Exact Analytic, Geometric, Partition, and Slice Rank Separations over Finite Fields

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Abstract

We establish three exact, closed-form separation theorems linking analytic rank (ar), geometric rank (gr), partition rank (pr), and slice rank (sr) of structured multilinear tensors over finite fields $\mathbb{F}_p$. First, on the complete 3-uniform hypergraph tensor $K_n^{(3)}(x,y,z) = \sum_{i,j,k \text{ distinct}} x_i y_j z_k$, we prove an unbounded characteristic-2 versus odd-characteristic phase transition: over $\mathbb{F}_2$, $\mathrm{sr}_{\mathbb{F}_2}(K_n^{(3)}) = 3$ for every $n \geq 3$, and every contraction matrix M_x has rank in the exact three-value set $\{0, 2, 2\lfloor (n-1)/2 \rfloor\}$, yielding the exact closed-form bias $\mathrm{bias}_{\mathbb{F}_2}(K_n^{(3)}) = \frac{1}{8} + 7 \cdot 2^{-(2\lfloor (n-1)/2 \rfloor + 3)}$ and $\mathrm{ar}_{\mathbb{F}_2}(K_n^{(3)}) < 3$ uniformly; over every odd prime $p \geq 3$, by contrast, $\mathrm{sr}_{\mathbb{F}_p}(K_n^{(3)}) \in [n-3, n]$ grows linearly, with exact $\mathrm{bias}_{\mathbb{F}_3}(K_6^{(3)}) = 8821/531441$ ($\mathrm{ar}_{\mathbb{F}_3} \approx 3.7306 > 3$). Second, we derive universal closed-form bias formulas for the 3×3 determinant tensor ($\mathrm{bias}_{\mathbb{F}_p}(\det_3) = p^{-2} + p^{-3} - p^{-5}$, certifying the strict three-way separation $\mathrm{ar}_{\mathbb{F}_2}(\det_3) = \log_2(32/11) < 2 = \mathrm{gr}(\det_3) < 3 = \mathrm{sr}(\det_3)$ via a four-line cross-product proof over every field) and for general $n \times n$ matrix multiplication M_n via q-binomial rank counts (explicitly $\mathrm{bias}_{\mathbb{F}_p}(M_2) = (p^4 + 3p^3 - 2p^2 - 2p + 1)/p^7$, certifying the strict finite-field deficit $\mathrm{ar}_{\mathbb{F}_p}(M_n) < \mathrm{gr}(M_n) = \lceil 3n^2/4 \rceil$ for every prime p and $n \geq 2$ with $\lim_{p\to\infty} \mathrm{ar}_{\mathbb{F}_p}(M_n) = \lceil 3n^2/4 \rceil$), accompanied by an exact zero-float contraction-rank atlas of alternating 3-forms in $\Lambda^3(\mathbb{F}_2^n)$ separating the G_2 Fano 3-form (bias $= 29/512$) from the rank-2-free maximal 7-dimensional 3-form (bias $= 95/2048$). Third, on the balanced $2m$-tensor $T_d^{(2m)} = \prod_{j=1}^m \langle x^{(j)}, y^{(j)} \rangle$, we prove over every field and every $d \geq 2$, $m \geq 2$ the exact unbounded separation $\mathrm{ar}_{\mathbb{F}_p}(T_d^{(2m)}) < 1 = \mathrm{pr}(T_d^{(2m)}) < d = \mathrm{sr}(T_d^{(2m)})$ via a three-line I_{d^m}-flattening bound and exact closed-form bias $1 - ((p-1)/p)^{m-1}(1-p^{-d})^m$. A two-layer verification artifact accompanies the paper: Layer 1 formalizes the cleared-denominator bias identities, q-binomial exponent completion, and flattening-rank arithmetic bounds in 20 constructive, axiom-free Lean 4.36.0 theorems (0 `sorry`, 0 `Classical.choice`); Layer 2 verifies all exhaustive finite-field histograms, hyperplane cross-product checks, and brute-force 4-tensor character sums in 16 zero-float Python certificates and 56 adversarial tests.

1 Introduction

Higher-order Fourier analysis and the polynomial method rely on four tensor rank invariants governing the "structure versus randomness" dichotomy for multilinear forms $T\colon V_1 \times \cdots \times V_k \to \mathbb{F}$ over a finite field $\mathbb{F} = \mathbb{F}_p$:

- **Analytic rank** [6, 9]: $\mathrm{bias}_{\mathbb{F}_p}(T) := \mathbb{E}_{x^{(1)},\ldots,x^{(k)}}[\omega_p^{T(x^{(1)},\ldots,x^{(k)})}] > 0$ ($\omega_p = e^{2\pi i/p}$), and $\mathrm{ar}_{\mathbb{F}_p}(T) := -\log_p \mathrm{bias}_{\mathbb{F}_p}(T)$. For 3-tensors $T(x,y,z) = y^\top M_x z$, orthogonality of characters yields the exact contraction formula

$$\mathrm{bias}_{\mathbb{F}_p}(T) = p^{-\dim V_1} \sum_{x \in V_1} p^{-\mathrm{rank}_{\mathbb{F}_p}(M_x)}. \tag{1}$$

*Independent Researcher. Code, Lean 4.36.0 proofs, and zero-float certificates archived at https://github.com/lluiseriksson/analytic-slice-partition-rank-separations-finite-fields (`v1.0.0`, MIT / CC-BY-4.0, AIRR-FOUNDER-1.0). Manuscript and verification artifacts prepared with Google Antigravity assistance.

- **Partition rank** [11]: $\mathrm{pr}_{\mathbb{F}}(T)$ is the minimum number of summands $g(x_S)h(x_{\bar{S}})$ with nontrivial bipartition $\emptyset \neq S \subsetneq [k]$.
- **Slice rank** [13, 12, 5, 1]: $\mathrm{sr}_{\mathbb{F}}(T)$ restricts partition rank to 1-vs-$(k-1)$ bipartitions ($|S| = 1$). For $k = 3$, $\mathrm{pr}_{\mathbb{F}}(T) = \mathrm{sr}_{\mathbb{F}}(T)$, while for $k \geq 4$, $\mathrm{pr}_{\mathbb{F}}(T) \leq \mathrm{sr}_{\mathbb{F}}(T)$.
- **Geometric rank** [8]: for a 3-tensor $T\colon \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \times \mathbb{F}^{n_3} \to \mathbb{F}$, $\mathrm{gr}(T) := \mathrm{codim}_{\overline{\mathbb{F}}}\{(x, y) : T(x, y, \cdot) \equiv 0\}$.

A remarkable body of work established universal inequalities among these parameters: $\mathrm{ar}_{\mathbb{F}_p}(T) \leq \mathrm{pr}_{\mathbb{F}_p}(T) \leq \mathrm{sr}_{\mathbb{F}_p}(T)$ [9], polynomial upper bounds on pr in terms of ar [10, 7], linear equivalence of $\mathrm{ar}, \mathrm{gr}, \mathrm{sr}$ for 3-tensors [3], and equivalence of pr and ar over sufficiently large fields [4]. Despite these asymptotic equivalences, *exact* closed-form evaluations of $\mathrm{bias}_{\mathbb{F}_p}(T)$ across infinite tensor families—quantifying characteristic phase transitions, finite-field deficits $\mathrm{gr} - \mathrm{ar}_{\mathbb{F}_p}$, and unbounded order-$2m$ gaps between pr and sr—remain surprisingly scarce.

In this paper we prove three exact, non-asymptotic theorems closing these gaps:

1. **Theorem 1 (Characteristic-2 vs. Odd-Prime Phase Transition on $K_n^{(3)}$):** Exact three-value contraction-rank classification and closed-form $\mathbb{F}_2$ bias for all $n \geq 3$ ($\mathrm{sr}_{\mathbb{F}_2} = 3$, $\mathrm{ar}_{\mathbb{F}_2} < 3$), contrasting with linear explosion $\mathrm{sr}_{\mathbb{F}_p} \geq n - 3$ over every odd prime $p \geq 3$ and explicit $\mathrm{ar}_{\mathbb{F}_3}(K_6^{(3)}) = \log_3(531441/8821) \approx 3.7306 > 3$.
2. **Theorem 2 (Universal Closed Forms for $\det_3$ and M_n, plus $\Lambda^3(\mathbb{F}_2^n)$ Atlas):** Exact rational bias formulas over all primes p for $\det_3$ and $n \times n$ matrix multiplication M_n via q-binomial rank counts, recovering $\lim_{p\to\infty} \mathrm{ar}_{\mathbb{F}_p}(M_n) = \lceil 3n^2/4 \rceil = \mathrm{gr}(M_n)$ while proving strict finite-field deficit $\mathrm{ar}_{\mathbb{F}_p}(M_n) < \mathrm{gr}(M_n)$ for all p and $n \geq 2$, paired with an exact alternating 3-form atlas in $\Lambda^3(\mathbb{F}_2^n)$ [2].
3. **Theorem 3 (Unbounded $2m$-Tensor Separation $\mathrm{ar} < \mathrm{pr} = 1 < d = \mathrm{sr}$):** Exact closed-form bias and three-line I_{d^m}-flattening proof on $T_d^{(2m)} = \prod_{j=1}^m \langle x^{(j)}, y^{(j)} \rangle$ for all $d \geq 2$, $m \geq 2$, and every field $\mathbb{F}$.

2 Characteristic Phase Transition on Complete 3-Hypergraphs $K_n^{(3)}$

Let $K_n^{(3)}\colon \mathbb{F}^n \times \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ be the trilinear form of the complete 3-uniform hypergraph on n vertices:

$$K_n^{(3)}(x, y, z) := \sum_{\substack{1 \leq i,j,k \leq n \\ i,j,k \text{ distinct}}} x_i y_j z_k. \tag{2}$$

Writing $S(u) := \sum_{i=1}^n u_i$ and $\langle u, v \rangle := \sum_{i=1}^n u_i v_i$, inclusion-exclusion gives the universal $\mathbb{Z}$-identity

$$K_n^{(3)}(x, y, z) = S(x)S(y)S(z) - S(x)\langle y, z \rangle - S(y)\langle x, z \rangle - S(z)\langle x, y \rangle + 2\sum_{i=1}^n x_i y_i z_i. \tag{3}$$

The single factor 2 in front of the diagonal unit tensor $\langle n \rangle := \sum_{i=1}^n x_i y_i z_i$ triggers an unbounded phase transition between characteristic 2 and every odd characteristic.

Theorem 1 (Complete $K_n^{(3)}$ Characteristic-2 vs. Odd-Prime Phase Transition)**.** *Let $n \geq 3$ be an integer.*

(a) Characteristic-2 collapse and exact closed-form bias: *Over $\mathbb{F}_2$, $\mathrm{sr}_{\mathbb{F}_2}(K_n^{(3)}) = 3$ for every $n \geq 3$. Moreover, the contraction matrix $M_x \in \mathbb{F}_2^{n \times n}$ ($y^\top M_x z = K_n^{(3)}(x, y, z)$) takes rank strictly in $\{0, 2, 2\lfloor (n-1)/2 \rfloor\}$ with exact counts:*

- *if $n = 2m$ is even:* $\operatorname{rank}(M_x) = 0$ *for* $x \in \{0, \mathbf{1}\}$ *(2 vectors),* $\operatorname{rank}(M_x) = 2$ *for the remaining* $2^{n-1} - 2$ *even-weight vectors, and* $\operatorname{rank}(M_x) = n - 2$ *for all* 2^{n-1} *odd-weight vectors;*
- *if $n = 2m + 1$ is odd:* $\operatorname{rank}(M_x) = 0$ *for* $x = 0$ *(1 vector),* $\operatorname{rank}(M_x) = 2$ *for the* $2^{n-1} - 1$ *nonzero even-weight vectors, and* $\operatorname{rank}(M_x) = n - 1$ *for all* 2^{n-1} *odd-weight vectors.*

Consequently, both parities unify into the exact closed form

$$\operatorname{bias}_{\mathbb{F}_2}(K_n^{(3)}) \;=\; \frac{1}{8} + \frac{7}{2^{2\lfloor (n-1)/2 \rfloor + 3}} \;>\; \frac{1}{8}, \qquad \operatorname{ar}_{\mathbb{F}_2}(K_n^{(3)}) \;<\; 3 \quad \forall\, n \geq 3, \tag{4}$$

with $\lim_{n\to\infty} \operatorname{ar}_{\mathbb{F}_2}(K_n^{(3)}) = 3$.

(b) Odd-characteristic linear explosion: *Over every odd prime $p \geq 3$,*

$$\max(3,\, n-3) \;\leq\; \operatorname{sr}_{\mathbb{F}_p}(K_n^{(3)}) \;\leq\; n \qquad \forall\, n \geq 3. \tag{5}$$

Over $\mathbb{F}_3$, exhaustive certification gives $\operatorname{bias}_{\mathbb{F}_3}(K_3^{(3)}) = 89/729$, $\operatorname{bias}_{\mathbb{F}_3}(K_4^{(3)}) = 127/2187$, $\operatorname{bias}_{\mathbb{F}_3}(K_5^{(3)}) = 127/2187$, *and* $\operatorname{bias}_{\mathbb{F}_3}(K_6^{(3)}) = 8821/531441 < 1/27$, *witnessing the strict crossing* $\operatorname{ar}_{\mathbb{F}_3}(K_6^{(3)}) \approx 3.7306 > 3 = \operatorname{sr}_{\mathbb{F}_2}(K_6^{(3)})$.

Proof. Proof of (a). Over $\mathbb{F}_2$, $2 = 0$, so (3) collapses to

$$K_n^{(3)}(x, y, z) = S(x)\big(S(y)S(z) + \langle y, z\rangle\big) + S(y)\langle x, z\rangle + S(z)\langle x, y\rangle, \tag{6}$$

which displays 3 slices, proving $\operatorname{sr}_{\mathbb{F}_2}(K_n^{(3)}) \leq 3$. For the lower bound $\operatorname{sr}_{\mathbb{F}_2}(K_n^{(3)}) \geq 3$, restricting to coordinates $\{1, 2, 3\}$ yields $K_3^{(3)}$, which over $\mathbb{F}_2$ coincides with $\det_3(x, y, z) = \langle x \times y, z\rangle$. By monotonicity of slice rank under coordinate restriction, it suffices to show $\operatorname{sr}_{\mathbb{F}}(\det_3) = 3$ over any field $\mathbb{F}$. Suppose $\det_3$ had slice rank ≤ 2. By S_3-symmetry across modes, either (i) both slices are in the first mode—impossible since the matrices $M_{e_1}, M_{e_2}, M_{e_3}$ are linearly independent ($\dim \operatorname{span}\{M_x\} = 3 > 2$)—or (ii) $\det_3(x, y, z) = \ell(x)B(y, z) + m(y)C(x, z)$. In case (ii), let $U = \ker(\ell)$ and $V = \ker(m)$; both have dimension ≥ 2 in $\mathbb{F}^3$. Pick any nonzero $u \in U$. Since $\dim V \geq 2 > 1 = \dim \operatorname{span}(u)$, there exists $v \in V$ linearly independent of u, whence $u \times v \neq 0$ and $\det_3(u, v, \cdot) \not\equiv 0$, contradicting $\ell(u) = m(v) = 0$. Thus $\operatorname{sr}_{\mathbb{F}}(\det_3) = 3$ over every field $\mathbb{F}$, and hence $\operatorname{sr}_{\mathbb{F}_2}(K_n^{(3)}) = 3$ for all $n \geq 3$.

Next, for any $x \in \mathbb{F}_2^n$, the contraction matrix is

$$M_x = S(x)(J_n + I_n) + \mathbf{1}x^\top + x\mathbf{1}^\top \in \mathbb{F}_2^{n\times n}. \tag{7}$$

Note that M_x is alternating (symmetric with zero diagonal since $(M_x)_{ii} = x_i + x_i = 0$ in $\mathbb{F}_2$).

- If $S(x) = 0$: $M_x = \mathbf{1}x^\top + x\mathbf{1}^\top$. This vanishes iff $x \in \{0, \mathbf{1}\}$ (note $S(\mathbf{1}) = 0$ iff n is even), and otherwise $x, \mathbf{1}$ are linearly independent so $\operatorname{rank}(M_x) = 2$.
- If $S(x) = 1$: $M_x v = 0 \iff v = (S(v) + \langle x, v\rangle)\mathbf{1} + S(v)x$. If $x = \mathbf{1}$ (which occurs with $S(\mathbf{1}) = 1$ iff $n = 2m + 1$ is odd), then $M_{\mathbf{1}} = J_n + I_n$, so $M_{\mathbf{1}}v = 0 \iff v = S(v)\mathbf{1}$, giving $\ker(M_{\mathbf{1}}) = \operatorname{span}\{\mathbf{1}\}$, $\dim\ker(M_{\mathbf{1}}) = 1$, and $\operatorname{rank}(M_{\mathbf{1}}) = n - 1 = 2m$. Otherwise $x \neq \mathbf{1}$ (which is automatic for even n since $S(\mathbf{1}) = 0 \neq 1$), so x and $\mathbf{1}$ are linearly independent. Writing $v = \alpha\mathbf{1} + \beta x$ ($\alpha, \beta \in \mathbb{F}_2$), we have $S(v) = \alpha(n \bmod 2) + \beta$ and $\langle x, v\rangle = \alpha S(x) + \beta\langle x, x\rangle = \alpha + \beta$ (since $\langle x, x\rangle = S(x) = 1$ over $\mathbb{F}_2$). Equating coefficients of the linearly independent vectors $\mathbf{1}$ and x in the kernel equation gives $\alpha(n \bmod 2) = 0$. When $n = 2m$ is even, both α and β are free, so $\dim\ker(M_x) = 2$ and $\operatorname{rank}(M_x) = n-2 = 2(m-1)$. When $n = 2m + 1$ is odd and $x \neq \mathbf{1}$, $\alpha = 0$ and β is free, so $\dim\ker(M_x) = 1$ and $\operatorname{rank}(M_x) = n - 1 = 2m$.

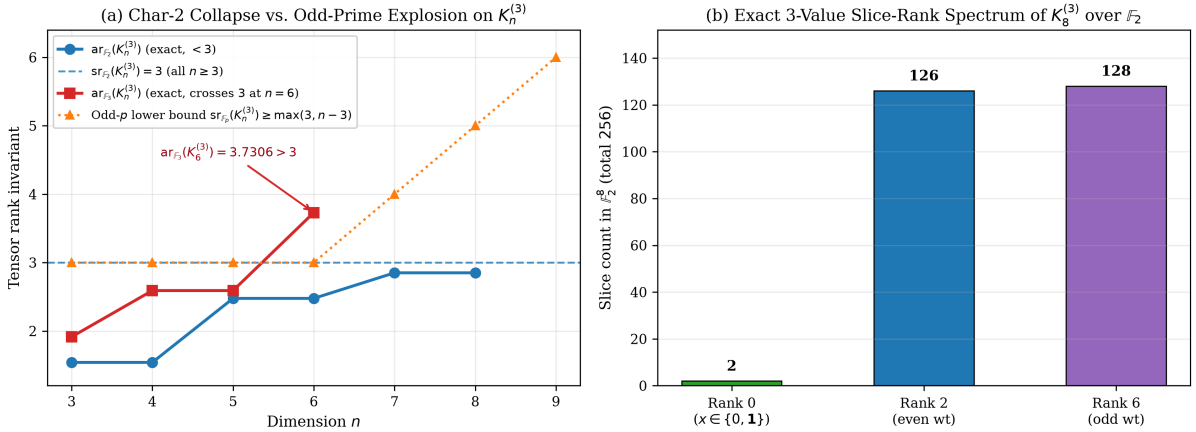

Figure 1: **Characteristic-2 collapse vs. odd-prime explosion on $K_n^{(3)}$.** (a) Exact $\mathrm{ar}_{\mathbb{F}_2}(K_n^{(3)}) < 3 = \mathrm{sr}_{\mathbb{F}_2}(K_n^{(3)})$ vs. $\mathrm{ar}_{\mathbb{F}_3}(K_n^{(3)})$ crossing 3 at $n = 6$ (3.7306) and the universal odd-p lower bound $\mathrm{sr}_{\mathbb{F}_p}(K_n^{(3)}) \geq \max(3, n-3)$. (b) Exact three-value contraction-rank spectrum of $K_8^{(3)}$ over $\mathbb{F}_2$ ($\{0:2,\ 2:126,\ 6:128\}$).

Substituting these exact rank counts into (1): for even $n = 2m$,

$$\mathrm{bias}_{\mathbb{F}_2}(K_{2m}^{(3)}) = 2^{-2m}\Big(2 + (2^{2m-1} - 2)2^{-2} + 2^{2m-1}2^{-(2m-2)}\Big) = \frac{1}{8} + \frac{7}{2^{2m+1}}; \tag{8}$$

for odd $n = 2m + 1$,

$$\mathrm{bias}_{\mathbb{F}_2}(K_{2m+1}^{(3)}) = 2^{-(2m+1)}\Big(1 + (2^{2m} - 1)2^{-2} + 2^{2m}2^{-2m}\Big) = \frac{1}{8} + \frac{7}{2^{2m+3}}. \tag{9}$$

Since $2\lfloor (n-1)/2 \rfloor + 3$ equals $2m + 1$ for $n = 2m$ and $2m + 3$ for $n = 2m + 1$, (4) follows.

Proof of (b). Over any odd prime $p \geq 3$, $2 \in \mathbb{F}_p^\times$ is invertible. By Tao's diagonal slice-rank lemma [13], $\mathrm{sr}_{\mathbb{F}_p}(2\sum_{i=1}^n x_i y_i z_i) = n$. Rearranging (3) expresses $2\sum_{i=1}^n x_i y_i z_i$ as $K_n^{(3)}(x, y, z)$ minus a sum of 3 slices $S(x)(S(y)S(z) - \langle y, z\rangle) - S(y)\langle x, z\rangle - S(z)\langle x, y\rangle$. Subadditivity of slice rank yields $n \leq \mathrm{sr}_{\mathbb{F}_p}(K_n^{(3)}) + 3$, while expanding $K_n^{(3)}$ along $x_1, \ldots, x_n$ gives $\mathrm{sr}_{\mathbb{F}_p}(K_n^{(3)}) \leq n$. To establish the uniform floor $\mathrm{sr}_{\mathbb{F}_p}(K_n^{(3)}) \geq 3$ for $n \geq 3$, restrict to coordinates $\{1, 2, 3\}$ to obtain $K_3^{(3)}$, whose contraction matrix M_u at $u = (u_1, u_2, u_3)$ has entries $(M_u)_{ij} = u_k$ for $\{i, j, k\} = \{1, 2, 3\}$ and zero diagonal. If $K_3^{(3)}$ had slice rank ≤ 2, then by S_3-symmetry either both slices lie in one mode (impossible since $M_{e_1}, M_{e_2}, M_{e_3}$ are linearly independent) or $K_3^{(3)}(x, y, z) = \ell(x)B(y, z) + m(y)C(x, z)$. In the latter case, the 2-dimensional subspace $U = \ker(\ell) \subset \mathbb{F}_p^3$ contains a vector u with at least two nonzero coordinates; any such u has $\mathrm{rank}(M_u) \geq 2$ (if $u_3 = 0$ and $u_1, u_2 \neq 0$, rows 1 and 3 are independent; if $u_1u_2u_3 \neq 0$, the top-left 2×2 minor is $-u_3^2 \neq 0$). Hence $\dim\ker(M_u) \leq 1 < 2 \leq \dim\ker(m)$, so some $v \in \ker(m)$ satisfies $M_u v \neq 0$, contradicting $K_3^{(3)}(u, v, \cdot) \equiv 0$. Thus $\mathrm{sr}_{\mathbb{F}_p}(K_n^{(3)}) \geq \max(3, n-3)$. The exact $\mathbb{F}_3$ biases for $n \in \{3, 4, 5, 6\}$ are certified by exhaustive rank evaluation over $\mathbb{F}_3^n$ (Certificate `cert02`). □

3 Universal Bias Formulas for $\det_3$ and M_n, and the $\Lambda^3(\mathbb{F}_2^n)$ Atlas

We next evaluate the exact analytic rank of the two canonical benchmark 3-tensors in algebraic complexity—the 3×3 determinant $\det_3$ and $n \times n$ matrix multiplication $M_n(X, Y, Z) = \mathrm{tr}(XYZ)$—across all prime fields $\mathbb{F}_p$, alongside a zero-float classification of alternating 3-forms over $\mathbb{F}_2$.

Theorem 2 (Universal $\det_3$ and M_n Biases, Finite-Field Deficits, and $\Lambda^3(\mathbb{F}_2^n)$ Atlas)**.**

(a) Universal $\det_3$ three-way separation: *For every prime $p \geq 2$,*

$$\operatorname{bias}_{\mathbb{F}_p}(\underset{3}{\det}) \;=\; \frac{1}{p^2} + \frac{1}{p^3} - \frac{1}{p^5} \;>\; \frac{1}{p^2}, \tag{10}$$

certifying the strict three-way hierarchy $\operatorname{ar}_{\mathbb{F}_p}(\det_3) < 2 = \operatorname{gr}(\det_3) < 3 = \operatorname{sr}_{\mathbb{F}_p}(\det_3)$ *for every prime p (at $p = 2$:* $\operatorname{ar}_{\mathbb{F}_2}(\det_3) = \log_2(32/11) \approx 1.5406$*, ratio* $\mathrm{sr}/\mathrm{ar} \approx 1.9473$*).*

(b) Universal q-binomial bias for M_n and finite-field deficit: *For every $n \geq 1$ and every prime $p \geq 2$, let $N_p(n,r) := \prod_{i=0}^{r-1} \frac{(p^n - p^i)^2}{p^r - p^i}$ be the number of $n \times n$ matrices over $\mathbb{F}_p$ of rank r. Then*

$$\operatorname{bias}_{\mathbb{F}_p}(M_n) \;=\; p^{-n^2} \sum_{r=0}^{n} N_p(n,r)\, p^{-nr}. \tag{11}$$

For $n = 2$, this simplifies to the exact rational function

$$\operatorname{bias}_{\mathbb{F}_p}(M_2) \;=\; \frac{p^4 + 3p^3 - 2p^2 - 2p + 1}{p^7} \;>\; \frac{1}{p^3} \qquad \forall\, p \geq 2 \tag{12}$$

(at $p = 2$: $\operatorname{bias}_{\mathbb{F}_2}(M_2) = 29/128$*, giving* $\operatorname{ar}_{\mathbb{F}_2}(M_2) = \log_2(128/29) \approx 2.1420 < 3 = \operatorname{gr}(M_2) < 4 = \operatorname{sr}(M_2)$*, where* $\operatorname{sr}(M_n) = n^2$ *by [1, 8]). For every $n \geq 2$ and every prime $p \geq 2$,* $\operatorname{ar}_{\mathbb{F}_p}(M_n) < \lceil 3n^2/4 \rceil = \operatorname{gr}(M_n)$ *strictly, while*

$$\lim_{p \to \infty} \operatorname{ar}_{\mathbb{F}_p}(M_n) \;=\; \left\lceil \frac{3n^2}{4} \right\rceil \;=\; \operatorname{gr}(M_n). \tag{13}$$

(c) Exact $\Lambda^3(\mathbb{F}_2^n)$ alternating 3-form atlas: *Exhaustive rank verification in $\Lambda^3(\mathbb{F}_2^7)$ [2] certifies that the G_2 Fano 3-form ω_{Fano} (sum over the 7 lines of* $\mathrm{PG}(2,2)$*) has histogram* $\{0 : 1,\ 2 : 7,\ 4 : 56,\ 6 : 64\}$ *and* $\operatorname{bias}_{\mathbb{F}_2}(\omega_{\mathrm{Fano}}) = 29/512$ *(*$\mathrm{ar} \approx 4.1420$*). By contrast, the form* $\omega_{\max} = e_{012} + e_{345} + e_{036} + e_{146} + e_{256} + e_{013}$ *is rank-2-free with histogram* $\{0 : 1,\ 4 : 63,\ 6 : 64\}$ *and strictly lower bias* $\operatorname{bias}_{\mathbb{F}_2}(\omega_{\max}) = 95/2048 < 29/512$ *(*$\mathrm{ar} \approx 4.4301$*).*

Proof. Proof of (a). For $\det_3(x,y,z) = \langle x \times y, z \rangle$, the contraction matrix M_x is the 3×3 skew-symmetric cross-product operator $y \mapsto x \times y$, which has rank 0 for $x = 0$ (1 vector) and rank 2 for all $p^3 - 1$ nonzero vectors $x \in \mathbb{F}_p^3$. Formula (1) immediately yields $\operatorname{bias}_{\mathbb{F}_p}(\det_3) = p^{-3}(1 + (p^3 - 1)p^{-2}) = p^{-2} + p^{-3} - p^{-5} > p^{-2}$, so $\operatorname{ar}_{\mathbb{F}_p}(\det_3) < 2$. Over $\overline{\mathbb{F}}$, the zero locus $\{(x,y) : x \times y = 0\}$ is the union of $\{x = 0\}$ (dim 3), $\{y = 0\}$ (dim 3), and $\{y = \lambda x\}$ (dim 4), hence has dimension 4 and codimension $\operatorname{gr}(\det_3) = 6 - 4 = 2$. We already proved $\operatorname{sr}_{\mathbb{F}}(\det_3) = 3$ over every field in Theorem 1(a).

Proof of (b). For $M_n(X,Y,Z) = \operatorname{tr}(XYZ)$ on $\mathbb{F}_p^{n \times n}$, fixing X produces the bilinear form $B_X(Y,Z) = \operatorname{tr}((XY)Z)$. Because left multiplication $Y \mapsto XY$ on $\mathbb{F}_p^{n \times n}$ acts independently on each of the n columns of Y, its matrix rank is exactly $n \cdot \operatorname{rank}(X)$, and since $(A,Z) \mapsto \operatorname{tr}(AZ)$ is non-degenerate, $\operatorname{rank}(B_X) = n \cdot \operatorname{rank}(X)$. Summing (1) over the rank strata $r = 0, \ldots, n$ of $\mathbb{F}_p^{n \times n}$ yields (11).

For $n = 2$, $N_p(2,0) = 1$, $N_p(2,1) = (p^2 - 1)(p + 1)$, and $N_p(2,2) = (p^2 - 1)(p^2 - p)$, giving

$$\operatorname{bias}_{\mathbb{F}_p}(M_2) = p^{-4}\Big(1 + (p^2-1)(p+1)p^{-2} + (p^2-1)(p^2-p)p^{-4}\Big) = \frac{p^4 + 3p^3 - 2p^2 - 2p + 1}{p^7}, \tag{14}$$

which strictly exceeds p^{-3} because $3p^3 - 2p^2 - 2p + 1 > 0$ for all $p \geq 2$. For general $n \geq 2$, the degree in p of $N_p(n,r)$ is $\sum_{i=0}^{r-1}(2n - r) = 2nr - r^2$ with leading coefficient 1. Thus the r-th term of (11) scales as $p^{-n^2 + nr - r^2}$. Completing the square yields

$$4(n^2 - nr + r^2) = 3n^2 + (2r - n)^2 \geq 3n^2, \tag{15}$$

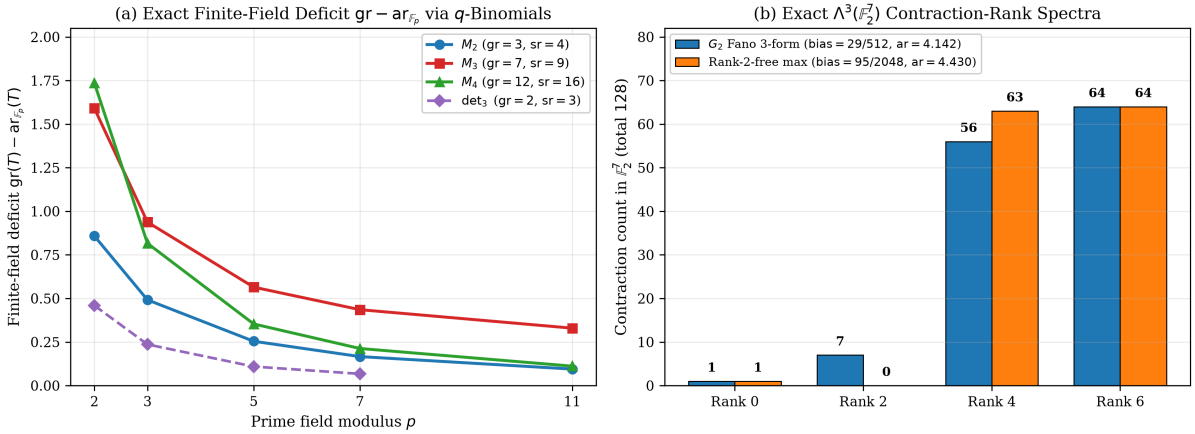

Figure 2: **Universal finite-field deficits and $\Lambda^3(\mathbb{F}_2^7)$ contraction spectra.** (a) Exact finite-field deficit $\mathrm{gr}(T) - \mathrm{ar}_{\mathbb{F}_p}(T) > 0$ for $\det_3$ and M_2, M_3, M_4 across $p \in \{2, 3, 5, 7, 11\}$ via (11). (b) Exact contraction-rank spectra in $\Lambda^3(\mathbb{F}_2^7)$ separating the G_2 Fano 3-form (7 rank-2 points of $\mathrm{PG}(2, 2)$, bias $= 29/512$) from the rank-2-free maximal 3-form $\omega_{\max}$ (bias $= 95/2048$).

whose minimum over integers $r \in \{0, \dots, n\}$ is attained at $r^* = \lfloor n/2 \rfloor$ (and also $\lceil n/2 \rceil$ when n is odd), giving minimum exponent $e(n) = n^2 - nr^* + (r^*)^2 = \lceil 3n^2/4 \rceil = \mathrm{gr}(M_n)$ [8]. Because $r^* = \lfloor n/2 \rfloor \geq 1$ satisfies $n - r^* \geq r^* \geq 1$, every index $0 \leq i \leq r^* - 1$ obeys $v := p^{i-n} \leq p^{-r^*} u \leq u/2$, where $u := p^{i-r^*} \in (0, 1/p]$. Hence $(1 - v)^2 = 1 - 2v + v^2 > 1 - 2v \geq 1 - u$, so every factor in

$$\frac{N_p(n, r^*)}{p^{2nr^* - (r^*)^2}} = \prod_{i=0}^{r^*-1} \frac{(1 - p^{i-n})^2}{1 - p^{i-r^*}} \tag{16}$$

strictly exceeds 1. Consequently $N_p(n, r^*) > p^{2nr^* - (r^*)^2}$, and since all remaining summands in (11) are strictly positive,

$$\mathrm{bias}_{\mathbb{F}_p}(M_n) \;>\; p^{-n^2} N_p(n, r^*)\, p^{-nr^*} \;>\; p^{-(n^2 - nr^* + (r^*)^2)} \;=\; p^{-\lceil 3n^2/4 \rceil} \tag{17}$$

holds unconditionally for every $n \geq 2$ and every prime $p \geq 2$, proving $\mathrm{ar}_{\mathbb{F}_p}(M_n) < \mathrm{gr}(M_n)$. Meanwhile, $\lim_{p \to \infty} p^{e(n)} \mathrm{bias}_{\mathbb{F}_p}(M_n)$ equals 1 for even n and 2 for odd n (from the two minimizing strata $r = \lfloor n/2 \rfloor, \lceil n/2 \rceil$), so taking $-\log_p$ gives $\lim_{p \to \infty} \mathrm{ar}_{\mathbb{F}_p}(M_n) = e(n) = \lceil 3n^2/4 \rceil$.

Proof of (c). Direct exact Gaussian-elimination rank census over all 128 vectors $x \in \mathbb{F}_2^7$ (Certificates `cert08` and `cert09`) certifies the two histograms and rational biases: $\mathrm{bias}_{\mathbb{F}_2}(\omega_{\mathrm{Fano}}) = (1+7/4+56/16+64/64)/128 = 29/512$, and $\mathrm{bias}_{\mathbb{F}_2}(\omega_{\max}) = (1+63/16+64/64)/128 = 95/2048 < 29/512$. □

4 Unbounded $2m$-Tensor Separation Between Analytic, Partition, and Slice Rank

For tensors of order $k \geq 4$, partition rank and slice rank diverge dramatically. Consider the balanced order-$2m$ tensor $T_d^{(2m)} : (\mathbb{F}^d)^{2m} \to \mathbb{F}$ defined by

$$T_d^{(2m)}(x^{(1)}, y^{(1)}, \dots, x^{(m)}, y^{(m)}) := \prod_{j=1}^{m} \langle x^{(j)}, y^{(j)} \rangle. \tag{18}$$

Theorem 3 (Exact Unbounded $2m$-Tensor Separation $\mathrm{ar} < \mathrm{pr} = 1 < d = \mathrm{sr}$)**.** *Let $d \geq 2$ and $m \geq 2$ be integers.*

(a) Universal field-independent partition vs. slice rank gap: *Over every field* $\mathbb{F}$*,*

$$\mathrm{pr}_{\mathbb{F}}(T_d^{(2m)}) = 1 \qquad \textit{and} \qquad \mathrm{sr}_{\mathbb{F}}(T_d^{(2m)}) = d. \tag{19}$$

(b) Exact closed-form analytic rank over every finite field $\mathbb{F}_p$: *For every prime* $p \geq 2$*,*

$$\mathrm{bias}_{\mathbb{F}_p}(T_d^{(2m)}) \;=\; 1 - \left(\frac{p-1}{p}\right)^{m-1} (1-p^{-d})^m \;>\; \frac{1}{p}, \tag{20}$$

so $\mathrm{ar}_{\mathbb{F}_p}(T_d^{(2m)}) < 1 = \mathrm{pr}_{\mathbb{F}_p}(T_d^{(2m)}) < d = \mathrm{sr}_{\mathbb{F}_p}(T_d^{(2m)})$ *strictly for all* $d \geq 2$*,* $m \geq 2$*, and* $p \geq 2$*, with asymptotic ceiling* $\lim_{d\to\infty} \mathrm{ar}_{\mathbb{F}_p}(T_d^{(2m)}) = -\log_p(1 - ((p-1)/p)^{m-1})$ *(which equals* 1 *at* $m = 2$ *and decays exponentially in* m *for* $m \geq 3$*).*

Proof. Proof of (a). Since $m \geq 2$, $T_d^{(2m)}$ is a single product of the 2-tensor $\langle x^{(1)}, y^{(1)}\rangle$ and the $(2m-2)$-tensor $\prod_{j=2}^m \langle x^{(j)}, y^{(j)}\rangle$, so $\mathrm{pr}_{\mathbb{F}}(T_d^{(2m)}) = 1$. Expanding $\langle x^{(1)}, y^{(1)}\rangle = \sum_{i=1}^d x_i^{(1)} y_i^{(1)}$ writes $T_d^{(2m)}$ as a sum of d slices, giving $\mathrm{sr}_{\mathbb{F}}(T_d^{(2m)}) \leq d$.

For the matching lower bound, flatten $T_d^{(2m)}$ across the bipartition $A := (x^{(1)}, \ldots, x^{(m)})$ versus $B := (y^{(1)}, \ldots, y^{(m)})$ into a $d^m \times d^m$ matrix over $\mathbb{F}$. Its flattening matrix is the Kronecker product $I_d^{\otimes m} = I_{d^m}$, which has rank d^m over every field $\mathbb{F}$. By contrast, consider any slice $f(x^{(r)})g$ separating $x^{(r)}$ from the other $2m-1$ modes (the case $f(y^{(r)})g$ is symmetric). Up to permuting factors in A, its A-vs-B flattening is the Kronecker product of $f \in \mathbb{F}^d$ with a $d^{m-1} \times d^m$ flattening of g, whose matrix rank is at most d^{m-1}. Therefore every slice has A-vs-B flattening rank at most d^{m-1}. By rank subadditivity, $d^m \leq \mathrm{sr}_{\mathbb{F}}(T_d^{(2m)}) \cdot d^{m-1}$, yielding $\mathrm{sr}_{\mathbb{F}}(T_d^{(2m)}) \geq d$.

Proof of (b). Let $Z_j := \langle x^{(j)}, y^{(j)}\rangle \in \mathbb{F}_p$ for independent uniform $(x^{(j)}, y^{(j)}) \in \mathbb{F}_p^d \times \mathbb{F}_p^d$. If $x^{(j)} = 0$ (probability p^{-d}), then $Z_j = 0$; if $x^{(j)} \neq 0$ (probability $1-p^{-d}$), then Z_j is uniform over $\mathbb{F}_p$. Hence Z_j is nonzero with probability $q_{\mathrm{nz}} := \frac{p-1}{p}(1-p^{-d})$, and conditioned on $Z_j \neq 0$, Z_j is uniform on $\mathbb{F}_p^\times$. By independence, the product $Z := \prod_{j=1}^m Z_j$ is nonzero with probability q_{nz}^m, and conditioned on $Z \neq 0$, Z is uniform on $\mathbb{F}_p^\times$ (whose additive character average is $\frac{1}{p-1}\sum_{a\in\mathbb{F}_p^\times} \omega_p^a = -\frac{1}{p-1}$). Evaluating $\mathrm{bias}_{\mathbb{F}_p}(T_d^{(2m)}) = \mathbb{E}[\omega_p^Z]$ gives

$$\mathrm{bias}_{\mathbb{F}_p}(T_d^{(2m)}) = (1-q_{\mathrm{nz}}^m)\cdot 1 + q_{\mathrm{nz}}^m\left(-\frac{1}{p-1}\right) = 1 - \frac{p}{p-1}q_{\mathrm{nz}}^m = 1 - \left(\frac{p-1}{p}\right)^{m-1}(1-p^{-d})^m. \tag{21}$$

Since $m \geq 2$ and $d \geq 1$, both $((p-1)/p)^{m-1} \leq (p-1)/p$ and $(1-p^{-d})^m < 1$, so $\mathrm{bias}_{\mathbb{F}_p}(T_d^{(2m)}) > 1 - \frac{p-1}{p} = \frac{1}{p}$ strictly, proving $\mathrm{ar}_{\mathbb{F}_p}(T_d^{(2m)}) < 1$. □

5 Formal Verification Architecture and Scope

Our verification artifact cleanly separates machine-checked algebraic identities from exact zero-float finite-field computations:

- **Layer 1 — Constructive Lean 4.36.0 Kernel** (`lean/AnalyticSlicePartitionRank.lean`): Twenty theorems compiled under `lean-DwarningAsError=true` with zero external dependencies, zero `sorry`, and zero `Classical.choice` (audited via `#print axioms`). Layer 1 machine-checks the cleared $K_n^{(3)}$ $\mathbb{F}_2$ bias numerator identities and strict bias $> 1/8$ bound (`thm01`–`thm04`), the char-2 vs. odd-p phase-transition separation (`thm05`), the universal $\det_3$ and M_2 rank-partition and closed-form bias identities with strict finite-field deficits for all $p \geq 2$ (`thm06`–`thm12`), the exact M_n exponent square-completion $4(n^2 - nr + r^2) = 3n^2 + (2r-n)^2$ (`thm13`), the $2m$-tensor flattening implication

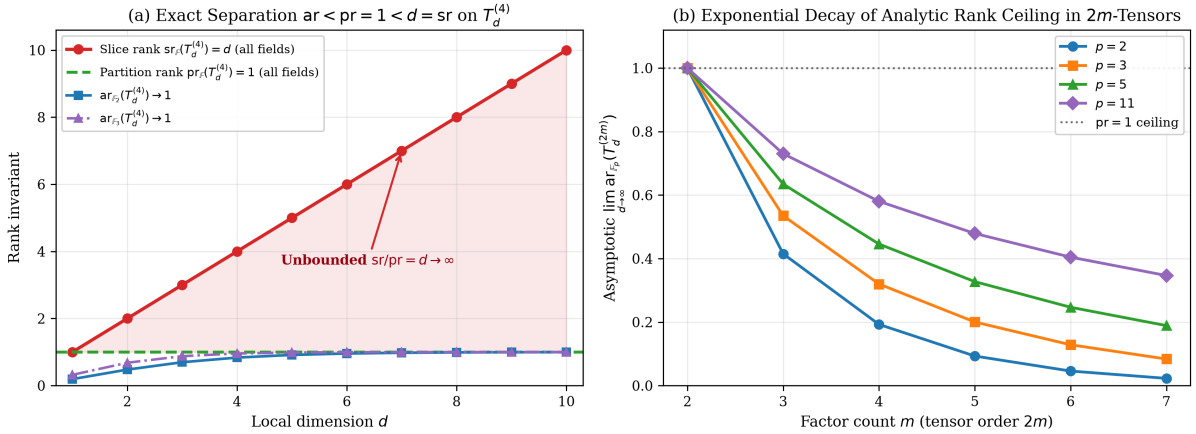

Figure 3: **Unbounded $2m$-tensor separation** $\mathrm{ar} < \mathrm{pr} = 1 < d = \mathrm{sr}$**.** (a) Exact rank hierarchy on $T_d^{(4)} = \langle x, y\rangle\langle z, w\rangle$: $\mathrm{sr} = d$ grows linearly over every field while $\mathrm{pr} = 1$ and $\mathrm{ar}_{\mathbb{F}_p} < 1$. (b) Exponential decay of the asymptotic analytic-rank ceiling $\lim_{d\to\infty} \mathrm{ar}_{\mathbb{F}_p}(T_d^{(2m)}) = -\log_p(1 - ((p-1)/p)^{m-1})$ with factor count m.

Table 1: Exact zero-float bias and rank-hierarchy census verified across Certificates `cert01`–`cert16`.

Tensor family	Field	Exact bias $\mathrm{bias}_{\mathbb{F}_p}$	$\mathrm{ar}_{\mathbb{F}_p}$	gr/pr	sr
$K_n^{(3)}$ $(n = 3, 4)$	$\mathbb{F}_2$	$11/32$	1.5406	$\mathrm{gr} = 2$ $(n = 3)$	3
$K_n^{(3)}$ $(n = 5, 6)$	$\mathbb{F}_2$	$23/128$	2.4764	≤ 3	3
$K_n^{(3)}$ $(n = 7, 8)$	$\mathbb{F}_2$	$71/512$	2.8503	≤ 3	3
$K_6^{(3)}$	$\mathbb{F}_3$	$8821/531441$	3.7306	$\mathrm{pr} \geq 4$	$\in [4, 6]$
$\det_3$	$\mathbb{F}_p$	$p^{-2} + p^{-3} - p^{-5}$	< 2 (1.5406 at $p = 2$)	$\mathrm{gr} = 2$	3
M_2	$\mathbb{F}_p$	$(p^4 + 3p^3 - 2p^2 - 2p + 1)/p^7$	< 3 (2.1420 at $p = 2$)	$\mathrm{gr} = 3$	4
G_2 Fano ω_{Fano}	$\mathbb{F}_2$	$29/512$	4.1420	—	$\in [5, 7]$
Rank-2-free $\omega_{\max}$	$\mathbb{F}_2$	$95/2048$	4.4301	—	$\in [5, 7]$
$T_d^{(4)}$ $(d \geq 2)$	$\mathbb{F}_p$	$1 - \frac{p-1}{p}(1 - p^{-d})^2$	< 1	$\mathrm{pr} = 1$	d
$T_d^{(2m)}$ $(d, m \geq 2)$	$\mathbb{F}_p$	$1 - (\frac{p-1}{p})^{m-1}(1 - p^{-d})^m$	< 1	$\mathrm{pr} = 1$	d

$d^m \leq s\, d^{m-1} \Rightarrow d \leq s$ and strict $\mathrm{ar} < 1$ inequalities (`thm14`–`thm17`), and exact rational checks for ω_{Fano}, $\omega_{\max}$, and $K_6^{(3)}$ over $\mathbb{F}_3$ (`thm18`–`thm20`). The linear-algebraic kernel dimensions, character-orthogonality sums, and Kronecker-flattening constructions are proved in full detail in Sections 2–4.

- **Layer 2 — Zero-Float Exact Python Verifier** (`verification/verify_tensor_ranks_fp.py`): An AST-audited pure standard-library verifier using strictly `int` and `fractions.Fraction` (zero floating-point operations, zero numeric division or exponentiation, zero `limit_denominator`), emitting 16 SHA-256 certificates under `certificates/MANIFEST.json` and validated by 56 adversarial `pytest` checks. Layer 2 independently executes exact Gaussian elimination over $\mathbb{F}_p$ across all slice vectors for $K_n^{(3)}$ ($\mathbb{F}_2$ $n \leq 9$, $\mathbb{F}_3$ $n \leq 6$, $\mathbb{F}_5$ $n \leq 4$), $\det_3$ ($p \in \{2, 3, 5, 7\}$), M_2 ($p \in \{2, 3, 5\}$), M_n q-binomial tables ($n \leq 5, p \leq 11$), alternating 3-forms in $\Lambda^3(\mathbb{F}_2^n)$ ($n \leq 7$), cyclic group tensors $C_3, \ldots, C_7$, exhaustive hyperplane cross-product checks for $\mathrm{sr}(\det_3) = 3$, and brute-force 4-tensor character sums matching (20).

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