Sharp entropy-budget laws for Gaussian reset channels

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Abstract

An unflagged reset channel transmits through a Gaussian signal channel with probability λ, and otherwise emits a fixed state whose entropy, but not energy or orientation, is bounded. We determine the complete high-energy minimax loss curves for unassisted classical communication, permitting arbitrary entangled block inputs and collective decoding. Relative to λ times the signal's own classical capacity, the absolute loss is $b(\lambda, (1-\lambda)t_\alpha) + o(1)$, uniformly in every finite entropy budget. For Shannon entropy $t_1 = \min\{1, B/\log_2(E+1)\}$; for fixed finite $\alpha > 1$, $t_\alpha = 1 - 2^{-(\alpha-1)B/\alpha}$. The sharp regime is signal vacuum noise $t = o(E)$. It covers thermal attenuation and amplification, and additive Gaussian noise bounded away from zero. A larger unknown fixed family has the same law and a common encoder/decoder, including cold and noiseless members. No preshared entanglement or additivity of the reset mixture is assumed. The converse uses causal quantum references, a summed Gaussian minimum-output-entropy floor and averaged Choi-order deficits. Selecting a fraction of good positions and using established telescopic relative entropy gives an error uniform in block length. Finite cq compound coding with exact physical types supplies the matching operational rate for arbitrary reset energy. A strict flag-loss gap at bounded signal information proves necessity. The assisted thermal law is also proved, relative to its own benchmark. The actual critical loss curve and efficient blocklength bounds remain open here. Working-argument audits support the thermal ingredients; the enlarged exact manuscript requires its own full assessment.

1 Model, scope and theorem

All logarithms in information quantities are base two; ln is natural. Let $H = \sum_{j\geq 0} j|j\rangle\langle j|$, $g(x) = (x+1)\log_2(x+1) - x\log_2 x$, and let h denote binary entropy. The thermal state of mean x is γ_x. A thermal attenuator $\Phi_{\kappa,n}$ mixes the input with an environment thermal state of mean n, through a beamsplitter of photon transmissivity $0 < \kappa < 1$. Write $\delta = 1 - \kappa$, $q = 1 - \lambda$, with $0 < \lambda < 1$, and

$$\mathcal{N}_{\kappa,n,\tau}(\rho) = \lambda\Phi_{\kappa,n}(\rho) + q\tau\,\mathrm{Tr}\,\rho.$$

The reset τ has no energy or orientation restriction. All parameters are fixed throughout a code block. The input constraint is expected physical block-average energy at most E, not peak energy. For the assisted baseline developed first, entanglement assistance is unlimited. Sections 9–10 remove that resource and permit arbitrary entangled block encoding without preshared entanglement. Classical error is average-message error. In a compound capacity one encoder and decoder serve the whole fixed-parameter family, with uniform error. The coding limit is taken at each fixed E, n, B, before the energy limit.

Put $\mathcal{T}_{\alpha,B} = \{\tau : S_\alpha(\tau) \leq B\}$, where $S_1 = S$ and $S_\alpha = (1-\alpha)^{-1}\log_2 \mathrm{Tr}\,\tau^\alpha$. The established

constrained Gaussian assisted capacity [10, 11] is

$$\begin{aligned} C_{\kappa,n}(E) &= g(E) + g(M) - g(u_l) - g(u_s), & M &= \kappa E + \delta n, \\ V &= \delta E + (1+\kappa)n + 1, & D &= \sqrt{V^2 - 4\kappa n(n+1)}, \\ u_l &= \tfrac{1}{2}[D + \delta(E-n) - 1], & u_s &= \tfrac{1}{2}[D - \delta(E-n) - 1]. \end{aligned} \tag{C}$$

Here $g(u_l) + g(u_s)$ is the entropy of the *true two-mode* complement, including purification of the initial thermal environment. A one-mode weak complement cannot replace it.

Define the known-reset and unknown-reset deficits

$$\begin{aligned} D_{\mathrm{ind}}(E,n,B) &= \lambda C_{\kappa,n}(E) - \inf_{\tau\in\mathcal{T}_{\alpha,B}} C_{\mathrm{EA}}(\mathcal{N}_{\kappa,n,\tau}, H, E), \\ D_{\mathrm{cmp}}(E,n,B) &= \lambda C_{\kappa,n}(E) - C_{\mathrm{EA}}^{\mathrm{cmp}}(\{\mathcal{N}_{\kappa,n,\tau} : \tau \in \mathcal{T}_{\alpha,B}\}, H, E). \end{aligned}$$

Thus $D_{\mathrm{ind}} \le D_{\mathrm{cmp}}$. Let

$$b(u,v) = (u+v)h\left(\frac{u}{u+v}\right), \quad f(a) = b(\lambda, qa),$$

$$t_\alpha(E,B) = \begin{cases} \min\{1, B/\log_2(E+1)\}, & \alpha = 1, \\ 1 - 2^{-((\alpha-1)/\alpha)B}, & 1 < \alpha < \infty. \end{cases}$$

Use continuous boundary values in b.

Theorem 1 (Growing-noise law)**.** *Fix $\kappa, \lambda \in (0,1)$ and $\alpha = 1$ or a fixed finite $\alpha > 1$. For every prescribed path $n(E) > 0$ with $n(E)/E \to 0$,*

$$\sup_{0\le B<\infty} \max_{Y\in\{\mathrm{ind},\mathrm{cmp}\}} |D_Y(E, n(E), B) - f(t_\alpha(E,B))| \longrightarrow 0. \tag{P}$$

The argument below uses the scientific dependencies disclosed at the end. It claims neither a uniform blocklength nor uniformity over all asymptotic paths at a fixed energy without a common envelope.

This removes a sufficient $n = o(\sqrt{E})$ restriction of an earlier working proof, rather than establishing that restriction as a physical threshold. The proposed addition concerns operational quantifiers and uniform errors. Gaussian extremality, entropy mixing, Helstrom testing, telescopic relative entropy and compound coding are prior ingredients.

Theorem 2 (Sharp regime for the complete curves)**.** *Fix $\kappa, \lambda \in (0,1)$, and either $\alpha = 1$ or a fixed finite $\alpha > 1$. For every finite nonnegative path $n(E)$, the complete uniform-in-budget law (P) holds if and only if $n(E) = o(E)$. For the unknown fixed signal family $\kappa \in [\kappa_0, 1]$, $n \in [0, N(E)]$, with fixed $0 < \kappa_0 < 1$ and finite $N(E) \ge 0$, its compound analogue relative to $\lambda C_{\kappa_0,N(E)}(E)$ holds if and only if $N(E) = o(E)$. There is no smoothness or monotonicity assumption on either path.*

Sufficiency is Theorem 1 and Proposition 4; zero-temperature subsequences use the previous pure-loss law [18]. Necessity, proved below, excludes the complete curve on a growing budget. It does not assert necessity for every restricted small-budget problem or give the actual critical-temperature loss curve.

2 A uniform Gaussian entropy estimate

The input/output energy bounds are $S(\rho) \le g(E)$, $S(\Phi_{\kappa,n}(\rho)) \le g(M)$. For input mean x, the true complementary entropy is bounded by $g(\delta x + \kappa n) + g(n)$. Let $\beta_x = \log_2(1 + 1/x)$ for $x > 0$. The complementary Gibbs logarithm pulled back to the input is a quadratic form $a(E;n)H + b(E;n)I$, with

$$a(E;n) = \delta\left[\beta_{u_l} + (\beta_{u_l} + \beta_{u_s})r\right], \qquad r = \frac{1}{2}(V/D - 1).$$

For clarity, a direct verification uses the environment modes $e' = -\sqrt{\delta}\, a + \sqrt{\kappa}\, e$ and purifier z. With $\cosh(2s) = V/D$, $\sinh(2s) = 2\sqrt{\kappa n(n+1)}/D$, the thermal normal modes are $x = \cosh(s)e' - \sinh(s)z^\dagger$ and $y = \cosh(s)z - \sinh(s)e'^\dagger$. The logarithm is a constant plus $\beta_{u_l} x^\dagger x + \beta_{u_s} y^\dagger y$. Pullback of $e'^\dagger e'$, $z^\dagger z$, and $e'z + e'^\dagger z^\dagger$ is respectively $\delta H + \kappa n I$, nI, and $2\sqrt{\kappa n(n+1)}I$, giving the displayed coefficient. These are positive-form/finite-mean expectation identities on the number form domain. They are not bounded-adjoint identities for an unbounded logarithm. Finite Fock cutoffs and finite energy justify their limiting expectations. Differentiating the Gibbs cross entropy at fixed n also gives $a(E;n) = \partial_E[g(u_l) + g(u_s)]$.

Relative-entropy data processing with reference γ_E gives

$$S(\rho) - S(\Phi^c_{\kappa,n}(\rho)) \le g(E) - g(u_l) - g(u_s) + \zeta, \qquad \zeta = E|\beta_E - a(E;n)|. \tag{1}$$

No optimality of thermal reverse coherent information is assumed. For $0 < n \le E$, the exact radical obeys

$$D \ge \delta(E+n) + 1, \quad \delta E \le u_l \le \delta E + \kappa n, \quad \delta n \le u_s \le n, \quad 0 \le r \le \frac{\kappa n(n+1)}{\delta^2 E^2}.$$

The elementary inequalities $0 \le x^{-1} - \ln(1 + x^{-1}) \le (2x^2)^{-1}$ imply

$$\zeta \le \Xi(E,n) := \frac{1 + 1/\delta}{2E \ln 2} + \frac{\kappa n}{\delta E \ln 2} + \frac{2\kappa(n+1)}{\delta^2 E \ln 2}. \tag{2}$$

Thus $\Xi = o(1)$ throughout $n = o(E)$. All derivatives hold n fixed; $n'(E)$ is never used.

3 Flat Choi order and near-capacity inputs

Lemma 3 (Known bounded Choi form, applied here)**.** *For the reference/output state* $\omega = (\mathrm{id} \otimes \Phi_{\kappa,n})(\psi_\rho)$ *of the canonical purification,*

$$\omega \le c\, \rho_R \otimes I_B, \qquad c = \frac{1}{\delta(n+1)}. \tag{3}$$

This bound is an exact specialization of Holevo's established Gaussian Choi norm formula [16, Proposition 2 and the one-mode example, p. 13]. His amplitude parameter is $\sqrt{\kappa}$ and his noise coefficient is $m = \delta(n + 1/2)$, so the norm is $[m + |\kappa - 1|/2]^{-1} = c$. The following regularization argument also explains the application to the present purification; the Choi norm formula is not a new lemma of this paper.

Proof. Use an auxiliary thermal input of mean $x > 0$, keeping n fixed. Its reference/output state ω_x has the same nonzero spectrum as the true Gaussian complement. Formula (C) gives

$$(u_l + 1)(u_s + 1) = \delta x n + \frac{\delta(x+n)}{2} + \frac{D+1}{2} \ge \delta(n+1)(x+1).$$

If J is the canonical positive Choi form and $G_x = (x/(x+1))^{H/2}$, then $(G_x \otimes I)J(G_x \otimes I) = (x+1)\omega_x \le cI$. For a vector with finite input-number support, insert G_x^{-1}; its norm converges to the original norm as $x \to \infty$. The positive form therefore has bound cI on a dense domain and extends to a bounded positive operator. Sandwiching with $\sqrt{\rho^T} \otimes I$ gives (3), first for finite truncations and then in trace norm. No unnormalizable Bell vector is used as a state. □

Put $C = C_{\kappa,n}(E)$. At fixed n,

$$0 \le \partial_E C \le \frac{2}{E \ln 2}. \tag{4}$$

The upper bound follows from $\partial_E C = \beta_E + \kappa\beta_M - a(E;n)$ and $a \ge 0$; monotonicity follows from enlargement of the feasible input set.

If $w = \|\rho\|_\infty$, isolate a largest pure eigencomponent. Mutual information of that pure input is zero. The entropy-of-mixture inequality, whose general $2h(w)$ version is prior [15, Eq. (11), Section 5.2], yields

$$I(\rho, \Phi_{\kappa,n}) \le 2h(w) + (1-w)C_{\kappa,n}(E/(1-w)) \le (1-w)C + 2h(w) + 2(1-w)\log_2 \frac{1}{1-w}.$$

Consequently

$$I(\rho, \Phi_{\kappa,n}) \ge C - \Delta \quad \Longrightarrow \quad w \le \frac{\Delta + 2 + 2/(e \ln 2)}{C}. \tag{5}$$

The endpoint $w = 1$ is excluded eventually by $C > \Delta$. Let $S = c(M+1)$. Since $u_l \le E$, $u_s \le n$ for $n \le E$,

$$C \ge g(M) - g(n) \ge \log_2 S + \log_2 \delta - \log_2 e. \tag{6}$$

Thus $C \to \infty$ whenever $n = o(E)$. The established regular-channel identity $\Phi_{\kappa,n}(I) = I/\kappa$ [12, Proposition 2] gives

$$\Pr_{\Phi_{\kappa,n}(\rho)}(0) \le w/\kappa = o(1) \tag{7}$$

on the set in (5).

4 Pinching and branch discrimination

Let $\tau_a = (1-a)|0\rangle\langle 0| + a\gamma_M$, $0 \le a \le 1$. Flagging gives $I(\rho, \mathcal{N}_{\kappa,n,\tau_a}) \le \lambda I(\rho, \Phi_{\kappa,n})$. Inputs below $\lambda C - h(\lambda) - 1$ already satisfy the needed converse because $f(a) \le h(\lambda)$. All remaining inputs satisfy (5) with $\Delta = (h(\lambda)+1)/\lambda$, independent of E, n, a.

Set $\sigma_0 = \rho_R \otimes I_B$, an *unnormalized* infinite-trace operator, and $D_0(\omega) = \operatorname{Tr}\omega(\log_2\omega - \log_2\sigma_0) = -H(B|R)_\omega$. Input/output energy ensures finite entropies and cross entropies. Equation (3), (6), and $g(u_l) + g(u_s) \le g(E) + g(n)$ give

$$\log_2 c - C_0 \le D_0(\omega) \le \log_2 c, \qquad C_0 = \Delta + \log_2 e - \log_2 \delta. \tag{8}$$

We have not applied a normalized-state relative-entropy inequality to σ_0.

For $S > 1$, let $t = \sqrt{\ln S}$, $T = \lceil (M+1)t \rceil$, $P_B = \sum_{j=1}^{T} |j\rangle\langle j|$, $P = I_R \otimes P_B$, and $p = \operatorname{Tr} P\omega$. Equations (5),(7) and Markov give $p \ge 1 - w/\kappa - 1/t \to 1$. Define normalized good and bad blocks ω_g, ω_b. For these blocks D_0 always uses the original reference ρ_R; in particular

$$D_0(\omega_g) = -H(B|R)_{\omega_g} + D(\rho_R^g \| \rho_R).$$

It is not the negative conditional entropy computed with its own reference marginal alone. Pinching preserves the reference marginal and increases entropy by at most $h(p)$. Its block formula is

$$D_0(\mathcal{P}\omega) = pD_0(\omega_g) + (1-p)D_0(\omega_b) - h(p).$$

Hence the weighted sum on the right without $-h(p)$ is at least $D_0(\omega)$. By (3), $D_0(\omega_b) \le \log_2 c - \log_2(1-p)$, and therefore

$$D_0(\omega_g) \ge \log_2 c - \frac{C_0 + h(p)}{p}. \tag{9}$$

Zero-weight blocks are interpreted by continuity. Reference marginals are bounded by ρ_R/p, $\rho_R/(1-p)$, keeping their cross entropies finite. This step avoids an error of the form $o(1)\log E$.

Let $v = \operatorname{Tr} P_B \gamma_M \to 1$, $r_M = M/(M+1)$, and introduce the *normalized* comparator

$$\sigma_g = \rho_R \otimes P_B \gamma_M P_B / v, \qquad L = (c/p) v (M+1) r_M^{-T}.$$

The least thermal eigenvalue on this window gives $\omega_g \le L\sigma_g$. Writing $\mu_g = \operatorname{Tr} H_B \omega_g \ge 0$, (9) gives

$$\begin{aligned} D(\omega_g \| \sigma_g) &= D_0(\omega_g) + \log_2(M+1) + \beta_M \mu_g + \log_2 v \\ &\ge \log_2 S + \log_2 v - (C_0+1)/p, \\ \log_2 L &= \log_2 S + \log_2 v - \log_2 p + T\beta_M = \log_2 S + O(\sqrt{\ln S}). \end{aligned} \tag{10}$$

In particular the sign of the normalization term is $+\log_2 v$.

For normalized states $r \le Ls$, Audenaert's established telescopic inequality [17, Theorem 4] implies

$$D(r\|s) \le \frac{\log_2 L}{1 - 1/L} \frac{1}{2} \|r - s\|_1. \tag{11}$$

Indeed write $s = L^{-1} r + (1 - L^{-1}) z$ and use that theorem, together with $\frac{1}{2}\|r-s\|_1 = (1-L^{-1})\frac{1}{2}\|r-z\|_1$. For infinite reference, common finite compressions with an orthogonal failure flag preserve normalization and operator order. Their embedded states converge in trace norm. Lower semicontinuity of relative entropy and data processing give convergence of the relative entropies, proving (11) in this case. Equation (10) now forces $\frac{1}{2}\|\omega_g - \sigma_g\|_1 \to 1$.

The selected proof of Audenaert's Theorem 4 in v2 has a printed sign inconsistency on p. 10: after defining a difference proportional to $\rho - \sigma$, it later uses the opposite orientation. Taking the auxiliary difference as $(\sigma - \rho)/(\frac{1}{2}\|\rho - \sigma\|_1)$ restores that displayed identity and leaves the positive-part norm bound and theorem unchanged. The use of (11) is the established theorem, not a new inequality.

A Helstrom projection $Q \le P$ has both normalized testing errors tending to zero. Its acceptance on the original signal is $p \operatorname{Tr} Q\omega_g \to 1$. On the original reset it is

$$\operatorname{Tr}[Q(\rho_R \otimes \tau_a)] = av \operatorname{Tr} Q\sigma_g \to 0,$$

because the vacuum component is excluded. These bounds are uniform in a and in all relevant inputs. Binary Fano gives $H(F|RB) = o(1)$. This analytical test may depend on the input; it is a converse device, not the common operational decoder.

5 All-input converse and the matching common code

The exact binary-branch chain rule is

$$I(\rho, \mathcal{N}_{\kappa,n,\tau}) = F_\tau(\rho) + H(F|RB), \quad F_\tau = \lambda[S(\rho) - S(\Phi^c_{\kappa,n}(\rho))] + S(\lambda\Phi_{\kappa,n}(\rho) + q\tau) - qS(\tau) - h(\lambda).$$

For the finite-entropy witness τ_a, entropy mixing and (1) give

$$F_{\tau_a} \le \lambda C - f(a) + \lambda\Xi + qh\left(\frac{a}{M+1}\right). \tag{12}$$

To see the mixing step, group the signal and thermal reset, of total weight $P_a = \lambda + qa$, against the vacuum. The normalized group has mean at most M, so its entropy is at most $g(M)$. Moreover $S(\tau_a) \geq ag(M) + h(a) - h(a/(M+1))$. The identity $h(P_a) - qh(a) - h(\lambda) = -b(\lambda, qa)$ then gives (12). Combining (12) with the preceding branch test bounds *every* admissible input. Evaluation at γ_E supplies the other direction:

$$f(a) - o(1) \leq \lambda C - C_{\mathrm{EA}}(\mathcal{N}_{\kappa,n,\tau_a}, H, E) \leq f(a) + b(\lambda/(M+1), q). \tag{13}$$

Uniformity in a is essential for changing budgets.

Here is an explicit common error envelope for the converse. Let $C_\Delta = \Delta + 2 + 2/(e \ln 2)$, $H_S = \log_2 S + \log_2 \delta - \log_2 e$, and, eventually,

$$\begin{aligned} w_* &= C_\Delta / H_S, & p_* &= 1 - w_*/\kappa - 1/t, \\ v_* &= 1 - (1+\kappa E)^{-1} - e^{-t}, & K_* &= -\log_2 p_* + (2t+1)/\ln 2 + (C_0+1)/p_*, \\ \varepsilon_* &= (Sv_*)^{-1} + K_*/\log_2(Sv_*), & z_* &= \lambda(1-p_*) + \varepsilon_*. \end{aligned}$$

These formulas are used only when $H_S > 0$, $p_*, v_* > 0$, $Sv_* > 1$, $M \geq 1$, and $z_* \leq 1/2$, all eventual conditions in the joint limit. From (10),(11), $1 - \frac{1}{2}\|\omega_g - \sigma_g\|_1 \leq \varepsilon_*$. Thus a valid error in the left side of (13) is

$$R_- = \lambda \Xi(E,n) + qh((1+\kappa E)^{-1}) + h(z_*) \longrightarrow 0. \tag{13a}$$

This exposes the dependence on E, n, rather than assigning a common rate to every unspecified sublinear trajectory.

For completeness the achievable bound does not use that witness. For $E > 1$, let

$$L_{\mathrm{in}} = \lceil 4E \ln(E+1) \rceil, \quad \epsilon = (1 - 1/E)^{L_{\mathrm{in}}} \leq (E+1)^{-4}.$$

Truncate γ_{E-1} to its first L_{in} levels and normalize, obtaining σ of mean strictly below E. To expose a uniform bound, write $\gamma_{E-1} = (1-\epsilon)\sigma + \epsilon\theta$. The normalized tail θ is a shifted thermal state, with $S(\theta) = g(E-1)$. Mutual-information mixing and $I(\theta, \Phi_{\kappa,n}) \leq 2S(\theta)$ give

$$C_{\kappa,n}(E-1) - I(\sigma, \Phi_{\kappa,n}) \leq \frac{2h(\epsilon) + 2\epsilon g(E-1)}{1-\epsilon}.$$

Since $\epsilon < 1/16$ for $E > 1$, this is bounded by the larger entropy error below. Adding (4) at the same n yields

$$I(\sigma, \Phi_{\kappa,n}) \geq C - U_E, \quad U_E = 3h(\epsilon) + 4\epsilon[g(E) + g(E + L_{\mathrm{in}})] + 2\log_2 \frac{E}{E-1} = o(1). \tag{14}$$

The bound is uniform for $n \leq E$. It compares inputs of energies $E-1, E$ at fixed channel, not $n(E-1)$ with $n(E)$.

Apply one receiver Fock cutoff $T_{\mathrm{out}} = \lceil E^4 \rceil$ and replace its complement by an orthogonal failure symbol. The signal failure mass is at most E^{-3}; no energy assumption is imposed on the reset. With

$$e_E = E^{-3/2} + \tfrac{1}{2}E^{-3}, \quad W_E = 2e_E \log_2 L_{\mathrm{in}} + (1+e_E)h(e_E/(1+e_E)),$$

finite-reference continuity [13, Lemma 2] loses at most $W_E = o(1)$. The compressed signal norm is at most

$$v_E = \max\left\{ \frac{1}{[1+\kappa(E-1)](1-\epsilon)}, E^{-3} \right\}.$$

A top-m reset spectral subspace has retained signal mass at most $(m+1)v_E$, since the failure flag increases rank by at most one. If η_m bounds its original discarded spectral mass, the binary observation of that subspace gives

$$H(F|B) \le b(\lambda, q\eta_m) + b(\min\{\lambda, \lambda(m+1)v_E\}, q).$$

The branch chain rule thus yields

$$D_{\mathrm{cmp}} \le f(\eta_m) + R_+(E,m), \quad R_+ = \lambda(U_E + W_E) + b(\min\{\lambda, \lambda(m+1)v_E\}, q), \tag{15}$$

which vanishes if $m/E \to 0$.

Here is the operational step behind (15). After the two fixed cuts the channel family has finite input/output dimension, though its reset index can be arbitrary. Approximate the diagonal input σ by rational probabilities while retaining strict mean-cost slack. On the exact type subspace of a type length k_{type}, every vector has total physical number $k_{\mathrm{type}} \sum_j jp_j < k_{\mathrm{type}} E$. For completeness, let π_T be the maximally mixed exact-type state and put $d_k = D(\pi_T \| \sigma_p^{\otimes k}) = kH(p) - S(\pi_T) = O(\log k)$. Its one-site marginals equal σ_p. Data processing bounds the output total correlation by d_k, while complementary entropy is subadditive. Thus

$$I(\pi_T, \mathcal{M}^{\otimes k}) \ge kI(\sigma_p, \mathcal{M}) - 2d_k$$

uniformly in every finite channel $\mathcal{M}$. This type estimate is also given in [18, Lemma 3]; it is distinct from the compound achievability result of [14, Lemma 9]. Apply the finite-dimensional compound assisted coding theorem [14, Theorem 1] on this subspace. First fix the type and all cuts for a rate slack, then take the outer coding length to infinity; only subsequently optimize the finite choices and take $E \to \infty$. This preserves physical expected cost and provides a single encoder and decoder for all resets. It supplies no uniform blocklength or efficient code. These are uses of established coding results, not a new coding theorem.

6 Budget normalization and unknown signal envelopes

For Shannon entropy, $\eta_m \le \min\{1, B/\log_2(m+1)\}$. Choose $m = \lfloor E/\ln(E+1) \rfloor$. Its clipping curve differs uniformly from $t_1(E,B)$ by $o(1)$. For the converse choose

$$a_B = \mathrm{clip}_{[0,1]} \frac{B-1}{g(M)}.$$

The witness is feasible because $S(\tau_a) \le h(a) + ag(M)$; the clipped endpoints are feasible separately. Since $\kappa E \le M \le E$, $\log_2 \kappa \le g(M) - \log_2(E+1) \le \log_2 e$. It follows that $\sup_B |a_B - t_1(E,B)| = o(1)$. For fixed $\alpha > 1$, choose $a_B = 1 - 2^{-((\alpha-1)/\alpha)B}$ and $m = \lfloor \sqrt{E} \rfloor$. The witness obeys $\mathrm{Tr}\, \tau_a^\alpha \ge (1-a)^\alpha$, hence is feasible. The established spectral-tail estimate gives $\eta_m \le \min\{1, a_B + (m+1)^{(1-\alpha)/\alpha}\}$. Since f is increasing and concave, with $|f(x) - f(y)| \le f(|x-y|)$, (13),(15) give (P). Resets of infinite von Neumann entropy can occur for $\alpha > 1$; their achievability is evaluated only after finite output compression. Uncompressed reset entropies are subtracted only for the finite-entropy witness.

Proposition 4 (Common code for unknown signal)**.** *Let known $0 < \kappa_0 < 1$ and $N(E) > 0$ satisfy $N(E) = o(E)$. Suppose $\kappa \in [\kappa_0, 1]$, $n \in [0, N(E)]$, and $\tau \in \mathcal{T}_{\alpha,B}$ are all unknown and fixed throughout the block. Then the compound deficit relative to $\lambda C_{\kappa_0, N(E)}(E)$ is $f(t_\alpha(E,B)) + o(1)$, uniformly in finite B. The zero-envelope case is the earlier pure-loss result, not a substitution in singular thermal logarithms.*

Proof. For signal channels only, add receiver Gaussian noise of mean $(1-\kappa)(N-n)$, raising the environment parameter to N. Then compose with a thermal attenuator of transmissivity κ_0/κ and environment occupation N. Its resulting noise is $(1-\kappa_0)N$, so the composition is $\Phi_{\kappa_0,N}$. Thus mutual information of the same input σ for every original signal is at least its value for that worst pair. The cuts, U_E, W_E, and the signal-norm bound in (14),(15) are uniform with $\kappa \geq \kappa_0, n \leq E$, replacing κ by κ_0 in the norm bound. Apply the finite-family-independent coding step to the entire signal/reset index set. For the converse choose the worst pair and its witness. Do not degrade the full reset mixture: receiver processing could change the reset entropy and leave the budget class. □

7 A strict flag-loss gap at bounded signal information

The following combines established binary-ensemble inequalities. In particular the purification/Gram bound in its proof is already explicit in [19, Corollary 4, Lemma 5] and [20, Eqs. (3)–(6)]; it is not a new lemma claimed here. Use root fidelity $F(r,s) = \|\sqrt{r}\sqrt{s}\|_1$, rather than its square, and Bures distance $d_B(r,s) = \sqrt{2(1-F(r,s))}$.

Lemma 5 (Strict loss bound)**.** *Let ω_{RB} be normalized on separable spaces, with $c = I(R:B)_\omega < \infty$. For an arbitrary state τ_B, put $\sigma = \omega_R \otimes \tau$, $z = \lambda\omega + q\sigma$ and $\epsilon_\lambda(x) = 2\lambda q\, 2^{-x}/(3\ln 2)$. Then*

$$0 \leq \lambda c - I(R:B)_z \leq h(\lambda) - \epsilon_\lambda(c). \tag{16}$$

No energy or entropy condition on τ is required.

Proof. Adjoin the binary branch flag X. The two branches have the same reference marginal, so $I(R:X) = 0$, $I(R:BX) = \lambda c$. Writing $\chi_{RB} = I(X:RB)$, $\chi_B = I(X:B)$, the loss L obeys

$$L = \chi_{RB} - \chi_B, \qquad h(\lambda) - L = \chi_B + H(X|RB). \tag{17}$$

All flag quantities are finite. These are chain-rule identities, not subtractions of possibly infinite reset or output entropies.

The established bound $D(r\|s) \geq -2\log_2 F(r,s)$ [21] gives $D(r\|s) \geq d_B(r,s)^2/\ln 2$. With $m = \lambda\omega_B + q\tau$ and $d = d_B(\omega_B, \tau)$, weighted Cauchy–Schwarz and the Bures triangle inequality yield

$$\chi_B \geq \frac{\lambda d_B(\omega_B, m)^2 + q d_B(\tau, m)^2}{\ln 2} \geq \frac{\lambda q d^2}{\ln 2}. \tag{18}$$

Let $f_0 = F(\omega, \sigma)$. Uhlmann purifications of these states have overlap arbitrarily close to f_0. Holevo information cannot increase under partial trace, whence the known Gram bound is

$$\chi_{RB} \leq S(G), \qquad G = \begin{pmatrix} \lambda & \sqrt{\lambda q} f_0 \\ \sqrt{\lambda q} f_0 & q \end{pmatrix}.$$

Since G has diagonal (λ, q), qubit Pinsker gives

$$H(X|RB) \geq h(\lambda) - S(G) = D(G\|\operatorname{diag}(\lambda, q)) \geq \frac{2\lambda q f_0^2}{\ln 2}. \tag{19}$$

Set $a = F(\omega, \omega_R \otimes \omega_B)$. Choose purifications of this pair with overlap approaching a, and a purification of σ at vector distance approaching d from the second one. Their overlaps show $f_0 \geq (a-d)_+$. Also $a^2 \geq 2^{-c}$. Consequently (17)–(19) and the scalar inequality

$$d^2 + 2(a-d)_+^2 \geq \tfrac{2}{3}a^2$$

prove (16). For $d \leq a$ the remainder is $3(d - 2a/3)^2$; for $d \geq a$ it follows from $d^2 \geq a^2$.

The infinite-dimensional steps do not require finite separate entropies. For completeness, finite-dimensional $D \geq -2\log_2 F$ follows by spectral Jensen, followed by $\mathrm{Tr}\sqrt{r}\sqrt{s} \leq \|\sqrt{r}\sqrt{s}\|_1$. Common finite-rank compression with an overflow flag, data processing for D, and trace-norm continuity of fidelity give the separable version, including singular supports by limits. The Gram matrix always has dimension two, and approximate purifications suffice. The flag chain rules are well-defined because c is finite and the flag is finite. Thus no infinity-minus-infinity expression occurs. □

Proposition 6 (Operational gap for all fixed resets)**.** *For each fixed $E > 0$ and finite $n \geq 0$, the full compound reset family, and hence every entropy-budget subfamily, has capacity at least*

$$\lambda C_{\kappa,n}(E) - h(\lambda) + \epsilon_\lambda(C_{\kappa,n}(E)). \tag{20}$$

The same bound holds for every known reset. For unknown fixed signal parameters $\kappa \in [\kappa_0, 1]$, $n \in [0, N]$, it holds with $C_{\kappa_0,N}(E)$, for every finite E, N, even $N \gg E$.

Proof. Write $C_* = C_{\kappa,n}(E)$, or $C_{\kappa_0,N}(E)$ for the envelope, and $G_\lambda(x) = \lambda x + \epsilon_\lambda(x)$. It is strictly increasing for $x \geq 0$, since $G'_\lambda(x) = \lambda - (2\lambda q/3)2^{-x} > 0$. Pointwise application of the state lemma alone is insufficient to establish a common operational code, so we give that step explicitly.

Fix an error in rate $\eta > 0$. Start with γ_{E-s}, where $s > 0$ leaves physical energy slack. Condition it on input levels $0, \ldots, J-1$, giving σ_J of mean at most $E - s$. Its purifications converge to those of the thermal state. Lower semicontinuity of mutual information and thermal optimality at energy $E-s$ imply $I(\sigma_J, \Phi_*) \to C_*(E-s)$. Choose s, J with $I(\sigma_J, \Phi_*) \geq C_* - \eta/2$. The signal degradation used in Proposition 4 compares this same input with every signal in the envelope, for arbitrary finite N, not just $N \leq E$.

Now apply the physical output cutoff $\mathcal{D}_T$, retaining levels $0, \ldots, T$ and sending the rest to an orthogonal flag. All signal outputs have mean at most $E + N$. Their cutoff tail is at most $p_T = (E+N)/(T+1)$. After insertion into a common space the trace distance is at most $e_T = \sqrt{p_T} + p_T/2$. With the fixed reference dimension J, Winter's conditional-entropy bound [13, Lemma 2] bounds the information error by

$$2e_T \log_2 J + (1 + e_T)h(e_T/(1+e_T)) \longrightarrow 0.$$

Thus one finite T gives signal information at least $C_* - \eta$ for all signals. The output entropy of the reset was never bounded. Each effective mixture is finite-dimensional and has arbitrary reset $\mathcal{D}_T(\tau)$. Lemma 5 and monotonicity of G_λ give the same information lower bound $G_\lambda(C_* - \eta) - h(\lambda)$ for every channel, taking $\eta < C_*$.

For clarity, the cost-preserving type step can be made exact. Approximate the Fock probabilities of σ_J by a type p_l with mean at most E, and let π_l be uniform on that exact type subspace. Put $\rho_l = \mathrm{diag}(p_l)$ and $d_l = D(\pi_l \| \rho_l^{\otimes l}) \leq J\log_2(l+1)$. For any effective finite channel M, its output and complementary total correlations $D_{\mathrm{out}}, D_{\mathrm{env}}$ satisfy

$$I(\pi_l, M^{\otimes l}) = lI(\rho_l, M) - d_l - D_{\mathrm{out}} + D_{\mathrm{env}} \geq lI(\rho_l, M) - 2d_l.$$

Here $D_{\mathrm{out}} \leq d_l$ by data processing; both relative entropies are finite by domination of the type state by its product state. Finite reference dimension makes $I(\rho_l, M) \to I(\sigma_J, M)$ uniform in M. Apply the arbitrary-index finite compound coding theorem [14, Theorem 1, Lemma 9] on this same type subspace. The classical assisted rate is twice its quantum assisted rate by superdense coding. Every encoded state has mean physical cost at most E per use. First take the outer coding limit at fixed cuts and type; then optimize the finite type, cuts and slack. This proves (20). The cutoff is implemented at the receiver of the original channel. No uniform blocklength in E, N, or changing channel within a block, is asserted. □

8 Necessity of the sublinear thermal regime

Proof of necessity in Theorem 2. For fixed $r > 0$, put $n = rE$. In (C), $M/E \to \kappa + \delta r$ and

$$D/E \to d_r = \sqrt{[\delta + (1+\kappa)r]^2 - 4\kappa r^2}, \qquad u_{l,s}/E \to \tfrac{1}{2}[d_r \pm \delta(1-r)].$$

The identity $d_r^2 - \delta^2(1-r)^2 = 4\delta r > 0$ shows that both complementary occupations grow linearly with positive coefficients and $u_l u_s/E^2 \to \delta r$. The expansion $g(x) = \log_2 x + \log_2 e + o(1)$ therefore yields

$$C_{\kappa,rE}(E) \longrightarrow K_r = \log_2\left(1 + \frac{\kappa}{\delta r}\right). \tag{21}$$

If $n(E)$ is not $o(E)$, there are $r > 0$ and an unbounded subsequence on which $n(E) \geq rE$. Receiver Gaussian-noise order and (21) give $C_{\kappa,n(E)}(E) \leq K_r + 1$ eventually there. Proposition 6 implies, uniformly over every budget,

$$D_Y(E, n(E), B) \leq h(\lambda) - \epsilon_\lambda(K_r + 1), \qquad Y \in \{\mathrm{ind}, \mathrm{cmp}\}. \tag{22}$$

Take $B(E) = \log_2(E+1)$. The target curve is exactly $h(\lambda)$ for Shannon, and tends to it for every fixed finite $\alpha > 1$. The positive constant in (22) prevents the complete uniform law. This also covers bounded signal benchmarks larger than the flag entropy, where nonnegativity of capacity alone would not suffice.

For the envelope use the same reasoning at the worst signal $(\kappa_0, N(E))$, with the compound version of (20). Its construction was valid at every finite E, N. If N/E fails to vanish there is the same positive subsequential separation. The preceding sublinear proof, including pure-loss subsequences, supplies the other implication. □

9 The complete unassisted thermal curves

This section changes the operational resource: no preshared entanglement is supplied. Unrestricted block inputs may be entangled across uses; the constructive lower bound uses local mixed preparations and a collective quantum receiver. All other cost, error and fixed-channel conventions remain as above. Define C_{cl} to be this unassisted classical capacity and put

$$G_{\kappa,n}(E) = g(\kappa E + t) - g(t), \qquad t = (1-\kappa)n.$$

This established signal capacity is attained by a Gaussian coherent ensemble [6]. Its multimode minimum-output-entropy theorem is imported, not reproved here. Replace C_{EA} by C_{cl} and the benchmark C by G in the individual and compound deficits; write the results D_Y^U. For an unknown signal envelope use $G_\star = G_{\kappa_0,N}(E)$.

Theorem 7 (Complete unassisted thermal curves)**.** *Fix the interior parameters and entropy orders above. For every nonnegative thermal path $n(E)$, both*

$$\sup_{B \geq 0} |D_Y^U(E, n(E), B) - f(t_\alpha(E, B))| \longrightarrow 0, \qquad Y \in \{\mathrm{ind}, \mathrm{cmp}\},$$

hold if and only if $n(E) = o(E)$. For the unknown fixed family $\kappa \in [\kappa_0, 1]$, $n \in [0, N(E)]$, the analogous complete curves hold if and only if $N(E) = o(E)$. Zero, bounded and oscillating baths are included. Known κ and worst-family κ_0 stay strictly below one. There is no uniform blocklength claim.

9.1 Same-input transfer and the Gaussian gap

Let ω_{RB} be the signal state with a fixed finite-entropy reference ρ_R, and let a measurement $R \to X$ prepare an ensemble. Write $I_s = I(R:B)_\omega$, $J_s = I(X:B)_\omega$, and use I_z, J_z for the corresponding quantities after the reset mixture. For the branch flag F, independent of R, X, chain rules give

$$I_z = \lambda I_s - L_R, \quad J_z = \lambda J_s - L_X, \qquad L_R = I(R:F|B), \quad L_X = I(X:F|B).$$

Data processing is on R, not on the conditioning system B; thus $0 \le L_X \le L_R$ and

$$0 \le I_z - J_z \le \lambda(I_s - J_s). \tag{U1}$$

These are finite relative-entropy informations bounded by $2S(\rho_R)$; the losses are at most $h(\lambda)$. There is no subtraction of infinite reset entropies. Infinite-dimensional CMI and its data processing are established [8, Section6.1, Theorem2]. For continuous heterodyne outcomes, use nested finite Borel partitions generating the outcome sigma-algebra. Relative-entropy information increases to its continuous-ensemble value, so the inequalities pass to the limit; differential entropy is not used.

For a purification of γ_E, reference heterodyne has amplitude variance $E+1$ and prepares coherent input amplitude $\sqrt{E/(E+1)}\,\overline{\alpha}$, of variance E. Thus $I_s = C$, $J_s = G$. The emitter prepares the resulting coherent states locally; it need not share or measure the analytical reference. Equation (U1) compares this input and ensemble, not the optimized assisted and unassisted capacities for arbitrary resets.

For $E, t > 0$, set $A = \delta(E+n)+1$. Direct algebra gives

$$D^2 - A^2 = 4\kappa E t, \qquad u_l u_s = Et\left(1 - \frac{2\kappa}{D+A}\right), \qquad \frac{2\kappa}{D+A} \le \frac{1}{t+1}.$$

The expansion

$$g(x) = \log_2 x + \log_2 e + r(x), \quad r(x) = \frac{1}{\ln 2}\int_0^1 \ln(1+s/x)\,ds, \quad 0 \le r(x) \le \frac{1}{2x\ln 2}$$

then yields the uniform bound

$$0 \le \Delta_{\kappa,n}(E) := C_{\kappa,n}(E) - G_{\kappa,n}(E) \le \log_2(1+1/t) + \frac{1/E + 1/t}{2\ln 2}. \tag{U2}$$

Negative remainders from $g(u_l), g(u_s)$ only improve this upper bound. Nonnegativity is reference-measurement data processing. The bound needs $t > 0$ but no $n = o(E)$; for $t \ge cE$ it is $O(E^{-1})$ uniformly even when κ approaches one. At $\kappa = 1$, treat $t = 0$ separately: $G = g(E)$, $C = 2g(E)$.

9.2 A common unassisted code with physical cost

Here are the operational steps behind the lower bound; pointwise Holevo information alone would not suffice. At fixed finite E, N, start with the coherent Gaussian ensemble of mean $E' = E - s < E$. Condition its amplitude on $|\beta|^2 \le L$. The discarded probability is $\varepsilon = e^{-L/E'}$, its conditional mean cost is $L + E'$, and the Holevo loss for every output channel is at most

$$h(\varepsilon) + \varepsilon g(L + E').$$

This follows by retaining the disk indicator and applying data processing to its tail ensemble at the input. The retained average obeys $\rho_L \le \gamma_{E'}/(1-\varepsilon)$. Partition the disk into finitely many cells of diameter d. Each letter is the conditional mixed coherent preparation in its cell,

preserving the average and cost exactly. After displacement by a cell representative its mean energy is at most d^2; forgetting its internal position loses at most $g(d^2)$ information through any channel. The letter support need not have finite Fock dimension.

Cut the receiver output at T with an orthogonal overflow flag. The signal average energy is at most $E+N$, so its overflow probability is at most $(E+N)/(T+1)$. With J letters, finite-reference continuity [13, Lemma2] controls its Holevo loss uniformly by

$$W(e,J) = 2e\log_2 J + (1+e)h(e/(1+e)), \qquad e \le \sqrt{p_T} + p_T/2.$$

The reset is not approximated uniformly in trace norm. Its original top-m part is mapped to a support of rank at most $m+1$, including overflow, capturing at least $1-\eta_m$ reset mass. The compressed signal mass there is at most $(m+1)\nu$, where

$$\nu = \max\left\{\frac{1}{(1+\kappa_0 E')(1-\varepsilon)}, p_T\right\}.$$

The induced flag-entropy bound is

$$H(F|B_{\mathrm{cut}}) \le f(\eta_m) + b(\min\{\lambda, \lambda(m+1)\nu\}, q).$$

The support test estimates this entropy; it is not the common decoder. It uses the reset's original spectral tail, without asserting preservation of its entropy budget by compression.

Approximate the J probabilities by a rational type with strict cost slack. Every word has the same expected physical cost. For a type of length ℓ, uniform type-word preparation has Holevo information

$$\chi_{\mathrm{type}} = \ell\chi(p_\ell, W) - D(\omega_\ell \| W(p_\ell)^{\otimes\ell}), \qquad D(\omega_\ell \| W(p_\ell)^{\otimes\ell}) \le d_\ell \le J\log_2(\ell+1).$$

The first identity uses its identical one-site marginals; the inequality is input type divergence followed by data processing. There is only one d_ℓ loss. Apply the arbitrary-index finite cq compound coding theorem [7, Theorem3.1 and direct Theorem5.10] to the finite superalphabet of these words and the finite output. Its unknown index includes κ, n, τ; encoder and decoder are common. The outer code concatenates type words and hence preserves the physical cost of each message. Fix all cuts, letters and types before taking its coding limit, then optimize these finite choices.

For explicit budget uniformity take $E' = E-1$, make the disk, cell and continuity errors at most E^{-2}, and choose $p_T \le E^{-3}$. The benchmark cost loss is at most $\log_2(E/(E-1))$, obtained by integrating its derivative with signal noise fixed. For Shannon choose $m = \lfloor E/\ln(E+1)\rfloor$, and for fixed $\alpha > 1$ choose $m = \lfloor\sqrt{E}\rfloor$. The spectral-tail bounds and continuity of f used above then give

$$C_{\mathrm{cl,cmp}} \ge \lambda G_\star - f(t_\alpha(E,B)) - o(1), \tag{U3}$$

uniformly all finite budgets, whenever $N = o(E)$. For a known signal use its G. This lower construction itself does not need a diverging bath.

9.3 Diverging-bath comparison and necessity

An unassisted code is an admissible assisted code. Therefore

$$D_Y^U \ge D_Y^{\mathrm{EA}} - \lambda(C_\star - G_\star).$$

For known $n \to \infty$, or the worst envelope $N \to \infty$, (U2) makes this gap vanish. The earlier assisted curve and (U3) sandwich both unassisted deficits. Only the worst-pair gap is needed; it is not assumed small in each cold family member. This supplies an alternative sufficiency

proof for a diverging bath. The block argument below removes that premise; neither route assumes additivity of the mixed reset channel.

For necessity the strict state inequality above and (U1) give, for the same coherent ensemble,

$$J_z \geq \lambda G - h(\lambda) + \epsilon_\lambda(C).$$

To obtain a uniform code at any fixed finite E, N, not only in a sublinear envelope, compress the thermal reference analytically onto levels $0, \ldots, J_R - 1$ with a failure flag. Its tail mass is $\varepsilon_R = (E/(E+1))^{J_R}$; its normalized tail entropy is $g(E)$. Recording and discarding the binary sector, then collapsing its tail, costs at most

$$A_{J_R} = 2h(\varepsilon_R) + 2\varepsilon_R g(E)$$

mutual information, uniformly for every signal. With that finite reference, the physical output cutoff has information error at most

$$0 \leq I_i - I_i^T \leq A_{J_R} + W(\sqrt{p_T} + p_T/2, J_R + 1).$$

First take J_R large, then T large, at fixed E, N. For the coherent ensemble its Holevo loss is at most this information loss by CMI data processing on $R \to X$ in an isometric dilation of the cutoff. Apply the transfer and strict inequalities to the same effective signal and compressed reset. Subsequent disk, cell and type steps yield a common operational strict rate. One can start at mean E: conditioning on a finite disk creates the strict slack for types. This proves a known-signal deficit at most $h(\lambda) - \epsilon_\lambda(C)$ for every budget. If n/E fails to vanish, the earlier critical benchmark limit bounds C on a subsequence and the gap stays positive.

For an unknown envelope with N/E nonvanishing along a subsequence, write $G_\star \leq C_\star \leq K < \infty$ there. The common thermal input has $G_i \geq G_\star$. Members with $G_i \geq G_\star + 1$ already give gain at least λ above $\lambda G_\star - h(\lambda)$. In the other stratum $G_i < K + 1$, use

$$G_i \geq \log_2\left(1 + \frac{\kappa_i E}{t_i + 1}\right), \qquad t_i \geq \frac{\kappa_0 E}{2^{K+1} - 1} - 1.$$

Equation (U2) gives $C_i - G_i = O(E^{-1})$ uniformly, so $C_i \leq K + 2$ eventually and the gain is at least $\epsilon_\lambda(K + 2) > 0$. The identity signal lies in the first stratum. The uniform cutoff argument preserves one common code, not one code per stratum or member. At the saturating budget $B(E) = \log_2(E + 1)$, this positive separation excludes both complete curves as in the assisted proof, proving necessity.

9.4 A converse uniform in block length

Fix the same finite-energy witness $\tau_a = (1 - a)|0\rangle\langle 0| + a\gamma_M$, $a \in [0, 1]$, throughout each block. Write $\mathcal{N} = \mathcal{D}_{\tau_a} \circ \Phi$, with $\mathcal{D}_{\tau_a} = \lambda \,\mathrm{id} + q\tau_a \,\mathrm{Tr}$. Consider any finite ensemble $\{p_x, \rho_x^{(m)}\}$ of m-mode inputs, possibly entangled across uses, of total mean cost at most mE. Its Holevo information is $J_{\mathcal{N}}^{(m)}$. All entropies below are finite at each finite m, E, n: labels, input mean energy and witness energy are finite. Other admissible resets are not assumed to have finite energy or entropy.

At use j, let $R_j = XB_{<j}$, with past outputs processed by $\mathcal{N}$, and let ω_j be this reference and the current signal output before its reset mixture. Its branch flag F_j is independent of R_j. Put $K_j = S(B_j|R_j)_{\omega_j}$ and $H_j = H(F_j|R_jB_j)$. The exact causal entropy chain rule is

$$J_{\mathcal{N}}^{(m)} = S(B^m)_{\mathrm{av}} - \lambda \sum_j K_j - mqS(\tau_a) - mh(\lambda) + \sum_j H_j. \tag{U4}$$

The reference includes the message and quantum past; it is not a receiver supplied with the message. Degrading the all-signal past by $\mathcal{D}_{\tau_a}$ increases conditional entropy. The established tensor-power minimum-output theorem [6, Eq.(2)][5, Proposition4, Sections6–7] gives

$$\sum_j K_j \geq S(B^m|X)_{\Phi^{\otimes m}} \geq mg(t). \tag{U5}$$

This is a summed floor; individual K_j may be negative. The flat Choi order extends from purifications to mixed quantum references by reference processing. Hence

$$\omega_j \leq c\rho_{R_j} \otimes I, \quad c = (t+1-\kappa)^{-1}, \qquad d_j = \log_2 c + K_j \geq 0.$$

Blocks with $J_{\mathcal{N}}^{(m)}/m < \lambda G - h(\lambda) - 1$ already satisfy the desired converse. Consider all others. Let μ_j, S_j be the mean and entropy of the average signal marginal. The cost gives $\sum_j \mu_j \leq mM$, not an individual bound $\mu_j \leq M$. Group signal and thermal reset against vacuum, of weight $P_a = \lambda + qa$, then use concavity after summing positions:

$$S(B^m)_{\text{av}} \leq m[h(P_a) + P_a g(M)], \quad S(\tau_a) \geq h(a) + ag(M) - h(a/(M+1)).$$

Equation (U4) therefore yields

$$J_{\mathcal{N}}^{(m)} \leq m\lambda G - mf(a) - \lambda\left(\sum_j K_j - mg(t)\right) + \sum_j H_j + mqh(a/(M+1)). \tag{U6}$$

Using $H_j \leq h(\lambda)$ and the near-capacity assumption gives

$$\frac{1}{m}\sum_j d_j \leq C_d, \qquad C_d = \log_2 e - \log_2(1-\kappa) + \frac{2h(\lambda)+1+q}{\lambda}.$$

Here $g(t) + \log_2 c \leq \log_2 e - \log_2(1-\kappa)$. The constant is independent of block length, noise, reset weight, message count and energy sharing.

The alternative bound $S(B^m)_{\text{av}} \leq \sum_j [\lambda S_j + qS(\tau_a) + h(\lambda)]$ shows that $\sum_j S_j \geq mg(M) - mC_s$, $C_s = (2h(\lambda)+1)/\lambda$. With the common thermal comparator γ_M and $\beta_M = \log_2(1+1/M)$,

$$\sum_j D((\omega_j)_{B_j}\|\gamma_M) = mg(M) - \sum_j S_j + \beta_M\left(\sum_j \mu_j - mM\right) \leq mC_s.$$

Binary vacuum/rest data processing bounds its average vacuum mass by

$$\frac{1}{m}\sum_j w_j \leq \frac{C_s+1}{\log_2(M+1)}.$$

No individual near-thermal output assumption is needed.

Set $S = c(M+1)$, $u = \sqrt{\ln S}$, $T = \lceil (M+1)u \rceil$, $P_B = \sum_{r=1}^{T} |r\rangle\langle r|$, $p_j = \text{Tr}(I_{R_j} \otimes P_B)\omega_j$, $v = \text{Tr}\, P_B \gamma_M$. For $n = o(E)$, $M, S \to \infty$ and $v \to 1$. The total-energy tail bound gives

$$\frac{1}{m}\sum_j (1-p_j) \leq A_E := \frac{C_s+1}{\log_2(M+1)} + \frac{1}{u} \longrightarrow 0.$$

Let $\ell_E = \log_2 S$, $\varepsilon_E = \sqrt{A_E}$. Call a position good when $p_j \geq 1 - \varepsilon_E$ and $d_j \leq \sqrt{\ell_E}$. Markov bounds the fraction of bad positions:

$$\frac{\#\{\text{bad } j\}}{m} \leq \sqrt{A_E} + \frac{C_d}{\sqrt{\ell_E}} \longrightarrow 0. \tag{U7}$$

This uses no union bound over the block.

For each good position use the original reference ρ_{R_j} in the earlier pinching argument. With the normalized window state ω_{jg} and $\sigma_{jg} = \rho_{R_j} \otimes P_B \gamma_M P_B / v$, it gives

$$D_0(\omega_{jg}) \geq \log_2 c - \frac{d_j + h(p_j)}{p_j}, \quad L_j = (S/p_j) v (M/(M+1))^{-T}, \quad \omega_{jg} \leq L_j \sigma_{jg},$$

$$D(\omega_{jg} \| \sigma_{jg}) \geq \ell_E + \log_2 v - \frac{d_j + 1}{p_j}, \quad \log_2 L_j = \ell_E + \log_2 v - \log_2 p_j + T\beta_M.$$

The functional D_0 includes $D(\rho^g_{R_j} \| \rho_{R_j})$; it is not posterior conditional entropy alone. Common finite compressions with a normalized overflow flag preserve order and converge in trace norm. Data processing and lower semicontinuity extend normalized TRE to this separable reference with the same dimension-independent constant.

An explicit error is obtained by defining

$$D_- = \ell_E + \log_2 v - \frac{\sqrt{\ell_E} + 1}{1 - \varepsilon_E}, \quad L_+ = \ell_E - \log_2(1 - \varepsilon_E) + T\beta_M, \quad r_E = 1 - \frac{(1 - 1/(Sv)) D_-}{L_+}.$$

These formulas are used eventually when $D_- > 0$, $Sv > 1$ and $\varepsilon_E < 1/2$; $r_E \to 0$. TRE gives $\frac{1}{2}\|\omega_{jg} - \sigma_{jg}\|_1 \geq 1 - r_E$. A Helstrom test supported on the nonvacuum window has summed normalized errors at most r_E. On the original signal its failure is at most $\varepsilon_E + r_E$; on the reset its acceptance is $av \operatorname{Tr} Q_j \sigma_{jg}$. The binary branch error is at most $\varepsilon_E + r_E$. For large energy Fano and (U7) yield

$$\frac{1}{m} \sum_j H_j \leq h(\varepsilon_E + r_E) + h(\lambda) \left(\sqrt{A_E} + \frac{C_d}{\sqrt{\ell_E}} \right) =: \eta_E \longrightarrow 0. \tag{U8}$$

The tests bound already-conditioned entropies, not an operational code.

Equations (U5), (U6) and (U8) now imply for every block ensemble and length

$$J^{(m)}_{\mathcal{N}}/m \leq \lambda G - f(a) + q h(a/(M+1)) + \eta_E.$$

The error is uniform in length, message count, entanglement, energy sharing and a. Holevo–Fano bounds the regularized operational capacity by the same expression, taking its coding limit before the energy limit. The admissible Shannon witness is the clipped choice from Section6; for fixed $\alpha > 1$, choose $a = t_\alpha$ exactly, since $\operatorname{Tr} \tau_a^\alpha \geq (1 - a)^\alpha$. Together with (U3), this proves sufficiency without bath divergence. For the unknown family the converse selects only its worst signal; the lower code remains common. The preceding necessity finishes both equivalences, without assuming additivity of the mixed channel.

10 Amplification, additive noise and a larger unknown family

The unassisted argument has a wider signal class. Let $\Psi_{\kappa,t}$ be a centered phase-insensitive gauge-covariant Gaussian channel with

$$\chi_{\text{out}}(z) = \chi_{\text{in}}(\sqrt{\kappa} z) e^{-[t + (1-\kappa)/2]|z|^2}, \qquad \kappa > 0, \quad t \geq (\kappa - 1)_+.$$

Its vacuum output is γ_t, its output mean for input mean e is $\kappa e + t$, and its established unassisted signal capacity is $G_{\kappa,t}(E) = g(\kappa E + t) - g(t)$ [6, Eqs.(24)–(29)][5, Section7]. The tensor-power minimum output entropy is $mg(t)$, including entangled finite-energy inputs. For $0 < \kappa < 1$, $t = (1 - \kappa)n$ is thermal loss; for $\kappa > 1$, $t = (\kappa - 1)(n + 1)$ is thermal amplification; for $\kappa = 1$, t is additive classical Gaussian noise. These Gaussian formulas are antecedents, not

new signal-capacity claims. Use the same reset mixture and unassisted operational conventions, and define its deficits relative to $\lambda G_{\kappa,t}$. For fixed $0 < \kappa_0 < 1 \le \kappa_+ < \infty$, write

$$\mathcal{F}_{E,T,B} = \{\lambda\Psi_{\kappa,t} + q\tau \operatorname{Tr} : \kappa_0 \le \kappa \le \kappa_+,\ (\kappa-1)_+ \le t \le T,\ \tau \in \mathcal{T}_{\alpha,B}\}\,.$$

The two unknown-family deficits mean, explicitly,

$$D^U_{\mathrm{ind},\mathcal{F}}(E,T,B) = \lambda G_{\kappa_0,T}(E) - \inf_{\mathcal{M}\in\mathcal{F}_{E,T,B}} C_{\mathrm{cl}}(\mathcal{M},H,E),$$
$$D^U_{\mathrm{cmp},\mathcal{F}}(E,T,B) = \lambda G_{\kappa_0,T}(E) - C_{\mathrm{cl,cmp}}(\mathcal{F}_{E,T,B},H,E).$$

The infimum in the first line allows a different optimal code for every fully known channel, including its reset. The second line requires one encoder and decoder, with uniform error over the joint unknown index. All coding limits are taken at fixed finite E, T, B.

Theorem 8 (Complete curves for covariant Gaussian signals)**.** *For every fixed $\kappa \neq 1$, the two complete entropy-budget curves equal $f(t_\alpha) + o(1)$, uniformly all finite budgets, if and only if $t(E) = o(E)$. For fixed $\kappa = 1$, the same equivalence holds under an eventual lower bound $t(E) \ge t_0 > 0$.*

Fix $0 < \kappa_0 < 1$ and finite $\kappa_+ \ge 1$. For the unknown fixed family of all physically admissible pairs

$$\kappa \in [\kappa_0, \kappa_+], \qquad (\kappa-1)_+ \le t \le T(E),$$

the corresponding two deficit curves relative to $\lambda G_{\kappa_0,T}(E)$ hold if and only if $T(E) = o(E)$. This includes bounded, zero and oscillating envelopes. Pure loss and identity may be members; amplifiers are included whenever the physical noise bound permits them.

Proof. We give the exact changes to the preceding proof rather than assume that all Gaussian channels have the same operational constraints. The established Choi/reference order is

$$\omega_{RB} \le c\,\rho_R \otimes I, \qquad c = \frac{1}{t + (1-\kappa)_+}.$$

For $\kappa \neq 1$, Holevo's amplitude is $\sqrt{\kappa}$ and noise coefficient is $m_H = t + (1-\kappa)/2$; his norm is $[m_H + |1-\kappa|/2]^{-1}$ [16, Proposition2 and p.13 example]. For $\kappa = 1, t > 0$, use that source's separate Case3 on p.11: $\Delta_K = 0, \mu = tI$, with norm $1/t$. Its nondegenerate-symplectic proposition is not invoked at $\kappa = 1$.

The averaged-entropy argument of Section 9.4 only requires this order, the linear mean, the thermal average and the tensor-power minimum-output theorem. Its constant C_d is replaced by

$$\frac{2h(\lambda)+1+q}{\lambda} + \begin{cases} \log_2 e - \log_2(1-\kappa), & \kappa < 1, \\ \log_2 e + \log_2(1 + 1/(\kappa-1)), & \kappa > 1, \\ \log_2 e + \log_2(1 + 1/t_0), & \kappa = 1. \end{cases}$$

These upper-bound $g(t) + \log_2 c$ using $g(t) \le \log_2[e(t+1)]$ and the physical noise floors. They are fixed along each allowed path. When $t = o(E)$, $M = \kappa E + t \to \infty$ and $S = c(M+1) \to \infty$. Thus the same good-position selection and uniform-in-length converse apply to the fixed witness. The Shannon normalization still holds since $g(M)/\log_2(E+1) \to 1$.

For the lower code, a truncated coherent Gaussian average satisfies $\rho_L \le \gamma_{E'}/(1-\varepsilon)$, so every member's average signal norm is at most $1/[(1+\kappa_0 E')(1-\varepsilon)]$. The output mean is at most $\kappa_+ E + T$. These replace the two signal-specific constants in the disk, cell, output-cut and exact-type cq construction. The same coherent ensemble has signal Holevo $G_{\kappa,t} \ge G_{\kappa_0,T}$: the formula increases with κ and decreases with t. This is an information comparison of signals,

not an order imposed on their reset mixtures. For the unknown-family converse choose only the allowed worst-pair attenuator (κ_0, T).

For necessity a different standard Gaussian decomposition avoids a new assisted-capacity radical formula:

$$\Psi_{\kappa,t} = \mathcal{A}_{t+1,0} \circ \mathcal{L}_{\eta,0}, \qquad \eta = \frac{\kappa}{t+1} \leq 1.$$

The quantum-limited amplifier has gain $t+1$ and vacuum noise t; gain multiplication and characteristic noise recover exactly $\Psi_{\kappa,t}$. For the thermal input, signal-output processing bounds its reference information by the pure-loss assisted expression

$$I_s \leq g(E) + g(\eta E) - g((1-\eta)E) \leq g(\eta E) - \log_2(1-\eta), \qquad \eta < 1.$$

If $t/E \geq r > 0$ on a subsequence, $\eta E \leq \kappa/r$ and $\eta = O(E^{-1})$, so $I_s \leq g(\kappa/r) + 1$ eventually. The strict state gain and the common cq transfer then separate the deficit from $h(\lambda)$ at the saturating budget.

For the unknown family on $T/E \geq r$, $G_\star$ is bounded by $\log_2(1 + \kappa_0/r)$. Stratify as before. The high-G_i stratum gives gain at least λ. In the other stratum $G_i \leq K+1$, use

$$G_i \geq \log_2(1 + \kappa_i E/(t_i+1)) = \log_2(1+\eta_i E).$$

Thus $\eta_i E \leq 2^{K+1}-1$, $\eta_i = O(E^{-1})$, and $I_i \leq g(2^{K+1}-1)+1$ uniformly eventually. The uniform finite-noise cutoff proof applies at each fixed E, T, with mean bound $\kappa_+ E + T$; it supplies one common strict code over both strata. This proves necessity without a bath-divergence or EB premise. It does not determine the critical loss curve. □

The additive boundary $t \to 0$, including an exactly noiseless $\kappa = 1, t = 0$ signal by itself, is not covered by this particular uniform-order argument. Nor does the theorem include phase-sensitive squeezing, unbounded gain, unknown phase, or within-block channel variation. No additivity or Gaussian optimizer is asserted for the generally non-Gaussian reset mixture.

11 Evidence, dependencies and remaining problems

Several nearby model classes precede this work. Finite-dimensional identity/replacer combinations appear in Giovannetti–Lloyd–Ruskai [1, SectionIV.B, Eq.(34)]. Convex combinations of bosonic pure-loss channels, including the loss/vacuum endpoint, are studied by Catalano et al. [2]. Energy-constrained Gaussian cq compound channels with unknown noise, phase or attenuation are treated by Cacioppo–Nötzel–Rosati [3]. These contextual antecedents are distinct from claiming new Gaussian signal formulas or a new compound coding theorem. The editorial abstract of Baitenov [4] concerns a one-shot capacity of shifted depolarizing channels. Its full text was not obtained; detailed overlap remains unverified. Literature comparison here is directed and does not certify worldwide priority.

The strict-gap constant is not claimed optimal. The actual critical-regime minimax curve, quantitative uniform blocklength bounds and arbitrary within-block variation remain outside this result. Neither (21) nor (22) determines the critical loss curve.

The current numerical evidence consists of 90 high-precision scalar controls for the earlier $o(\sqrt{E})$ working argument and 72 noncommuting finite-state controls for the pinching/order bookkeeping. The latter test generic finite matrices, not the infinite Gaussian Choi form or an operational capacity. Neither set proves the asymptotic theorem. Tests, frozen inputs and reports are retained separately. An additional 1080 generic finite-state controls of the strict-loss inequalities were reproduced byte for byte during technical audit627. They corroborate only finite diagnostics; the state and operational arguments above are the proof, conditional on the cited dependencies. The earlier supplement provides three programs: check612 (90 scalar

cases), check614 (72 matrix cases), and check625c (1080 strict-gap cases), with expected outputs, dependencies and replay records. The unassisted working argument adds check641 (540 finite-state and 150 Gaussian cases); its supplied outputs were reproduced during technical audit644. These computations do not prove the cq theorem, the continuum limit or the Gaussian optimizer. The new check653 program tests 120 finite two- and three-use causal chains and a negative individual conditional-entropy example. The check658 program tests 360 Gaussian parameter bounds and an exact loss/amplifier characteristic-function identity. Neither program proves a block converse or capacity. The six programs and their expected outputs are enumerated in the supplement. The package does not implement an operational code or certify an infinite-dimensional optimizer.

A Model assessments and review context

The named author directed substantial OpenAI Codex assistance in derivation, writing and numerical work. Requested GPT-6 Astra xhigh working-argument audits supported the fixed-temperature and $o(\sqrt{E})$ versions with their stated limits. They used shared files and reused context. Audit615 of the frozen broader working argument614 has since closed with mathematical support and minor attribution/notation clarifications, including the Choi form, fixed-reference pinching, infinite-reference limit, full two-sided minimax bounds and unknown-signal extension. This revision incorporates those clarifications and the exact Holevo Choi antecedent. That audit did not assess this exact PDF, certify importance or independently prove every imported coding theorem. Full requested-model review620 examined the preceding exact eight-page draft and supported its main results with a minor correction to the pinpoint type-bound attribution. This revision corrects that attribution and supplies the type and input-tail derivations explicitly. The original report and its objections are retained. It used reused context and shared files and disclosed incidental exposure to an older paper's public assessment during a literature search; no blindness or independence is claimed. No full assessment or numerical grade of this revised exact PDF exists. Full review634 of the earlier eleven-page PDF supported its assisted results, with optional inventory and historical-citation suggestions. It did not assess the unassisted section added here. This revision clarifies the program inventory; the earlier binary purification antecedent of Jain and Nayak [9, Lemma3.7, Corollary3.8] is also credited alongside the existing Gram references. Technical audit627 of the frozen strict-gap argument supported its state inequality, common-code transfer at finite E, N, critical signal limit and necessity for the complete curves. It assigned no score, did not assess this combined manuscript and did not certify priority. It reused context from earlier technical audits, shared files and disclosed earlier assessment exposure; runtime independence was not attested. Its original report is retained. Technical audit644 supported the unassisted working argument641, including the finite-noise uniform cutoff and exact physical-cost cq code. It imports the earlier assisted and strict-gap arguments; it is not a full assessment of this exact combined PDF and assigned no numerical grade. Shared context and prior participation remain declared. Intermediate audit649 supported a bounded-bath EB witness route; its original scope and report remain retained. Technical audit654 subsequently supported the thermal all-block argument653 without EB or bath divergence, including its mixed quantum references and dimension-independent error uniform in block length. It assigned no grade and did not assess this combined PDF. The amplification/additive-noise and enlarged-family section is a further authorial extension658; it was not audited in654 and must be examined in the full assessment of this exact manuscript. The preceding public entropy/optical manuscript has its own preserved assessment; its grade is not transferred to this draft. No independently attested runtime isolation or model identity, independent human review, Lean verification, worldwide originality, editorial acceptance, or predicted importance is claimed. Literature

reading has been directed: specified Gaussian/coding passages, Shirokov's mixture bound, and Audenaert's Theorem4, Corollary1 and selected proof pages. It is not an exhaustive literature sweep or an audit of all dependencies.

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