delta\left[\beta_{u_l} + (\beta_{u_l} + \beta_{u_s})r\right], \qquad r = \frac{1}{2}(V/D - 1).$$

For clarity, a direct verification uses the environment modes $e' = -\sqrt{\delta}\, a + \sqrt{\kappa}\, e$ and purifier z. With $\cosh(2s) = V/D$, $\sinh(2s) = 2\sqrt{\kappa n(n+1)}/D$, the thermal normal modes are $x =$

$\cosh(s)e' - \sinh(s)z^\dagger$ and $y = \cosh(s)z - \sinh(s)e'^\dagger$. The logarithm is a constant plus $\beta_{u_l} x^\dagger x + \beta_{u_s} y^\dagger y$. Pullback of $e'^\dagger e'$, $z^\dagger z$, and $e'z + e'^\dagger z^\dagger$ is respectively $\delta H + \kappa n I$, nI, and $2\sqrt{\kappa n(n+1)}I$, giving the displayed coefficient. These are positive-form/finite-mean expectation identities on the number form domain. They are not bounded-adjoint identities for an unbounded logarithm. Finite Fock cutoffs and finite energy justify their limiting expectations. Differentiating the Gibbs cross entropy at fixed n also gives $a(E;n) = \partial_E[g(u_l) + g(u_s)]$.

Relative-entropy data processing with reference γ_E gives

$$S(\rho) - S(\Phi^c_{\kappa,n}(\rho)) \le g(E) - g(u_l) - g(u_s) + \zeta, \qquad \zeta = E|\beta_E - a(E;n)|. \tag{1}$$

No optimality of thermal reverse coherent information is assumed. For $0 < n \le E$, the exact radical obeys

$$D \ge \delta(E+n) + 1, \quad \delta E \le u_l \le \delta E + \kappa n, \quad \delta n \le u_s \le n, \quad 0 \le r \le \frac{\kappa n(n+1)}{\delta^2 E^2}.$$

The elementary inequalities $0 \le x^{-1} - \ln(1+x^{-1}) \le (2x^2)^{-1}$ imply

$$\zeta \le \Xi(E,n) := \frac{1 + 1/\delta}{2E \ln 2} + \frac{\kappa n}{\delta E \ln 2} + \frac{2\kappa(n+1)}{\delta^2 E \ln 2}. \tag{2}$$

Thus $\Xi = o(1)$ throughout $n = o(E)$. All derivatives hold n fixed; $n'(E)$ is never used.

3 Flat Choi order and near-capacity inputs

Lemma 3 (Known bounded Choi form, applied here)**.** *For the reference/output state* $\omega = (\mathrm{id} \otimes \Phi_{\kappa,n})(\psi_\rho)$ *of the canonical purification,*

$$\omega \le c\, \rho_R \otimes I_B, \qquad c = \frac{1}{\delta(n+1)}. \tag{3}$$

This bound is an exact specialization of Holevo's established Gaussian Choi norm formula [7, Proposition 2 and the one-mode example, p. 13]. His amplitude parameter is $\sqrt{\kappa}$ and his noise coefficient is $m = \delta(n + 1/2)$, so the norm is $[m + |\kappa - 1|/2]^{-1} = c$. The following regularization argument also explains the application to the present purification; the Choi norm formula is not a new lemma of this paper.

Proof. Use an auxiliary thermal input of mean $x > 0$, keeping n fixed. Its reference/output state ω_x has the same nonzero spectrum as the true Gaussian complement. Formula (C) gives

$$(u_l + 1)(u_s + 1) = \delta x n + \frac{\delta(x+n)}{2} + \frac{D+1}{2} \ge \delta(n+1)(x+1).$$

If J is the canonical positive Choi form and $G_x = (x/(x+1))^{H/2}$, then $(G_x \otimes I)J(G_x \otimes I) = (x+1)\omega_x \le cI$. For a vector with finite input-number support, insert G_x^{-1}; its norm converges to the original norm as $x \to \infty$. The positive form therefore has bound cI on a dense domain and extends to a bounded positive operator. Sandwiching with $\sqrt{\rho^T} \otimes I$ gives (3), first for finite truncations and then in trace norm. No unnormalizable Bell vector is used as a state. □

Put $C = C_{\kappa,n}(E)$. At fixed n,

$$0 \le \partial_E C \le \frac{2}{E \ln 2}. \tag{4}$$

The upper bound follows from $\partial_E C = \beta_E + \kappa\beta_M - a(E;n)$ and $a \ge 0$; monotonicity follows from enlargement of the feasible input set.

If $w = \|\rho\|_\infty$, isolate a largest pure eigencomponent. Mutual information of that pure input is zero. The entropy-of-mixture inequality, whose general $2h(w)$ version is prior [6, Eq. (11), Section 5.2], yields

$$I(\rho, \Phi_{\kappa,n}) \le 2h(w) + (1-w)C_{\kappa,n}(E/(1-w)) \le (1-w)C + 2h(w) + 2(1-w)\log_2 \frac{1}{1-w}.$$

Consequently

$$I(\rho, \Phi_{\kappa,n}) \ge C - \Delta \quad \Longrightarrow \quad w \le \frac{\Delta + 2 + 2/(e \ln 2)}{C}. \tag{5}$$

The endpoint $w = 1$ is excluded eventually by $C > \Delta$. Let $S = c(M+1)$. Since $u_l \le E$, $u_s \le n$ for $n \le E$,

$$C \ge g(M) - g(n) \ge \log_2 S + \log_2 \delta - \log_2 e. \tag{6}$$

Thus $C \to \infty$ whenever $n = o(E)$. The established regular-channel identity $\Phi_{\kappa,n}(I) = I/\kappa$ [3, Proposition 2] gives

$$\Pr_{\Phi_{\kappa,n}(\rho)}(0) \le w/\kappa = o(1) \tag{7}$$

on the set in (5).

4 Pinching and branch discrimination

Let $\tau_a = (1-a)|0\rangle\langle 0| + a\gamma_M$, $0 \le a \le 1$. Flagging gives $I(\rho, \mathcal{N}_{\kappa,n,\tau_a}) \le \lambda I(\rho, \Phi_{\kappa,n})$. Inputs below $\lambda C - h(\lambda) - 1$ already satisfy the needed converse because $f(a) \le h(\lambda)$. All remaining inputs satisfy (5) with $\Delta = (h(\lambda)+1)/\lambda$, independent of E, n, a.

Set $\sigma_0 = \rho_R \otimes I_B$, an *unnormalized* infinite-trace operator, and $D_0(\omega) = \operatorname{Tr} \omega(\log_2 \omega - \log_2 \sigma_0) = -H(B|R)_\omega$. Input/output energy ensures finite entropies and cross entropies. Equation (3), (6), and $g(u_l) + g(u_s) \le g(E) + g(n)$ give

$$\log_2 c - C_0 \le D_0(\omega) \le \log_2 c, \qquad C_0 = \Delta + \log_2 e - \log_2 \delta. \tag{8}$$

We have not applied a normalized-state relative-entropy inequality to σ_0.

For $S > 1$, let $t = \sqrt{\ln S}$, $T = \lceil (M+1)t \rceil$, $P_B = \sum_{j=1}^{T} |j\rangle\langle j|$, $P = I_R \otimes P_B$, and $p = \operatorname{Tr} P\omega$. Equations (5),(7) and Markov give $p \ge 1 - w/\kappa - 1/t \to 1$. Define normalized good and bad blocks ω_g, ω_b. For these blocks D_0 always uses the original reference ρ_R; in particular

$$D_0(\omega_g) = -H(B|R)_{\omega_g} + D(\rho_R^g \| \rho_R).$$

It is not the negative conditional entropy computed with its own reference marginal alone. Pinching preserves the reference marginal and increases entropy by at most $h(p)$. Its block formula is

$$D_0(\mathcal{P}\omega) = pD_0(\omega_g) + (1-p)D_0(\omega_b) - h(p).$$

Hence the weighted sum on the right without $-h(p)$ is at least $D_0(\omega)$. By (3), $D_0(\omega_b) \le \log_2 c - \log_2(1-p)$, and therefore

$$D_0(\omega_g) \ge \log_2 c - \frac{C_0 + h(p)}{p}. \tag{9}$$

Zero-weight blocks are interpreted by continuity. Reference marginals are bounded by ρ_R/p, $\rho_R/(1-p)$, keeping their cross entropies finite. This step avoids an error of the form $o(1)\log E$.

Let $v = \operatorname{Tr} P_B \gamma_M \to 1$, $r_M = M/(M+1)$, and introduce the *normalized* comparator

$$\sigma_g = \rho_R \otimes P_B \gamma_M P_B / v, \qquad L = (c/p)v(M+1)r_M^{-T}.$$

The least thermal eigenvalue on this window gives $\omega_g \le L\sigma_g$. Writing $\mu_g = \operatorname{Tr} H_B\omega_g \ge 0$, (9) gives

$$\begin{aligned} D(\omega_g\|\sigma_g) &= D_0(\omega_g) + \log_2(M+1) + \beta_M\mu_g + \log_2 v \\ &\ge \log_2 S + \log_2 v - (C_0+1)/p, \\ \log_2 L &= \log_2 S + \log_2 v - \log_2 p + T\beta_M = \log_2 S + O(\sqrt{\ln S}). \end{aligned} \tag{10}$$

In particular the sign of the normalization term is $+\log_2 v$.

For normalized states $r \le Ls$, Audenaert's established telescopic inequality [8, Theorem 4] implies

$$D(r\|s) \le \frac{\log_2 L}{1-1/L}\,\frac{1}{2}\|r-s\|_1. \tag{11}$$

Indeed write $s = L^{-1}r + (1-L^{-1})z$ and use that theorem, together with $\frac{1}{2}\|r-s\|_1 = (1-L^{-1})\frac{1}{2}\|r-z\|_1$. For infinite reference, common finite compressions with an orthogonal failure flag preserve normalization and operator order. Their embedded states converge in trace norm. Lower semicontinuity of relative entropy and data processing give convergence of the relative entropies, proving (11) in this case. Equation (10) now forces $\frac{1}{2}\|\omega_g - \sigma_g\|_1 \to 1$.

The selected proof of Audenaert's Theorem 4 in v2 has a printed sign inconsistency on p. 10: after defining a difference proportional to $\rho - \sigma$, it later uses the opposite orientation. Taking the auxiliary difference as $(\sigma-\rho)/(\frac{1}{2}\|\rho-\sigma\|_1)$ restores that displayed identity and leaves the positive-part norm bound and theorem unchanged. The use of (11) is the established theorem, not a new inequality.

A Helstrom projection $Q \le P$ has both normalized testing errors tending to zero. Its acceptance on the original signal is $p\operatorname{Tr} Q\omega_g \to 1$. On the original reset it is

$$\operatorname{Tr}[Q(\rho_R \otimes \tau_a)] = av\operatorname{Tr} Q\sigma_g \to 0,$$

because the vacuum component is excluded. These bounds are uniform in a and in all relevant inputs. Binary Fano gives $H(F|RB) = o(1)$. This analytical test may depend on the input; it is a converse device, not the common operational decoder.

5 All-input converse and the matching common code

The exact binary-branch chain rule is

$$I(\rho,\mathcal{N}_{\kappa,n,\tau}) = F_\tau(\rho) + H(F|RB), \quad F_\tau = \lambda[S(\rho) - S(\Phi^c_{\kappa,n}(\rho))] + S(\lambda\Phi_{\kappa,n}(\rho) + q\tau) - qS(\tau) - h(\lambda).$$

For the finite-entropy witness τ_a, entropy mixing and (1) give

$$F_{\tau_a} \le \lambda C - f(a) + \lambda\Xi + qh\left(\frac{a}{M+1}\right). \tag{12}$$

To see the mixing step, group the signal and thermal reset, of total weight $P_a = \lambda + qa$, against the vacuum. The normalized group has mean at most M, so its entropy is at most $g(M)$. Moreover $S(\tau_a) \ge ag(M) + h(a) - h(a/(M+1))$. The identity $h(P_a) - qh(a) - h(\lambda) = -b(\lambda, qa)$ then gives (12). Combining (12) with the preceding branch test bounds *every* admissible input. Evaluation at γ_E supplies the other direction:

$$f(a) - o(1) \le \lambda C - C_{\mathrm{EA}}(\mathcal{N}_{\kappa,n,\tau_a}, H, E) \le f(a) + b(\lambda/(M+1), q). \tag{13}$$

Uniformity in a is essential for changing budgets.

Here is an explicit common error envelope for the converse. Let $C_\Delta = \Delta + 2 + 2/(e \ln 2)$, $H_S = \log_2 S + \log_2 \delta - \log_2 e$, and, eventually,

$$\begin{aligned} w_* &= C_\Delta / H_S, & p_* &= 1 - w_*/\kappa - 1/t, \\ v_* &= 1 - (1+\kappa E)^{-1} - e^{-t}, & K_* &= -\log_2 p_* + (2t+1)/\ln 2 + (C_0+1)/p_*, \\ \varepsilon_* &= (S v_*)^{-1} + K_*/\log_2(S v_*), & z_* &= \lambda(1-p_*) + \varepsilon_*. \end{aligned}$$

These formulas are used only when $H_S > 0$, $p_*, v_* > 0$, $S v_* > 1$, $M \geq 1$, and $z_* \leq 1/2$, all eventual conditions in the joint limit. From (10),(11), $1 - \frac{1}{2}\|\omega_g - \sigma_g\|_1 \leq \varepsilon_*$. Thus a valid error in the left side of (13) is

$$R_- = \lambda \Xi(E, n) + q h((1+\kappa E)^{-1}) + h(z_*) \longrightarrow 0. \tag{13a}$$

This exposes the dependence on E, n, rather than assigning a common rate to every unspecified sublinear trajectory.

For completeness the achievable bound does not use that witness. For $E > 1$, let

$$L_{\text{in}} = \lceil 4E \ln(E+1) \rceil, \quad \epsilon = (1 - 1/E)^{L_{\text{in}}} \leq (E+1)^{-4}.$$

Truncate γ_{E-1} to its first L_{in} levels and normalize, obtaining σ of mean strictly below E. To expose a uniform bound, write $\gamma_{E-1} = (1-\epsilon)\sigma + \epsilon\theta$. The normalized tail θ is a shifted thermal state, with $S(\theta) = g(E-1)$. Mutual-information mixing and $I(\theta, \Phi_{\kappa,n}) \leq 2S(\theta)$ give

$$C_{\kappa,n}(E-1) - I(\sigma, \Phi_{\kappa,n}) \leq \frac{2h(\epsilon) + 2\epsilon g(E-1)}{1-\epsilon}.$$

Since $\epsilon < 1/16$ for $E > 1$, this is bounded by the larger entropy error below. Adding (4) at the same n yields

$$I(\sigma, \Phi_{\kappa,n}) \geq C - U_E, \quad U_E = 3h(\epsilon) + 4\epsilon[g(E) + g(E + L_{\text{in}})] + 2\log_2 \frac{E}{E-1} = o(1). \tag{14}$$

The bound is uniform for $n \leq E$. It compares inputs of energies $E-1, E$ at fixed channel, not $n(E-1)$ with $n(E)$.

Apply one receiver Fock cutoff $T_{\text{out}} = \lceil E^4 \rceil$ and replace its complement by an orthogonal failure symbol. The signal failure mass is at most E^{-3}; no energy assumption is imposed on the reset. With

$$e_E = E^{-3/2} + \tfrac{1}{2}E^{-3}, \quad W_E = 2e_E \log_2 L_{\text{in}} + (1+e_E) h(e_E/(1+e_E)),$$

finite-reference continuity [4, Lemma 2] loses at most $W_E = o(1)$. The compressed signal norm is at most

$$v_E = \max\left\{ \frac{1}{[1+\kappa(E-1)](1-\epsilon)}, E^{-3} \right\}.$$

A top-m reset spectral subspace has retained signal mass at most $(m+1)v_E$, since the failure flag increases rank by at most one. If η_m bounds its original discarded spectral mass, the binary observation of that subspace gives

$$H(F|B) \leq b(\lambda, q\eta_m) + b(\min\{\lambda, \lambda(m+1)v_E\}, q).$$

The branch chain rule thus yields

$$D_{\text{cmp}} \leq f(\eta_m) + R_+(E, m), \quad R_+ = \lambda(U_E + W_E) + b(\min\{\lambda, \lambda(m+1)v_E\}, q), \tag{15}$$

which vanishes if $m/E \to 0$.

Here is the operational step behind (15). After the two fixed cuts the channel family has finite input/output dimension, though its reset index can be arbitrary. Approximate the diagonal input σ by rational probabilities while retaining strict mean-cost slack. On the exact type subspace of a type length k_{type}, every vector has total physical number $k_{\text{type}} \sum_j j p_j < k_{\text{type}} E$. For completeness, let π_T be the maximally mixed exact-type state and put $d_k = D(\pi_T \| \sigma_p^{\otimes k}) = kH(p) - S(\pi_T) = O(\log k)$. Its one-site marginals equal σ_p. Data processing bounds the output total correlation by d_k, while complementary entropy is subadditive. Thus

$$I(\pi_T, \mathcal{M}^{\otimes k}) \geq kI(\sigma_p, \mathcal{M}) - 2d_k$$

uniformly in every finite channel $\mathcal{M}$. This type estimate is also given in [9, Lemma 3]; it is distinct from the compound achievability result of [5, Lemma 9]. Apply the finite-dimensional compound assisted coding theorem [5, Theorem 1] on this subspace. First fix the type and all cuts for a rate slack, then take the outer coding length to infinity; only subsequently optimize the finite choices and take $E \to \infty$. This preserves physical expected cost and provides a single encoder and decoder for all resets. It supplies no uniform blocklength or efficient code. These are uses of established coding results, not a new coding theorem.

6 Budget normalization and unknown signal envelopes

For Shannon entropy, $\eta_m \leq \min\{1, B/\log_2(m+1)\}$. Choose $m = \lfloor E/\ln(E+1) \rfloor$. Its clipping curve differs uniformly from $t_1(E, B)$ by $o(1)$. For the converse choose

$$a_B = \text{clip}_{[0,1]} \frac{B-1}{g(M)}.$$

The witness is feasible because $S(\tau_a) \leq h(a) + ag(M)$; the clipped endpoints are feasible separately. Since $\kappa E \leq M \leq E$, $\log_2 \kappa \leq g(M) - \log_2(E+1) \leq \log_2 e$. It follows that $\sup_B |a_B - t_1(E, B)| = o(1)$. For fixed $\alpha > 1$, choose $a_B = 1 - 2^{-((\alpha-1)/\alpha)B}$ and $m = \lfloor \sqrt{E} \rfloor$. The witness obeys $\operatorname{Tr} \tau_a^\alpha \geq (1-a)^\alpha$, hence is feasible. The established spectral-tail estimate gives $\eta_m \leq \min\{1, a_B + (m+1)^{(1-\alpha)/\alpha}\}$. Since f is increasing and concave, with $|f(x) - f(y)| \leq f(|x-y|)$, (13),(15) give (P). Resets of infinite von Neumann entropy can occur for $\alpha > 1$; their achievability is evaluated only after finite output compression. Uncompressed reset entropies are subtracted only for the finite-entropy witness.

Proposition 4 (Common code for unknown signal)**.** *Let known $0 < \kappa_0 < 1$ and $N(E) > 0$ satisfy $N(E) = o(E)$. Suppose $\kappa \in [\kappa_0, 1]$, $n \in [0, N(E)]$, and $\tau \in \mathcal{T}_{\alpha,B}$ are all unknown and fixed throughout the block. Then the compound deficit relative to $\lambda C_{\kappa_0, N(E)}(E)$ is $f(t_\alpha(E, B)) + o(1)$, uniformly in finite B. The zero-envelope case is the earlier pure-loss result, not a substitution in singular thermal logarithms.*

Proof. For signal channels only, add receiver Gaussian noise of mean $(1-\kappa)(N-n)$, raising the environment parameter to N. Then compose with a thermal attenuator of transmissivity κ_0/κ and environment occupation N. Its resulting noise is $(1-\kappa_0)N$, so the composition is $\Phi_{\kappa_0, N}$. Thus mutual information of the same input σ for every original signal is at least its value for that worst pair. The cuts, U_E, W_E, and the signal-norm bound in (14),(15) are uniform with $\kappa \geq \kappa_0, n \leq E$, replacing κ by κ_0 in the norm bound. Apply the finite-family-independent coding step to the entire signal/reset index set. For the converse choose the worst pair and its witness. Do not degrade the full reset mixture: receiver processing could change the reset entropy and leave the budget class. □

7 A strict flag-loss gap at bounded signal information

The following combines established binary-ensemble inequalities. In particular the purification/Gram bound in its proof is already explicit in [10, Corollary 4, Lemma 5] and [11, Eqs. (3)–(6)]; it is not a new lemma claimed here. Use root fidelity $F(r,s) = \|\sqrt{r}\sqrt{s}\|_1$, rather than its square, and Bures distance $d_B(r,s) = \sqrt{2(1-F(r,s))}$.

Lemma 5 (Strict loss bound)**.** *Let ω_{RB} be normalized on separable spaces, with $c = I(R:B)_\omega < \infty$. For an arbitrary state τ_B, put $\sigma = \omega_R \otimes \tau$, $z = \lambda\omega + q\sigma$ and $\epsilon_\lambda(x) = 2\lambda q\, 2^{-x}/(3\ln 2)$. Then*

$$0 \le \lambda c - I(R:B)_z \le h(\lambda) - \epsilon_\lambda(c). \tag{16}$$

No energy or entropy condition on τ is required.

Proof. Adjoin the binary branch flag X. The two branches have the same reference marginal, so $I(R:X) = 0$, $I(R:BX) = \lambda c$. Writing $\chi_{RB} = I(X:RB)$, $\chi_B = I(X:B)$, the loss L obeys

$$L = \chi_{RB} - \chi_B, \qquad h(\lambda) - L = \chi_B + H(X|RB). \tag{17}$$

All flag quantities are finite. These are chain-rule identities, not subtractions of possibly infinite reset or output entropies.

The established bound $D(r\|s) \ge -2\log_2 F(r,s)$ [12] gives $D(r\|s) \ge d_B(r,s)^2/\ln 2$. With $m = \lambda\omega_B + q\tau$ and $d = d_B(\omega_B, \tau)$, weighted Cauchy–Schwarz and the Bures triangle inequality yield

$$\chi_B \ge \frac{\lambda d_B(\omega_B, m)^2 + q d_B(\tau, m)^2}{\ln 2} \ge \frac{\lambda q d^2}{\ln 2}. \tag{18}$$

Let $f_0 = F(\omega, \sigma)$. Uhlmann purifications of these states have overlap arbitrarily close to f_0. Holevo information cannot increase under partial trace, whence the known Gram bound is

$$\chi_{RB} \le S(G), \qquad G = \begin{pmatrix} \lambda & \sqrt{\lambda q} f_0 \\ \sqrt{\lambda q} f_0 & q \end{pmatrix}.$$

Since G has diagonal (λ, q), qubit Pinsker gives

$$H(X|RB) \ge h(\lambda) - S(G) = D(G\|\operatorname{diag}(\lambda, q)) \ge \frac{2\lambda q f_0^2}{\ln 2}. \tag{19}$$

Set $a = F(\omega, \omega_R \otimes \omega_B)$. Choose purifications of this pair with overlap approaching a, and a purification of σ at vector distance approaching d from the second one. Their overlaps show $f_0 \ge (a-d)_+$. Also $a^2 \ge 2^{-c}$. Consequently (17)–(19) and the scalar inequality

$$d^2 + 2(a-d)_+^2 \ge \tfrac{2}{3}a^2$$

prove (16). For $d \le a$ the remainder is $3(d - 2a/3)^2$; for $d \ge a$ it follows from $d^2 \ge a^2$.

The infinite-dimensional steps do not require finite separate entropies. For completeness, finite-dimensional $D \ge -2\log_2 F$ follows by spectral Jensen, followed by $\operatorname{Tr}\sqrt{r}\sqrt{s} \le \|\sqrt{r}\sqrt{s}\|_1$. Common finite-rank compression with an overflow flag, data processing for D, and trace-norm continuity of fidelity give the separable version, including singular supports by limits. The Gram matrix always has dimension two, and approximate purifications suffice. The flag chain rules are well-defined because c is finite and the flag is finite. Thus no infinity-minus-infinity expression occurs. □

Proposition 6 (Operational gap for all fixed resets)**.** *For each fixed $E > 0$ and finite $n \ge 0$, the full compound reset family, and hence every entropy-budget subfamily, has capacity at least*

$$\lambda C_{\kappa,n}(E) - h(\lambda) + \epsilon_\lambda(C_{\kappa,n}(E)). \tag{20}$$

The same bound holds for every known reset. For unknown fixed signal parameters $\kappa \in [\kappa_0, 1]$, $n \in [0, N]$, it holds with $C_{\kappa_0,N}(E)$, for every finite E, N, even $N \gg E$.

Proof. Write $C_* = C_{\kappa,n}(E)$, or $C_{\kappa_0,N}(E)$ for the envelope, and $G_\lambda(x) = \lambda x + \epsilon_\lambda(x)$. It is strictly increasing for $x \geq 0$, since $G'_\lambda(x) = \lambda - (2\lambda q/3)2^{-x} > 0$. Pointwise application of the state lemma alone is insufficient to establish a common operational code, so we give that step explicitly.

Fix an error in rate $\eta > 0$. Start with γ_{E-s}, where $s > 0$ leaves physical energy slack. Condition it on input levels $0, \ldots, J-1$, giving σ_J of mean at most $E - s$. Its purifications converge to those of the thermal state. Lower semicontinuity of mutual information and thermal optimality at energy $E-s$ imply $I(\sigma_J, \Phi_*) \to C_*(E-s)$. Choose s, J with $I(\sigma_J, \Phi_*) \geq C_* - \eta/2$. The signal degradation used in Proposition 4 compares this same input with every signal in the envelope, for arbitrary finite N, not just $N \leq E$.

Now apply the physical output cutoff $\mathcal{D}_T$, retaining levels $0, \ldots, T$ and sending the rest to an orthogonal flag. All signal outputs have mean at most $E + N$. Their cutoff tail is at most $p_T = (E+N)/(T+1)$. After insertion into a common space the trace distance is at most $e_T = \sqrt{p_T} + p_T/2$. With the fixed reference dimension J, Winter's conditional-entropy bound [4, Lemma 2] bounds the information error by

$$2e_T \log_2 J + (1 + e_T) h(e_T/(1+e_T)) \longrightarrow 0.$$

Thus one finite T gives signal information at least $C_* - \eta$ for all signals. The output entropy of the reset was never bounded. Each effective mixture is finite-dimensional and has arbitrary reset $\mathcal{D}_T(\tau)$. Lemma 5 and monotonicity of G_λ give the same information lower bound $G_\lambda(C_* - \eta) - h(\lambda)$ for every channel, taking $\eta < C_*$.

For clarity, the cost-preserving type step can be made exact. Approximate the Fock probabilities of σ_J by a type p_l with mean at most E, and let π_l be uniform on that exact type subspace. Put $\rho_l = \mathrm{diag}(p_l)$ and $d_l = D(\pi_l \| \rho_l^{\otimes l}) \leq J \log_2(l+1)$. For any effective finite channel M, its output and complementary total correlations $D_{\mathrm{out}}, D_{\mathrm{env}}$ satisfy

$$I(\pi_l, M^{\otimes l}) = lI(\rho_l, M) - d_l - D_{\mathrm{out}} + D_{\mathrm{env}} \geq lI(\rho_l, M) - 2d_l.$$

Here $D_{\mathrm{out}} \leq d_l$ by data processing; both relative entropies are finite by domination of the type state by its product state. Finite reference dimension makes $I(\rho_l, M) \to I(\sigma_J, M)$ uniform in M. Apply the arbitrary-index finite compound coding theorem [5, Theorem 1, Lemma 9] on this same type subspace. The classical assisted rate is twice its quantum assisted rate by superdense coding. Every encoded state has mean physical cost at most E per use. First take the outer coding limit at fixed cuts and type; then optimize the finite type, cuts and slack. This proves (20). The cutoff is implemented at the receiver of the original channel. No uniform blocklength in E, N, or changing channel within a block, is asserted. □

8 Necessity of the sublinear thermal regime

Proof of necessity in Theorem 2. For fixed $r > 0$, put $n = rE$. In (C), $M/E \to \kappa + \delta r$ and

$$D/E \to d_r = \sqrt{[\delta + (1+\kappa)r]^2 - 4\kappa r^2}, \qquad u_{l,s}/E \to \tfrac{1}{2}[d_r \pm \delta(1-r)].$$

The identity $d_r^2 - \delta^2(1-r)^2 = 4\delta r > 0$ shows that both complementary occupations grow linearly with positive coefficients and $u_l u_s/E^2 \to \delta r$. The expansion $g(x) = \log_2 x + \log_2 e + o(1)$ therefore yields

$$C_{\kappa,rE}(E) \longrightarrow K_r = \log_2\left(1 + \frac{\kappa}{\delta r}\right). \tag{21}$$

If $n(E)$ is not $o(E)$, there are $r > 0$ and an unbounded subsequence on which $n(E) \geq rE$. Receiver Gaussian-noise order and (21) give $C_{\kappa,n(E)}(E) \leq K_r+1$ eventually there. Proposition 6 implies, uniformly over every budget,

$$D_Y(E, n(E), B) \leq h(\lambda) - \epsilon_\lambda(K_r + 1), \qquad Y \in \{\mathrm{ind}, \mathrm{cmp}\}. \tag{22}$$

Take $B(E) = \log_2(E + 1)$. The target curve is exactly $h(\lambda)$ for Shannon, and tends to it for every fixed finite $\alpha > 1$. The positive constant in (22) prevents the complete uniform law. This also covers bounded signal benchmarks larger than the flag entropy, where nonnegativity of capacity alone would not suffice.

For the envelope use the same reasoning at the worst signal $(\kappa_0, N(E))$, with the compound version of (20). Its construction was valid at every finite E, N. If N/E fails to vanish there is the same positive subsequential separation. The preceding sublinear proof, including pure-loss subsequences, supplies the other implication. □

9 Evidence, dependencies and remaining problems

The strict-gap constant is not claimed optimal. The actual critical-regime minimax curve, quantitative uniform blocklength bounds and arbitrary within-block variation remain outside this result. Neither (21) nor (22) determines the critical loss curve.

The current numerical evidence consists of 90 high-precision scalar controls for the earlier $o(\sqrt{E})$ working argument and 72 noncommuting finite-state controls for the pinching/order bookkeeping. The latter test generic finite matrices, not the infinite Gaussian Choi form or an operational capacity. Neither set proves the asymptotic theorem. Tests, frozen inputs and reports are retained separately. An additional 1080 generic finite-state controls of the strict-loss inequalities were reproduced byte for byte during technical audit627. They corroborate only finite diagnostics; the state and operational arguments above are the proof, conditional on the cited dependencies. The accompanying supplement provides the two unchanged programs, expected outputs, dependency instructions and a replay record. Both outputs were reproduced byte for byte during packaging. The package does not implement an operational code or certify an infinite-dimensional optimizer.

Model assessments and review context

The named author directed substantial OpenAI Codex assistance in derivation, writing and numerical work. Requested GPT-6 Astra xhigh working-argument audits supported the fixed-temperature and $o(\sqrt{E})$ versions with their stated limits. They used shared files and reused context. Audit615 of the frozen broader working argument614 has since closed with mathematical support and minor attribution/notation clarifications, including the Choi form, fixed-reference pinching, infinite-reference limit, full two-sided minimax bounds and unknown-signal extension. This revision incorporates those clarifications and the exact Holevo Choi antecedent. That audit did not assess this exact PDF, certify importance or independently prove every imported coding theorem. Full requested-model review620 examined the preceding exact eight-page draft and supported its main results with a minor correction to the pinpoint type-bound attribution. This revision corrects that attribution and supplies the type and input-tail derivations explicitly. The original report and its objections are retained. It used reused context and shared files and disclosed incidental exposure to an older paper's public assessment during a literature search; no blindness or independence is claimed. No full assessment or numerical grade of this revised exact PDF exists. Technical audit627 of the frozen strict-gap argument supported its state inequality, common-code transfer at finite E, N, critical signal limit and necessity for the complete curves. It assigned no score, did not assess

this combined manuscript and did not certify priority. It reused context from earlier technical audits, shared files and disclosed earlier assessment exposure; runtime independence was not attested. Its original report is retained. The preceding public entropy/optical manuscript has its own preserved assessment; its grade is not transferred to this draft. No independently attested runtime isolation or model identity, independent human review, Lean verification, worldwide originality, editorial acceptance, or predicted importance is claimed. Literature reading has been directed: specified Gaussian/coding passages, Shirokov's mixture bound, and Audenaert's Theorem4, Corollary1 and selected proof pages. It is not an exhaustive literature sweep or an audit of all dependencies.

References

[1] A. S. Holevo and R. F. Werner, *Evaluating capacities of Bosonic Gaussian channels*, arXiv:quant-ph/9912067v1 (1999), selected Sections IV.C, V.A.

[2] A. S. Holevo and M. E. Shirokov, *On the entanglement-assisted classical capacity of infinite-dimensional quantum channels*, arXiv:1210.6926v2 (2013), Theorem1.

[3] A. S. Holevo, *The entropy gain of infinite-dimensional quantum channels*, arXiv:1003.5765v1 (2010), Proposition2.

[4] A. Winter, *Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints*, arXiv:1507.07775v6 (2016), Lemma2.

[5] M. Berta, H. Gharibyan and M. Walter, *Entanglement-assisted capacities of compound quantum channels*, arXiv:1603.02282v2 (2017), Theorem1, Lemma9 and assisted-capacity equivalence.

[6] M. E. Shirokov, *Convergence conditions for the quantum relative entropy and other applications of the generalized quantum Dini lemma*, arXiv:2205.09108v2 (2022), selected Eq.(11), Section5.2.

[7] A. S. Holevo, *On the Choi-Jamiolkowski Correspondence in Infinite Dimensions*, arXiv:1004.0196v3 (2011), Proposition2 and one-mode example on p.13.

[8] K. M. R. Audenaert, *Telescopic Relative Entropy*, arXiv:1102.3040v2 (2011), Theorem4 and Corollary1.

[9] L. Eriksson, *Sharp entropy budgets for communication with unknown resets and optical loss*, AIRR.SCIENCE ARR-2026-7SWAJZ0X278GJ9NK, v1 (2026), Working paper. https://airr.science/papers/ARR-2026-7SWAJZ0X278GJ9NK/versions/v1/.

[10] W. Roga, M. Fannes and K. Życzkowski, *Universal bounds for the Holevo quantity, coherent information and the Jensen–Shannon divergence*, arXiv:1004.4782v1 (2010), Corollary4 and Lemma5.

[11] M. Fannes, F. de Melo, W. Roga and K. Życzkowski, *Matrices of fidelities for ensembles of quantum states and the Holevo quantity*, arXiv:1104.2271v1 (2011), Eqs.(3)–(6); that source uses squared fidelity.

[12] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr and M. Tomamichel, *On quantum Rényi entropies: a new generalization and some properties*, arXiv:1306.3142v4 (2013), order-one-half fidelity identity and Theorem7.