Entropy-budget minimax losses under growing thermal noise

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Abstract

We establish an extension of the entropy-budget minimax law for oscillator reset channels to a thermal attenuator in the transmitted branch. The sufficient regime is environment occupation $n = o(E)$, where E is the input energy, with the channel fixed throughout each code. The full Shannon and fixed super-Shannon curves have the same bounded absolute loss as for an identity or pure-loss signal, relative to the thermal channel's own entanglement-assisted benchmark. The key argument combines a bounded Choi form, a largest-eigenvalue bound for near-capacity inputs, output pinching, and Audenaert's established telescopic relative-entropy inequality. It avoids a continuity error proportional to $\log E$. Explicit input/output truncation errors give the matching common-code bound without an energy constraint on the reset. A separate extension treats unknown fixed signal parameters in a known sublinear-temperature envelope. This draft records the argument for further scrutiny. A working-argument audit supports the stated converse and unknown-signal extension; it is not a completed exact-PDF assessment or a claim of priority, importance, formal verification or editorial acceptance.

1 Model, scope and theorem

All logarithms in information quantities are base two; ln is natural. Let $H = \sum_{j\geq 0} j|j\rangle\langle j|$, $g(x) = (x+1)\log_2(x+1) - x\log_2 x$, and let h denote binary entropy. The thermal state of mean x is γ_x. A thermal attenuator $\Phi_{\kappa,n}$ mixes the input with an environment thermal state of mean n, through a beamsplitter of photon transmissivity $0 < \kappa < 1$. Write $\delta = 1 - \kappa$, $q = 1 - \lambda$, with $0 < \lambda < 1$, and

$$\mathcal{N}_{\kappa,n,\tau}(\rho) = \lambda\Phi_{\kappa,n}(\rho) + q\tau \operatorname{Tr}\rho.$$

The reset τ has no energy or orientation restriction. All parameters are fixed throughout a code block. The input constraint is expected physical block-average energy at most E, not peak energy. Entanglement assistance is unlimited. Classical error is average-message error. In a compound capacity one encoder and decoder serve the whole fixed-parameter family, with uniform error. The coding limit is taken at each fixed E, n, B, before the energy limit.

Put $\mathcal{T}_{\alpha,B} = \{\tau : S_\alpha(\tau) \leq B\}$, where $S_1 = S$ and $S_\alpha = (1-\alpha)^{-1}\log_2 \operatorname{Tr}\tau^\alpha$. The established constrained Gaussian assisted capacity [1, 2] is

$$\begin{aligned} C_{\kappa,n}(E) &= g(E) + g(M) - g(u_l) - g(u_s), & M &= \kappa E + \delta n, \\ V &= \delta E + (1+\kappa)n + 1, & D &= \sqrt{V^2 - 4\kappa n(n+1)}, \\ u_l &= \tfrac{1}{2}[D + \delta(E-n) - 1], & u_s &= \tfrac{1}{2}[D - \delta(E-n) - 1]. \end{aligned} \tag{C}$$

Here $g(u_l) + g(u_s)$ is the entropy of the *true two-mode* complement, including purification of the initial thermal environment. A one-mode weak complement cannot replace it.

Define the known-reset and unknown-reset deficits

$$D_{\mathrm{ind}}(E,n,B) = \lambda C_{\kappa,n}(E) - \inf_{\tau\in\mathcal{T}_{\alpha,B}} C_{\mathrm{EA}}(\mathcal{N}_{\kappa,n,\tau},H,E),$$
$$D_{\mathrm{cmp}}(E,n,B) = \lambda C_{\kappa,n}(E) - C_{\mathrm{EA}}^{\mathrm{cmp}}(\{\mathcal{N}_{\kappa,n,\tau}:\tau\in\mathcal{T}_{\alpha,B}\},H,E).$$

Thus $D_{\mathrm{ind}} \le D_{\mathrm{cmp}}$. Let

$$b(u,v) = (u+v)h\!\left(\frac{u}{u+v}\right), \quad f(a) = b(\lambda, qa),$$

$$t_\alpha(E,B) = \begin{cases} \min\{1, B/\log_2(E+1)\}, & \alpha = 1, \\ 1 - 2^{-((\alpha-1)/\alpha)B}, & 1 < \alpha < \infty. \end{cases}$$

Use continuous boundary values in b.

Theorem 1 (Growing-noise law)**.** *Fix $\kappa,\lambda\in(0,1)$ and $\alpha = 1$ or a fixed finite $\alpha > 1$. For every prescribed path $n(E) > 0$ with $n(E)/E \to 0$,*

$$\sup_{0\le B<\infty} \max_{Y\in\{\mathrm{ind},\mathrm{cmp}\}} |D_Y(E,n(E),B) - f(t_\alpha(E,B))| \longrightarrow 0. \tag{P}$$

The argument below uses the scientific dependencies disclosed at the end. It claims neither a uniform blocklength nor uniformity over all asymptotic paths at a fixed energy without a common envelope.

This removes a sufficient $n = o(\sqrt{E})$ restriction of an earlier working proof, rather than establishing that restriction as a physical threshold. The proposed addition concerns operational quantifiers and uniform errors. Gaussian extremality, entropy mixing, Helstrom testing, telescopic relative entropy and compound coding are prior ingredients.

2 A uniform Gaussian entropy estimate

The input/output energy bounds are $S(\rho) \le g(E)$, $S(\Phi_{\kappa,n}(\rho)) \le g(M)$. For input mean x, the true complementary entropy is bounded by $g(\delta x + \kappa n) + g(n)$. Let $\beta_x = \log_2(1 + 1/x)$ for $x > 0$. The complementary Gibbs logarithm pulled back to the input is a quadratic form $a(E;n)H + b(E;n)I$, with

$$a(E;n) = \delta\left[\beta_{u_l} + (\beta_{u_l} + \beta_{u_s})r\right], \qquad r = \frac{1}{2}(V/D - 1).$$

For clarity, a direct verification uses the environment modes $e' = -\sqrt{\delta}\,a + \sqrt{\kappa}\,e$ and purifier z. With $\cosh(2s) = V/D$, $\sinh(2s) = 2\sqrt{\kappa n(n+1)}/D$, the thermal normal modes are $x = \cosh(s)e' - \sinh(s)z^\dagger$ and $y = \cosh(s)z - \sinh(s)e'^\dagger$. The logarithm is a constant plus $\beta_{u_l}x^\dagger x + \beta_{u_s}y^\dagger y$. Pullback of $e'^\dagger e'$, $z^\dagger z$, and $e'z + e'^\dagger z^\dagger$ is respectively $\delta H + \kappa n I$, nI, and $2\sqrt{\kappa n(n+1)}I$, giving the displayed coefficient. These are positive-form/finite-mean expectation identities on the number form domain. They are not bounded-adjoint identities for an unbounded logarithm. Finite Fock cutoffs and finite energy justify their limiting expectations. Differentiating the Gibbs cross entropy at fixed n also gives $a(E;n) = \partial_E[g(u_l) + g(u_s)]$.

Relative-entropy data processing with reference γ_E gives

$$S(\rho) - S(\Phi_{\kappa,n}^c(\rho)) \le g(E) - g(u_l) - g(u_s) + \zeta, \qquad \zeta = E|\beta_E - a(E;n)|. \tag{1}$$

No optimality of thermal reverse coherent information is assumed. For $0 < n \le E$, the exact radical obeys

$$D \ge \delta(E+n) + 1, \quad \delta E \le u_l \le \delta E + \kappa n, \quad \delta n \le u_s \le n, \quad 0 \le r \le \frac{\kappa n(n+1)}{\delta^2 E^2}.$$

The elementary inequalities $0 \le x^{-1} - \ln(1+x^{-1}) \le (2x^2)^{-1}$ imply

$$\zeta \le \Xi(E,n) := \frac{1+1/\delta}{2E\ln 2} + \frac{\kappa n}{\delta E \ln 2} + \frac{2\kappa(n+1)}{\delta^2 E \ln 2}. \tag{2}$$

Thus $\Xi = o(1)$ throughout $n = o(E)$. All derivatives hold n fixed; $n'(E)$ is never used.

3 Flat Choi order and near-capacity inputs

Lemma 2 (Known bounded Choi form, applied here)**.** *For the reference/output state* $\omega = (\mathrm{id}\otimes\Phi_{\kappa,n})(\psi_\rho)$ *of the canonical purification,*

$$\omega \le c\,\rho_R \otimes I_B, \qquad c = \frac{1}{\delta(n+1)}. \tag{3}$$

This bound is an exact specialization of Holevo's established Gaussian Choi norm formula [7, Proposition 2 and the one-mode example, p. 13]. His amplitude parameter is $\sqrt{\kappa}$ and his noise coefficient is $m = \delta(n+1/2)$, so the norm is $[m + |\kappa - 1|/2]^{-1} = c$. The following regularization argument also explains the application to the present purification; the Choi norm formula is not a new lemma of this paper.

Proof. Use an auxiliary thermal input of mean $x > 0$, keeping n fixed. Its reference/output state ω_x has the same nonzero spectrum as the true Gaussian complement. Formula (C) gives

$$(u_l+1)(u_s+1) = \delta x n + \frac{\delta(x+n)}{2} + \frac{D+1}{2} \ge \delta(n+1)(x+1).$$

If J is the canonical positive Choi form and $G_x = (x/(x+1))^{H/2}$, then $(G_x\otimes I)J(G_x\otimes I) = (x+1)\omega_x \le cI$. For a vector with finite input-number support, insert G_x^{-1}; its norm converges to the original norm as $x\to\infty$. The positive form therefore has bound cI on a dense domain and extends to a bounded positive operator. Sandwiching with $\sqrt{\rho^T}\otimes I$ gives (3), first for finite truncations and then in trace norm. No unnormalizable Bell vector is used as a state. □

Put $C = C_{\kappa,n}(E)$. At fixed n,

$$0 \le \partial_E C \le \frac{2}{E\ln 2}. \tag{4}$$

The upper bound follows from $\partial_E C = \beta_E + \kappa\beta_M - a(E;n)$ and $a \ge 0$; monotonicity follows from enlargement of the feasible input set.

If $w = \|\rho\|_\infty$, isolate a largest pure eigencomponent. Mutual information of that pure input is zero. The entropy-of-mixture inequality, whose general $2h(w)$ version is prior [6, Eq. (11), Section 5.2], yields

$$I(\rho,\Phi_{\kappa,n}) \le 2h(w) + (1-w)C_{\kappa,n}(E/(1-w)) \le (1-w)C + 2h(w) + 2(1-w)\log_2\frac{1}{1-w}.$$

Consequently

$$I(\rho,\Phi_{\kappa,n}) \ge C - \Delta \quad\Longrightarrow\quad w \le \frac{\Delta + 2 + 2/(e\ln 2)}{C}. \tag{5}$$

The endpoint $w = 1$ is excluded eventually by $C > \Delta$. Let $S = c(M+1)$. Since $u_l \le E$, $u_s \le n$ for $n \le E$,

$$C \ge g(M) - g(n) \ge \log_2 S + \log_2\delta - \log_2 e. \tag{6}$$

Thus $C\to\infty$ whenever $n = o(E)$. The established regular-channel identity $\Phi_{\kappa,n}(I) = I/\kappa$ [3, Proposition 2] gives

$$\Pr_{\Phi_{\kappa,n}(\rho)}(0) \le w/\kappa = o(1) \tag{7}$$

on the set in (5).

4 Pinching and branch discrimination

Let $\tau_a = (1-a)|0\rangle\langle 0| + a\gamma_M$, $0 \le a \le 1$. Flagging gives $I(\rho, \mathcal{N}_{\kappa,n,\tau_a}) \le \lambda I(\rho, \Phi_{\kappa,n})$. Inputs below $\lambda C - h(\lambda) - 1$ already satisfy the needed converse because $f(a) \le h(\lambda)$. All remaining inputs satisfy (5) with $\Delta = (h(\lambda)+1)/\lambda$, independent of E, n, a.

Set $\sigma_0 = \rho_R \otimes I_B$, an *unnormalized* infinite-trace operator, and $D_0(\omega) = \operatorname{Tr}\omega(\log_2\omega - \log_2\sigma_0) = -H(B|R)_\omega$. Input/output energy ensures finite entropies and cross entropies. Equation (3), (6), and $g(u_l) + g(u_s) \le g(E) + g(n)$ give

$$\log_2 c - C_0 \le D_0(\omega) \le \log_2 c, \qquad C_0 = \Delta + \log_2 e - \log_2 \delta. \tag{8}$$

We have not applied a normalized-state relative-entropy inequality to σ_0.

For $S > 1$, let $t = \sqrt{\ln S}$, $T = \lceil (M+1)t \rceil$, $P_B = \sum_{j=1}^{T} |j\rangle\langle j|$, $P = I_R \otimes P_B$, and $p = \operatorname{Tr} P\omega$. Equations (5),(7) and Markov give $p \ge 1 - w/\kappa - 1/t \to 1$. Define normalized good and bad blocks ω_g, ω_b. For these blocks D_0 always uses the original reference ρ_R; in particular

$$D_0(\omega_g) = -H(B|R)_{\omega_g} + D(\rho_R^g \| \rho_R).$$

It is not the negative conditional entropy computed with its own reference marginal alone. Pinching preserves the reference marginal and increases entropy by at most $h(p)$. Its block formula is

$$D_0(\mathcal{P}\omega) = pD_0(\omega_g) + (1-p)D_0(\omega_b) - h(p).$$

Hence the weighted sum on the right without $-h(p)$ is at least $D_0(\omega)$. By (3), $D_0(\omega_b) \le \log_2 c - \log_2(1-p)$, and therefore

$$D_0(\omega_g) \ge \log_2 c - \frac{C_0 + h(p)}{p}. \tag{9}$$

Zero-weight blocks are interpreted by continuity. Reference marginals are bounded by ρ_R/p, $\rho_R/(1-p)$, keeping their cross entropies finite. This step avoids an error of the form $o(1)\log E$.

Let $v = \operatorname{Tr} P_B\gamma_M \to 1$, $r_M = M/(M+1)$, and introduce the *normalized* comparator

$$\sigma_g = \rho_R \otimes P_B\gamma_M P_B/v, \qquad L = (c/p)v(M+1)r_M^{-T}.$$

The least thermal eigenvalue on this window gives $\omega_g \le L\sigma_g$. Writing $\mu_g = \operatorname{Tr} H_B\omega_g \ge 0$, (9) gives

$$\begin{aligned} D(\omega_g\|\sigma_g) &= D_0(\omega_g) + \log_2(M+1) + \beta_M\mu_g + \log_2 v \\ &\ge \log_2 S + \log_2 v - (C_0+1)/p, \\ \log_2 L &= \log_2 S + \log_2 v - \log_2 p + T\beta_M = \log_2 S + O(\sqrt{\ln S}). \end{aligned} \tag{10}$$

In particular the sign of the normalization term is $+\log_2 v$.

For normalized states $r \le Ls$, Audenaert's established telescopic inequality [8, Theorem 4] implies

$$D(r\|s) \le \frac{\log_2 L}{1 - 1/L}\,\frac{1}{2}\|r - s\|_1. \tag{11}$$

Indeed write $s = L^{-1}r + (1-L^{-1})z$ and use that theorem, together with $\frac{1}{2}\|r-s\|_1 = (1-L^{-1})\frac{1}{2}\|r-z\|_1$. For infinite reference, common finite compressions with an orthogonal failure flag preserve normalization and operator order. Their embedded states converge in trace norm. Lower semicontinuity of relative entropy and data processing give convergence of the relative entropies, proving (11) in this case. Equation (10) now forces $\frac{1}{2}\|\omega_g - \sigma_g\|_1 \to 1$.

The selected proof of Audenaert's Theorem 4 in v2 has a printed sign inconsistency on p. 10: after defining a difference proportional to $\rho - \sigma$, it later uses the opposite orientation.

Taking the auxiliary difference as $(\sigma-\rho)/(\frac{1}{2}\|\rho-\sigma\|_1)$ restores that displayed identity and leaves the positive-part norm bound and theorem unchanged. The use of (11) is the established theorem, not a new inequality.

A Helstrom projection $Q \le P$ has both normalized testing errors tending to zero. Its acceptance on the original signal is $p\,\mathrm{Tr}\,Q\omega_g \to 1$. On the original reset it is

$$\mathrm{Tr}[Q(\rho_R \otimes \tau_a)] = av\,\mathrm{Tr}\,Q\sigma_g \to 0,$$

because the vacuum component is excluded. These bounds are uniform in a and in all relevant inputs. Binary Fano gives $H(F|RB) = o(1)$. This analytical test may depend on the input; it is a converse device, not the common operational decoder.

5 All-input converse and the matching common code

The exact binary-branch chain rule is

$$I(\rho,\mathcal{N}_{\kappa,n,\tau}) = F_\tau(\rho)+H(F|RB), \quad F_\tau = \lambda[S(\rho)-S(\Phi^c_{\kappa,n}(\rho))]+S(\lambda\Phi_{\kappa,n}(\rho)+q\tau)-qS(\tau)-h(\lambda).$$

For the finite-entropy witness τ_a, entropy mixing and (1) give

$$F_{\tau_a} \le \lambda C - f(a) + \lambda\Xi + qh\left(\frac{a}{M+1}\right). \tag{12}$$

To see the mixing step, group the signal and thermal reset, of total weight $P_a = \lambda + qa$, against the vacuum. The normalized group has mean at most M, so its entropy is at most $g(M)$. Moreover $S(\tau_a) \ge ag(M)+h(a)-h(a/(M+1))$. The identity $h(P_a)-qh(a)-h(\lambda) = -b(\lambda, qa)$ then gives (12). Combining (12) with the preceding branch test bounds *every* admissible input. Evaluation at γ_E supplies the other direction:

$$f(a) - o(1) \le \lambda C - C_{\mathrm{EA}}(\mathcal{N}_{\kappa,n,\tau_a}, H, E) \le f(a) + b(\lambda/(M+1), q). \tag{13}$$

Uniformity in a is essential for changing budgets.

Here is an explicit common error envelope for the converse. Let $C_\Delta = \Delta + 2 + 2/(e\ln 2)$, $H_S = \log_2 S + \log_2\delta - \log_2 e$, and, eventually,

$$\begin{aligned} w_* &= C_\Delta/H_S, & p_* &= 1 - w_*/\kappa - 1/t, \\ v_* &= 1-(1+\kappa E)^{-1} - e^{-t}, & K_* &= -\log_2 p_* + (2t+1)/\ln 2 + (C_0+1)/p_*, \\ \varepsilon_* &= (Sv_*)^{-1} + K_*/\log_2(Sv_*), & z_* &= \lambda(1-p_*)+\varepsilon_*. \end{aligned}$$

These formulas are used only when $H_S > 0$, $p_*, v_* > 0$, $Sv_* > 1$, $M \ge 1$, and $z_* \le 1/2$, all eventual conditions in the joint limit. From (10),(11), $1 - \frac{1}{2}\|\omega_g - \sigma_g\|_1 \le \varepsilon_*$. Thus a valid error in the left side of (13) is

$$R_- = \lambda\Xi(E,n) + qh((1+\kappa E)^{-1}) + h(z_*) \longrightarrow 0. \tag{13a}$$

This exposes the dependence on E, n, rather than assigning a common rate to every unspecified sublinear trajectory.

For completeness the achievable bound does not use that witness. For $E > 1$, let

$$L_{\mathrm{in}} = \lceil 4E\ln(E+1)\rceil, \quad \epsilon = (1-1/E)^{L_{\mathrm{in}}} \le (E+1)^{-4}.$$

Truncate γ_{E-1} to its first L_{in} levels and normalize, obtaining σ of mean strictly below E. Entropy mixing of the input, output and true two-mode complement, and (4) at the same n, yield

$$I(\sigma,\Phi_{\kappa,n}) \ge C - U_E, \quad U_E = 3h(\epsilon) + 4\epsilon[g(E)+g(E+L_{\mathrm{in}})] + 2\log_2\frac{E}{E-1} = o(1). \tag{14}$$

The bound is uniform for $n \le E$. It compares inputs of energies $E-1, E$ at fixed channel, not $n(E-1)$ with $n(E)$.

Apply one receiver Fock cutoff $T_{\text{out}} = \lceil E^4 \rceil$ and replace its complement by an orthogonal failure symbol. The signal failure mass is at most E^{-3}; no energy assumption is imposed on the reset. With

$$e_E = E^{-3/2} + \tfrac{1}{2}E^{-3}, \quad W_E = 2e_E \log_2 L_{\text{in}} + (1+e_E)h(e_E/(1+e_E)),$$

finite-reference continuity [4, Lemma 2] loses at most $W_E = o(1)$. The compressed signal norm is at most

$$v_E = \max\left\{\frac{1}{[1+\kappa(E-1)](1-\epsilon)}, E^{-3}\right\}.$$

A top-m reset spectral subspace has retained signal mass at most $(m+1)v_E$, since the failure flag increases rank by at most one. If η_m bounds its original discarded spectral mass, the binary observation of that subspace gives

$$H(F|B) \le b(\lambda, q\eta_m) + b(\min\{\lambda, \lambda(m+1)v_E\}, q).$$

The branch chain rule thus yields

$$D_{\text{cmp}} \le f(\eta_m) + R_+(E,m), \quad R_+ = \lambda(U_E + W_E) + b(\min\{\lambda, \lambda(m+1)v_E\}, q), \tag{15}$$

which vanishes if $m/E \to 0$.

Here is the operational step behind (15). After the two fixed cuts the channel family has finite input/output dimension, though its reset index can be arbitrary. Approximate the diagonal input σ by rational probabilities while retaining strict mean-cost slack. On the exact type subspace of a type length k_{type}, every vector has total physical number $k_{\text{type}} \sum_j j p_j < k_{\text{type}} E$. The standard type-subspace mutual-information estimate loses at most $2d_{k_{\text{type}}}$, where $d_{k_{\text{type}}} = O(\log k_{\text{type}})$, uniformly in the channel [5, Lemma 9]. Apply the finite-dimensional compound assisted coding theorem [5, Theorem 1] on this subspace. First fix the type and all cuts for a rate slack, then take the outer coding length to infinity; only subsequently optimize the finite choices and take $E \to \infty$. This preserves physical expected cost and provides a single encoder and decoder for all resets. It supplies no uniform blocklength or efficient code. These are uses of established coding results, not a new coding theorem.

6 Budget normalization and unknown signal envelopes

For Shannon entropy, $\eta_m \le \min\{1, B/\log_2(m+1)\}$. Choose $m = \lfloor E/\ln(E+1) \rfloor$. Its clipping curve differs uniformly from $t_1(E,B)$ by $o(1)$. For the converse choose

$$a_B = \text{clip}_{[0,1]} \frac{B-1}{g(M)}.$$

The witness is feasible because $S(\tau_a) \le h(a) + ag(M)$; the clipped endpoints are feasible separately. Since $\kappa E \le M \le E$, $\log_2 \kappa \le g(M) - \log_2(E+1) \le \log_2 e$. It follows that $\sup_B |a_B - t_1(E,B)| = o(1)$. For fixed $\alpha > 1$, choose $a_B = 1 - 2^{-((\alpha-1)/\alpha)B}$ and $m = \lfloor \sqrt{E} \rfloor$. The witness obeys $\text{Tr}\,\tau_a^\alpha \ge (1-a)^\alpha$, hence is feasible. The established spectral-tail estimate gives $\eta_m \le \min\{1, a_B + (m+1)^{(1-\alpha)/\alpha}\}$. Since f is increasing and concave, with $|f(x)-f(y)| \le f(|x-y|)$, (13),(15) give (P). Resets of infinite von Neumann entropy can occur for $\alpha > 1$; their achievability is evaluated only after finite output compression. Uncompressed reset entropies are subtracted only for the finite-entropy witness.

Proposition 3 (Common code for unknown signal)**.** *Let known $0 < \kappa_0 < 1$ and $N(E) > 0$ satisfy $N(E) = o(E)$. Suppose $\kappa \in [\kappa_0, 1]$, $n \in [0, N(E)]$, and $\tau \in \mathcal{T}_{\alpha,B}$ are all unknown and fixed throughout the block. Then the compound deficit relative to $\lambda C_{\kappa_0,N(E)}(E)$ is $f(t_\alpha(E,B)) + o(1)$, uniformly in finite B. The zero-envelope case is the earlier pure-loss result, not a substitution in singular thermal logarithms.*

Proof. For signal channels only, add receiver Gaussian noise of mean $(1-\kappa)(N-n)$, raising the environment parameter to N. Then compose with a thermal attenuator of transmissivity κ_0/κ and environment occupation N. Its resulting noise is $(1-\kappa_0)N$, so the composition is $\Phi_{\kappa_0,N}$. Thus mutual information of the same input σ for every original signal is at least its value for that worst pair. The cuts, U_E, W_E, and the signal-norm bound in (14),(15) are uniform with $\kappa \geq \kappa_0, n \leq E$, replacing κ by κ_0 in the norm bound. Apply the finite-family-independent coding step to the entire signal/reset index set. For the converse choose the worst pair and its witness. Do not degrade the full reset mixture: receiver processing could change the reset entropy and leave the budget class. □

7 Boundary, evidence and unresolved status

The sublinear-temperature hypothesis is sufficient, not proved necessary in general. At $\kappa = \lambda = 1/2, n = E$, Eq. (C) gives $C_{\kappa,E}(E) \to 1$ bit. Indeed $u_l = u_s = (\sqrt{2E(E+1)+1}-1)/2$, and $g(x) = \log_2 x + \log_2 e + o(1)$. The flagged benchmark therefore tends to $1/2$ bit whereas $f(1) = 1$ bit. Nonnegativity of capacity excludes the proposed constant loss law in that regime. This is a boundary example, not a full critical-regime solution.

The current numerical evidence consists of 90 high-precision scalar controls for the earlier $o(\sqrt{E})$ working argument and 72 noncommuting finite-state controls for the pinching/order bookkeeping. The latter test generic finite matrices, not the infinite Gaussian Choi form or an operational capacity. Neither set proves the asymptotic theorem. Tests, frozen inputs and reports are retained separately. The accompanying supplement provides the two unchanged programs, expected outputs, dependency instructions and a replay record. Both outputs were reproduced byte for byte during packaging. The package does not implement an operational code or certify an infinite-dimensional optimizer.

Model assessments and review context

The named author directed substantial OpenAI Codex assistance in derivation, writing and numerical work. Requested GPT-6 Astra xhigh working-argument audits supported the fixed-temperature and $o(\sqrt{E})$ versions with their stated limits. They used shared files and reused context. Audit615 of the frozen broader working argument614 has since closed with mathematical support and minor attribution/notation clarifications, including the Choi form, fixed-reference pinching, infinite-reference limit, full two-sided minimax bounds and unknown-signal extension. This revision incorporates those clarifications and the exact Holevo Choi antecedent. That audit did not assess this exact PDF, certify importance or independently prove every imported coding theorem. No full assessment or numerical grade of this exact manuscript exists. The preceding public entropy/optical manuscript has its own preserved assessment; its grade is not transferred to this draft. No independently attested runtime isolation or model identity, independent human review, Lean verification, worldwide originality, editorial acceptance, or predicted importance is claimed. Literature reading has been directed: specified Gaussian/coding passages, Shirokov's mixture bound, and Audenaert's Theorem4, Corollary1 and selected proof pages. It is not an exhaustive literature sweep or an audit of all dependencies.

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