

Exact Classical Rate–Distortion for Phase-Lifted Generalized Coherent States

Cartan-product Laplace rigidity, universal radial envelopes,
and Slater-determinant transitions

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Abstract

Let V_λ be a finite-dimensional irreducible unitary representation of a compact connected semisimple group, and let X be uniform on the phase-lifted orbit of a highest-weight vector. We determine the classical Shannon rate–distortion function of X under squared ambient loss when the description alphabet is arbitrary standard Borel and the decoder is not restricted to the orbit. The known integer-index generalized Lieb–Wehrl theorem has a decisive information-theoretic consequence: a Cartan-product identity makes coherent rays maximize the overlap Laplace transform at every field and every fixed report norm. The full vector problem therefore reduces exactly to

$$C_\lambda(s) = e^{-s} \max_{0 \leq b \leq 1} e^{-sb^2} L_\lambda(2sb), \quad L_\lambda(\kappa) = \sum_{m \geq 0} \frac{(\kappa^2/4)^m}{(m!)^2 \dim V_{m\lambda}},$$

and $R_\lambda(D) = \sup_{s \geq 0} \{-sD - \log C_\lambda(s)\}$. Covariant tilted channels attain exposed points, while a revealed flag mixing active radii attains every nonexposed point. We derive a universal representation-dimensional criterion for discontinuous directional- information onset and a sharp root-system high-fidelity constant. For $V_\lambda = \Lambda^k \mathbb{C}^n$, the sources are phased fermionic Slater determinants. Their normalizer is an explicit ${}_{k-1}F_k$ function, their squared Plücker overlap is a product of independent beta variables, and every interior family $2 \leq k \leq n-2$ with $n \geq 6$ has a discontinuous first activation. A separate phase-invariant corollary gives the intrinsic Plücker-distortion frontier on the projective coherent orbit. These are classical compression theorems for amplitude coordinates, not quantum rate–distortion coding or a statement about click-only Born data.

1 Introduction

Generalized coherent states form the closed projective orbit of a highest-weight ray. They include spin coherent states, bosonic product states, fermionic Slater determinants, and Plücker embeddings of complex Grassmannians. Their localization properties are governed by Wehrl and Rényi–Wehrl moments. Sugita proved that coherent states maximize every integer Husimi moment for compact semisimple groups [1]; the quadratic equality set is the highest-weight orbit characterized by the Kostant–Lichtenstein quadrics [2, 3]. Those results are inputs here, not claims of priority.

We ask a different question. Draw a coherent state uniformly, attach an independent phase, and encode the resulting vector using completely classical memory. How many nats are required to reconstruct it with a prescribed squared Euclidean error if the decoder may report any vector, not merely another coherent state? The permission to leave the orbit is substantive. Conditional

means live in the coherent-state orbitope and can shrink radially; their competition produces linear faces and discontinuous information onsets absent from a same-manifold codebook problem.

Rate–distortion on invariant groups and high-resolution bounds on Grassmann manifolds are classical subjects [4–6]. For $V_\lambda = \mathbb{C}^n$ the phase-lifted source is exactly the real unit sphere S^{2n-1} , and our $k = 1$ formula reduces, with radius one and natural logarithms, to the exact sphere solution of Dytso and Cardone [7]. Quantum rate–distortion instead optimizes quantum communication resources and quantum distortion observables [8, 9]; that is not the task considered here.

The paper’s contribution begins after the known coherent-state moment extremum. We prove:

1. the all-field fixed-radius Laplace order obtained by positive phase–Bessel resummation of Sugita’s integer-moment extremum, together with its exact equality classification;
2. an exact classical Shannon frontier over the full coherent-state orbitope, including covariant equality and flagged attainment at nonexposed distortions;
3. a universal criterion $\dim V_{2\lambda} < \frac{1}{2}(\dim V_\lambda)^2$ forcing a discontinuous first activation;
4. a high-fidelity constant expressed directly by positive roots and the highest weight;
5. an explicit all- (n, k) Slater-determinant specialization with a hypergeometric normalizer, beta-product overlap law, and a complete classification of the sign responsible for first-order onset.

The scalar formula is exact even when the radial maximizer is not unique. We do not claim a unique positive branch or exclude later radial exchanges for every representation. That distinction keeps the result independent of unproved phase-diagram assumptions.

2 Coherent orbits and the unrestricted problem

Let G be a compact connected semisimple Lie group, let (π, V_λ) be a nontrivial finite-dimensional irreducible unitary representation with dominant highest weight λ , and choose a unit highest-weight vector v_λ . We use the underlying real Hilbert structure

$$(a, x)_{\mathbb{R}} = \operatorname{Re} \langle a, x \rangle, \quad \|a\|^2 = \langle a, a \rangle.$$

Definition 2.1 (Phase-lifted coherent orbit). *The source manifold and its orbitope are*

$$\mathcal{M}_\lambda = \{e^{i\theta}\pi(g)v_\lambda : \theta \in \mathbb{R}/2\pi\mathbb{Z}, g \in G\}, \quad \mathcal{B}_\lambda = \operatorname{conv}(\mathcal{M}_\lambda).$$

The probability μ_λ is the pushforward of normalized Haar measure on G and normalized Lebesgue measure on the phase circle.

The independent phase makes $\mathbb{E}X = 0$ for $X \sim \mu_\lambda$. It also gives $0 \in \mathcal{B}_\lambda$, and convexity gives $\|a\| \leq 1$ for every $a \in \mathcal{B}_\lambda$.

Let Z take values in an arbitrary standard-Borel space and let $a : Z \rightarrow V_\lambda$ be Borel. We define

$$R_\lambda(D) = \inf I(X; Z), \quad \mathbb{E}\|X - a(Z)\|^2 \leq D, \quad (2.1)$$

where the infimum is over all joint laws with X -marginal μ_λ . Mutual information and logarithms are in nats.

Lemma 2.2 (Posterior-mean reduction). *The infimum in (2.1) is unchanged if the decoder is required to lie in $\mathcal{B}_\lambda$. More precisely, every decoder can be replaced by $\hat{X}(Z) = \mathbb{E}[X \mid Z] \in \mathcal{B}_\lambda$ without increasing distortion or mutual information.*

Proof. Finite dimensionality makes the Bochner conditional expectation well-defined. Conditional least squares gives

$$\mathbb{E}[\|X - a(Z)\|^2 \mid Z] = \mathbb{E}[\|X - \hat{X}(Z)\|^2 \mid Z] + \|a(Z) - \hat{X}(Z)\|^2.$$

The conditional mean belongs to the closed convex hull of the essential range of X , which is $\mathcal{B}_\lambda$. It is a measurable function of Z , so data processing cannot increase mutual information. $\square$

This unrestricted formulation is stronger than a codebook restricted to $\mathcal{M}_\lambda$. The report can be a proper mixture of coherent states, but [theorem 3.2](#) will show that only one radial coherent direction is needed in the dual optimum.

3 Cartan products and all-field Laplace rigidity

Let $V_{m\lambda}$ denote the irreducible Cartan component generated by $v_\lambda^{\otimes m}$ inside $V_\lambda^{\otimes m}$, let $P_{m\lambda}$ be its orthogonal projector, and write

$$d_m = d_m(\lambda) = \dim V_{m\lambda}, \quad d_0 = 1.$$

Lemma 3.1 (Cartan moment identity). *For every $a \in V_\lambda$ and integer $m \geq 1$,*

$$\int_G |\langle a, \pi(g)v_\lambda \rangle|^{2m} dg = \frac{\|P_{m\lambda}a^{\otimes m}\|^2}{d_m} \leq \frac{\|a\|^{2m}}{d_m}. \quad (3.1)$$

For $a \neq 0$, equality at $m = 2$ holds exactly when $[a]$ lies on the projective highest-weight orbit.

Proof. The vectors $(\pi(g)v_\lambda)^{\otimes m}$ span $V_{m\lambda}$. The operator

$$T_m = \int_G |(\pi(g)v_\lambda)^{\otimes m}\rangle \langle (\pi(g)v_\lambda)^{\otimes m}| dg$$

vanishes on the orthogonal complement and commutes with the irreducible G -action on $V_{m\lambda}$. Schur's lemma makes it a scalar multiple of $P_{m\lambda}$; its trace is one, so $T_m = P_{m\lambda}/d_m$. Contracting with $a^{\otimes m}$ gives the identity and the projection bound. For $m = 2$, equality says $a^{\otimes 2} \in V_{2\lambda}$. The highest-weight quadratic criterion identifies precisely the coherent rays [2]. This is also the projection proof underlying the integer-index generalized Lieb–Wehrl theorem [1]. $\square$

Define the entire even function

$$L_\lambda(\kappa) = \sum_{m=0}^{\infty} \frac{(\kappa^2/4)^m}{(m!)^2 d_m}. \quad (3.2)$$

Convergence follows already from $d_m \geq 1$ and comparison with $I_0(|\kappa|)$.

Theorem 3.2 (All-field coherent-ray rigidity). *For every $a \in V_\lambda$ and real κ ,*

$$\int_{\mathcal{M}_\lambda} e^{\kappa \operatorname{Re}\langle a, x \rangle} d\mu_\lambda(x) \leq L_\lambda(\kappa \|a\|). \quad (3.3)$$

Equality holds for every scalar multiple of a coherent vector. If $\kappa \|a\| \neq 0$, equality forces a to lie on a coherent ray.

Proof. For $z \in \mathbb{C}$, phase averaging gives

$$\frac{1}{2\pi} \int_0^{2\pi} (\operatorname{Re}(e^{i\theta} z))^{2m} d\theta = \frac{\binom{2m}{m}}{4^m} |z|^{2m},$$

and all odd moments vanish. Expand the exponential, apply [theorem 3.1](#), and use $\binom{2m}{m}/(2m)! = 1/(m!)^2$. Every term is nonnegative. Coherent a saturates each projection inequality. Conversely, equality of the sum at a nonzero argument forces equality of its $m = 2$ term, and the final claim follows from [theorem 3.1](#). $\square$

The theorem is an all-field consequence of known integer moment rigidity. It is useful because exponential loss duals require the entire transform, not a fixed Rényi index.

4 Exact Shannon rate–distortion

For $s \geq 0$ and $a \in \mathcal{B}_\lambda$ of norm b ,

$$\int e^{-s\|x-a\|^2} d\mu_\lambda(x) = e^{-s(1+b^2)} \int e^{2s \operatorname{Re}\langle a, x \rangle} d\mu_\lambda(x). \quad (4.1)$$

Because $bv_\lambda \in \mathcal{B}_\lambda$ for $0 \leq b \leq 1$, [theorem 3.2](#) makes the supremum exact.

Theorem 4.1 (Universal coherent-orbit RDF). *Put*

$$C_\lambda(s) = e^{-s} \max_{0 \leq b \leq 1} \left\{ e^{-sb^2} L_\lambda(2sb) \right\}. \quad (4.2)$$

Then

$$\boxed{R_\lambda(D) = \sup_{s \geq 0} \{-sD - \log C_\lambda(s)\}.} \quad (4.3)$$

This identity holds for arbitrary standard-Borel classical descriptions and all $D \geq 0$. In particular, $R_\lambda(D) = 0$ for $D \geq 1$.

At every exposed point, an active radius b is attained by a covariant tilted channel supported on $b\mathcal{M}_\lambda$. At a nonexposed point, a revealed independent flag mixing at most two active radii attains the frontier.

Proof. We give the standard-Borel converse and equality construction explicitly. By [theorem 2.2](#), assume from the outset that every report a_z lies in $\mathcal{B}_\lambda$. Regular conditional laws $\nu_z = P_{X|Z=z}$ exist. For every report a_z and $s \geq 0$, the entropy variational inequality gives, with extended values,

$$D(\nu_z \| \mu_\lambda) + s \mathbb{E}_{\nu_z} \|X - a_z\|^2 \geq -\log \int e^{-s\|x-a_z\|^2} d\mu_\lambda(x) \geq -\log C_\lambda(s). \quad (4.4)$$

Averaging and using $I(X; Z) = \mathbb{E}_Z D(P_{X|Z} \| \mu_\lambda)$ yields $I(X; Z) \geq -sD - \log C_\lambda(s)$. Optimization in s proves the lower bound. This is the constant Shannon dual written without a hidden finite-alphabet or absolute-continuity assumption [\[10, 11\]](#).

Fix a maximizing radius b and let Y be Haar on $\mathcal{M}_\lambda$. Define the reverse kernel

$$\frac{dP_{X|Y=y}}{d\mu_\lambda}(x) = \frac{e^{-s\|x-by\|^2}}{C_\lambda(s)}. \quad (4.5)$$

The denominator is correct because b is active. Averaging the numerator over y gives a $G \times U(1)$ -invariant function of x and is therefore constant on the transitive source orbit. Its integral is $C_\lambda(s)$, so the X -marginal of the joint law is exactly μ_λ . Reporting $A = bY$, direct evaluation gives

$$I(X; Y) = -sD_b - \log C_\lambda(s), \quad D_b = \mathbb{E} \|X - bY\|^2. \quad (4.6)$$

For an interior active radius, differentiation of $K_\lambda(2sb) - sb^2$ gives

$$b = K'_\lambda(2sb). \quad (4.7)$$

The gradient of the log partition at a fixed-norm coherent maximizer is radial, and its radial component is $K'_\lambda(2sb)$; hence $\mathbb{E}[X | Y = y] = by$. Thus the displayed report is also the posterior mean. At $b = 0$ the same statement follows from phase symmetry; $b = 1$ is excluded at finite field below. Thus every distortion in $-\partial \log C_\lambda(s)$ is attained. If several radii are active, choose an independent revealed flag J and use the corresponding kernel conditionally on J . Each component has the same X -marginal, hence $I(X; J) = 0$ and both distortion and conditional mutual information mix linearly. One-dimensional convexity requires at most two radii.

At $s = 0$, $b = 0$ gives $D = 1$ and zero rate. As $s \rightarrow \infty$, active radii approach one and the distortion approaches zero; this follows also from the high-field estimate in [theorem 6.2](#). Convex closure supplies the endpoints and completes Fenchel equality. $\square$

An equivalent primal form is useful for geometry. Define the Cramér transform and its squared-radius parametrization by

$$j_\lambda(b) = \sup_{\kappa \geq 0} \{\kappa b - K_\lambda(\kappa)\}, \quad g_\lambda(u) = j_\lambda(\sqrt{u}), \quad 0 \leq u < 1. \quad (4.8)$$

Corollary 4.2 (Convex-envelope form). *For $0 \leq D \leq 1$,*

$$R_\lambda(D) = (\text{co } g_\lambda)(1 - D), \quad (4.9)$$

where co denotes the lower convex envelope. Every chord is attained by time sharing between two covariant coherent tilts.

Proof. For a posterior law ν with mean a and $b = \|a\|$, entropy variation with the field $\kappa a/b$ and [theorem 3.2](#) gives $D(\nu \| \mu_\lambda) \geq j_\lambda(b)$. Conditional means satisfy $\mathbb{E}\|X - \mathbb{E}[X | Z]\|^2 = 1 - \mathbb{E}b(Z)^2$. The function $\text{co } g_\lambda$ is nonnegative, convex, vanishes at zero, and is therefore nondecreasing. Jensen's inequality consequently gives

$$\mathbb{E}g_\lambda(b(Z)^2) \geq \mathbb{E}(\text{co } g_\lambda)(b(Z)^2) \geq (\text{co } g_\lambda)(1 - D),$$

which is the converse. Conversely, a coherent exponential tilt of parameter κ has mean $K'_\lambda(\kappa)y$, divergence $j_\lambda(K'_\lambda(\kappa))$, and the invariant source marginal after Haar mixing of y . Strict convexity makes K'_λ increasing; phase symmetry gives $K'_\lambda(0) = 0$. The high-field estimate in [theorem 6.1](#) gives $K_\lambda(\kappa)/\kappa \rightarrow 1$, and convexity then forces $K'_\lambda(\kappa) \rightarrow 1$ without differentiating its remainder. Hence it maps $\kappa \in [0, \infty)$ onto $b \in [0, 1)$, so all points of g_λ are attained. A revealed binary flag mixing two such joint laws preserves the same source marginal, is independent of X , and attains every lower-hull chord. $\square$

Remark 4.3 (Finite-field boundary). *The endpoint $b = 1$ is not active at finite $s > 0$. Writing $K = \log L$ and differentiating the radial objective gives $2s[K'(2sb) - b]$. Under every finite coherent tilt the overlap is strictly less than one almost surely, so $K'(2s) < 1$; moving left from $b = 1$ strictly increases the objective.*

5 A representation-dimensional phase criterion

The radial problem is

$$\phi_s(b) = K_\lambda(2sb) - sb^2, \quad K_\lambda = \log L_\lambda. \quad (5.1)$$

The zero report always has value zero. Its quadratic stability changes at $s_0 = \dim V_\lambda$.

Proposition 5.1 (Fourth-cumulant criterion). *Let $N = d_1 = \dim V_\lambda$. For $Y = \text{Re} \langle v_\lambda, X \rangle$,*

$$\mathbb{E}Y^2 = \frac{1}{2N}, \quad \text{cum}_4(Y) = \frac{3}{8} \left(\frac{1}{d_2} - \frac{2}{N^2} \right). \quad (5.2)$$

If $d_2 < N^2/2$, the first global activation of a nonzero report occurs at some $s_c < N$ with a radius bounded away from zero. Hence R_λ has a linear coexistence face adjacent to $D = 1$ and a discontinuous onset of directional information.

Proof. The moments follow from [theorem 3.1](#) and phase averaging. Since

$$K_\lambda(t) = \frac{t^2}{4N} + \frac{\text{cum}_4(Y)}{24}t^4 + O(t^6),$$

the coefficient of b^2 in (5.1) is $s(s/N - 1)$. Thus $b = 0$ is a strict local maximum for $0 < s < N$. At $s = N$ the quadratic term vanishes; if $\text{cum}_4(Y) > 0$, every sufficiently small nonzero b has $\phi_N(b) > 0$. For small $s > 0$, zero is the unique global maximizer: $|\text{Re} \langle a, X \rangle| \leq \|a\|$ and

Hoeffding's lemma give $K_\lambda(2sb) \leq 2s^2b^2$, so $\phi_s(b) < 0$ for $0 < s < 1/2$ and $b > 0$. Compactness and continuity therefore give an attained first contact $0 < s_c < N$.

Were positive maximizers at contact to converge to zero, the locally uniform expansion at $s_c < N$ would make $\phi_s(b) < 0$ for all sufficiently small nonzero b , a contradiction. The contact radius is therefore positive. Mixing the zero and positive covariant channels in [theorem 4.1](#) produces the linear face. $\square$

This criterion proves existence, not uniqueness, of coexistence. A global one-fold theorem requires additional information about the full dimension sequence d_m and is deliberately not assumed.

6 Root-system high-fidelity asymptotics

Let ρ be the Weyl vector and let

$$\Phi_\lambda^+ = \{\alpha > 0 : \langle \lambda, \alpha^\vee \rangle > 0\}, \quad p = |\Phi_\lambda^+|, \quad A_\lambda = \prod_{\alpha \in \Phi_\lambda^+} \frac{\langle \rho, \alpha^\vee \rangle}{\langle \lambda, \alpha^\vee \rangle}. \quad (6.1)$$

The projective coherent orbit has real dimension $2p$; its phase lift has dimension $q = 2p + 1$.

Lemma 6.1 (Bessel-weighted polynomial saddle). *Suppose d_m is given by Weyl's formula for $V_{m\lambda}$. Then, with $a = p + 1/2$,*

$$L_\lambda(\kappa) = \frac{2^p A_\lambda}{\sqrt{2\pi}} e^\kappa \kappa^{-a} (1 + o(1)), \quad (6.2)$$

Proof. Weyl's dimension formula is

$$d_m = \prod_{\alpha > 0} \frac{m \langle \lambda, \alpha^\vee \rangle + \langle \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} = \frac{m^p}{A_\lambda} (1 + O(m^{-1})). \quad (6.3)$$

Put $z = \kappa/2$ and let M_z have probability masses

$$\Pr\{M_z = m\} = \frac{z^{2m}}{(m!)^2 I_0(2z)}. \quad (6.4)$$

If N_1, N_2 are independent Poisson(z) variables, then $M_z \stackrel{d}{=} N_1 \mid \{N_1 = N_2\}$. For fixed $0 < \epsilon < 1$, Poisson Chernoff bounds and Stirling's lower bound at $m_0 = \lfloor z \rfloor$ give

$$\Pr(|N_1 - z| > \epsilon z) \leq 2e^{-c_\epsilon z}, \quad \Pr(N_1 = N_2) \geq \Pr(N_1 = m_0)^2 \geq cz^{-1}. \quad (6.5)$$

Consequently

$$\Pr(|M_z - z| > \epsilon z) \leq Cze^{-c_\epsilon z}. \quad (6.6)$$

The Weyl asymptotic in (6.3) is uniform on every tail interval. On $|M_z - z| \leq \epsilon z$, it squeezes z^p/d_{M_z} between quantities converging to A_λ as first $z \rightarrow \infty$ and then $\epsilon \downarrow 0$. On the complement, $d_m^{-1} \leq 1$ and (6.6) is exponentially smaller than z^{-p} ; the $m = 0$ mass is also exponentially small. Hence

$$\mathbb{E}d_{M_z}^{-1} \sim A_\lambda z^{-p}. \quad (6.7)$$

Finally $L_\lambda(2z) = I_0(2z)\mathbb{E}d_{M_z}^{-1}$ and $I_0(2z) \sim e^{2z}/\sqrt{4\pi z}$, which proves (6.2) with the displayed constant. This drifting-saddle argument is uniform and does not interchange an unbounded limit with a merely pointwise Weyl estimate. $\square$

Proposition 6.2 (Sharp high-fidelity constant). *As $D \downarrow 0$,*

$$\boxed{R_\lambda(D) = \frac{q}{2} \log \frac{q}{D} - \frac{q}{2} - \log \left(\frac{2^p A_\lambda}{\sqrt{2\pi}} \right) + o(1).} \quad (6.8)$$

Proof. By [theorem 6.1](#), $K(\kappa) = \kappa - (q/2) \log \kappa + \log(2^p A_\lambda / \sqrt{2\pi}) + o(1)$. Put $a = q/2$, $C = 2^p A_\lambda / \sqrt{2\pi}$, and $t = \delta\kappa$. Uniformly for t in compact subsets of $(0, \infty)$,

$$\kappa(1 - \delta) - K(\kappa) = a \log(1/\delta) + a \log t - t - \log C + o(1).$$

The deterministic function $a \log t - t$ has its unique maximum at $t = a$. The same asymptotic excludes $t \rightarrow \infty$ and excludes $t \rightarrow 0$ whenever $\kappa \rightarrow \infty$, while bounded κ gives only $O(1)$ and cannot compete with the diverging value $a \log(1/\delta)$. Hence $\delta\kappa_\delta \rightarrow a$ and direct Legendre optimization gives

$$j_\lambda(1 - \delta) = \frac{q}{2} \log \frac{q/2}{\delta} - \frac{q}{2} - \log \left(\frac{2^p A_\lambda}{\sqrt{2\pi}} \right) + o(1).$$

Since $D = 1 - b^2 = 2\delta + o(\delta)$, this is (6.8) on the deterministic-radius curve. Lower convexification does not improve its leading term. Indeed, for a mixed squared radius U put $\Delta = 1 - U$ and assume $\mathbb{E}\Delta = D$. On $A_D = \{\Delta \leq \sqrt{D}\}$, Markov gives $P(A_D) = 1 - o(1)$ and the preceding asymptotic is uniform. Conditional Jensen for $-\log \Delta$ yields

$$P(A_D) \mathbb{E}[-\log \Delta \mid A_D] \geq P(A_D) \log \frac{P(A_D)}{\mathbb{E}[\Delta; A_D]} \geq P(A_D) \log \frac{P(A_D)}{D} = \log(1/D) + o(1).$$

The complement contributes nonnegatively. This gives the matching lower bound, while the deterministic choice gives the upper bound. No differentiation of the $o(1)$ saddle remainder is used. $\square$

7 Complex Grassmannians and Slater determinants

Take $G = SU(n)$, $V_\lambda = \Lambda^k \mathbb{C}^n$, and $\lambda = \omega_k$, with $1 \leq k \leq n/2$ by duality. A coherent vector is a unit decomposable form $u_1 \wedge \cdots \wedge u_k$: a phased fermionic Slater determinant [\[12\]](#). The projective orbit is the Plücker embedding of $\text{Gr}_\mathbb{C}(k, n)$.

Proposition 7.1 (Rectangular dimensions and normalizer). *Let $p = k(n - k)$ and $N = \binom{n}{k}$. Then*

$$d_m(n, k) = \prod_{i=1}^k \frac{(n - k + i)_m}{(i)_m}, \quad (7.1)$$

and

$$\boxed{L_{n,k}(\kappa) = {}_{k-1}F_k \left(\begin{matrix} 2, 3, \dots, k \\ n - k + 1, n - k + 2, \dots, n \end{matrix}; \frac{\kappa^2}{4} \right).} \quad (7.2)$$

For $k = 1$ the empty numerator gives ${}_0F_1(; n; \kappa^2/4)$, the complex sphere normalizer.

Proof. Apply the Weyl dimension formula to the rectangular highest weight (m^k) . Substitution into (3.2) and cancellation of $(1)_m = m!$ gives (7.2). Formula (7.1) is invariant under $k \leftrightarrow n - k$, as required by Hodge duality. $\square$

Proposition 7.2 (Exact overlap product). *If X, Y are independent Haar Slater determinants and $T = |\langle X, Y \rangle|^2$, then*

$$T \stackrel{d}{=} \prod_{i=1}^k B_i, \quad B_i \text{ independent, } B_i \sim \text{Beta}(i, n - k). \quad (7.3)$$

Proof. The Cartan identity gives

$$\mathbb{E}T^m = \frac{1}{d_m(n, k)} = \prod_{i=1}^k \frac{(i)_m}{(n - k + i)_m}.$$

The right side is exactly the moment product of the beta variables in (7.3). Both laws are supported on $[0, 1]$, where the Hausdorff moment problem is determinate. This representation-theoretic derivation of the beta-product identity is included for the Slater specialization and is not itself presented as a priority claim. $\square$

The fourth cumulant has a simple complete classification.

Theorem 7.3 (Slater activation threshold). *For the phased Slater source,*

$$\frac{d_2(n, k)}{\binom{n}{k}^2} = \frac{n + 1}{(k + 1)(n - k + 1)}. \quad (7.4)$$

Consequently:

1. for $k = 1$, the fourth cumulant is negative;
2. for $(n, k) = (4, 2)$ it is negative;
3. for $(n, k) = (5, 2)$ it vanishes, and the sixth cumulant is $-1/11200$;
4. for every $2 \leq k \leq n - 2$ with $n \geq 6$, it is positive and the first activation is discontinuous at some $s_c < \binom{n}{k}$.

Proof. The hook formula for the rectangle (2^k) gives (7.4). By [theorem 5.1](#), positivity is equivalent to

$$2(n + 1) < (k + 1)(n - k + 1).$$

For $2 \leq k \leq n - 2$, the right side is minimized at $k = 2$ or $n - 2$ and is $3(n - 1)$. The inequality is strict exactly for $n \geq 6$; equality at the remaining interior boundary is $(5, 2)$ up to duality. Direct use of $d_1 = 10, d_2 = 50, d_3 = 175$ in the phase-averaged moments gives

$$\text{cum}_6(Y) = \mathbb{E}Y^6 - 15\mathbb{E}Y^4\mathbb{E}Y^2 + 30(\mathbb{E}Y^2)^3 = -\frac{1}{11200}.$$

The case $(4, 2)$ and $k = 1$ follow directly from (7.4). $\square$

The high-fidelity constant is also explicit:

$$A_{n,k} = \prod_{i=1}^k \frac{\Gamma(n - k + i)}{\Gamma(i)}, \quad q = 2k(n - k) + 1. \quad (7.5)$$

Thus [theorem 6.2](#) gives a closed rate offset for every particle number and one-particle dimension.

8 The phase-invariant quotient

Global phase is part of the source in [theorem 4.1](#). When only the projective coherent state matters, use the intrinsic Plücker distortion

$$\delta([x], [y]) = 1 - |\langle x, y \rangle|^2. \quad (8.1)$$

Here reproduction is restricted to the same projective homogeneous space; there is no radial orbitope report.

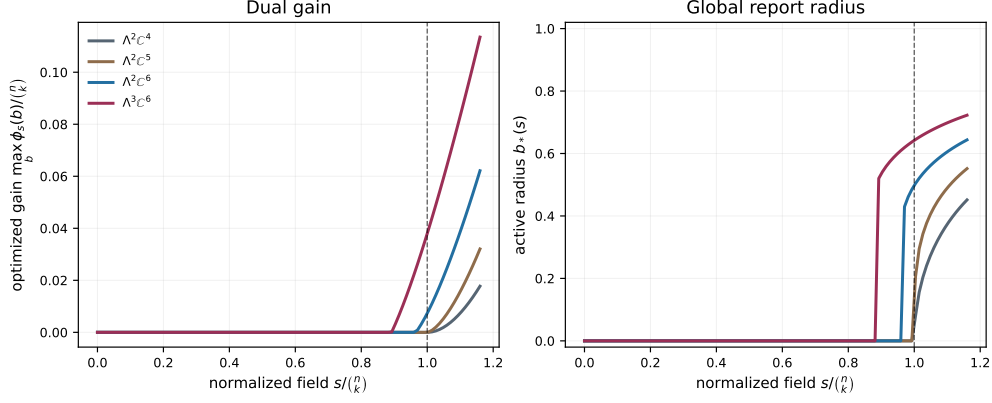


Figure 1: Scalar dual envelope for representative phased Slater families. The plotted quantity is $\max_b \phi_s(b)$; dots mark the local stability field $s_0 = \binom{n}{k}$. Interior families with $n \geq 6$ already have a positive-radius global phase at s_0 , as proved by [theorem 7.3](#). The plot is diagnostic and is not used in any proof.

Corollary 8.1 (Projective coherent-orbit frontier). *Let $[X]$ be Haar on the compact projective highest-weight orbit $G/K_\lambda \simeq G_{\mathbb{C}}/P_\lambda$, and put $N = \dim V_\lambda$. With reproduction restricted to this same orbit, under (8.1),*

$$R_\lambda^{\text{proj}}(D) = \sup_{s \geq 0} \{-sD + s - \log H_\lambda(s)\}, \quad H_\lambda(s) = \sum_{m=0}^{\infty} \frac{s^m}{m! d_m}. \quad (8.2)$$

for $0 < D < 1 - 1/N$. Moreover $R_\lambda^{\text{proj}}(D) = 0$ for $D \geq 1 - 1/N$, and the value at $D = 0$ is the limiting value $+\infty$ for a positive-dimensional orbit. For $\text{Gr}_{\mathbb{C}}(k, n)$, substitute (7.1).

Proof. For fixed $[y]$, invariance makes

$$\mathbb{E} e^{-s\delta([X], [y])} = e^{-s} \mathbb{E} e^{s| \langle X, y \rangle |^2} = e^{-s} H_\lambda(s)$$

independent of $[y]$. The constant Shannon dual is therefore feasible, and the covariant exponential reverse channel has the Haar source marginal and saturates it. Its distortion is $D(s) = 1 - H'_\lambda(s)/H_\lambda(s)$; strict log-convexity makes $D(s)$ continuous and strictly decreasing from $1 - 1/N$ to zero, so every stated interior distortion is covered. This is the same invariant-partition mechanism used for compact groups [4], applied here directly to the transitive projective orbit; the explicit dimension sequence is the representation-theoretic specialization. $\square$

The quotient result answers an intrinsic subspace-compression question. It must not be conflated with [theorem 4.1](#), whose decoder may report any point of the orbitope and whose radial shrinkage causes coexistence.

9 Reproducibility and claim boundaries

The lightweight replay accompanying this paper checks:

- 728 exact rectangular-dimension and Hodge-complement identities;
- the hook identity (7.4) and every quartic sign for $n \leq 12$;
- the exceptional cumulants, including $-1/11200$ at $(5, 2)$;
- the $k = 1$ ${}_0F_1$ reduction to relative error below 2.1×10^{-15} ;

- representative high-field constants and scalar activation diagnostics.

These checks replay algebra and numerics; they do not certify [theorems 3.1](#) and [4.1](#) or the uniform saddle proof. Those arguments are analytic and are stated in full above.

Several boundaries are essential. First, the integer moment maximization is due to Sugita; our new use is the all-field transform and exact classical RDF. Second, the phase-lifted source is a classical random amplitude vector. It is not a quantum message, so no quantum coding rate is claimed. Third, the squared ambient distortion is not click-only Born calibration. Fourth, the discontinuity criterion proves a first coexistence event but not a universal one-fold or no-reentrance theorem. Finally, the intrinsic quotient corollary restricts reproduction to the homogeneous space and is formally a different operational problem.

10 Conclusion

The Cartan product turns a large family of apparently high-dimensional compression problems into one scalar envelope. The key geometric fact is simultaneous: coherent rays maximize every integer overlap moment and hence the complete Laplace transform at every field. Conditional means and Shannon duality then promote that rigidity to an exact unrestricted rate–distortion theorem for every phase-lifted generalized coherent-state orbit.

Representation theory remains visible in the operational answer. The full dimension sequence $\dim V_{m\lambda}$ is the scalar partition function; $\dim V_{2\lambda}$ decides whether directional information turns on discontinuously; positive roots determine the exact high-fidelity offset. For fermionic Slater determinants these data become elementary rectangular dimensions, a beta-product overlap, and a sharp all- (n, k) transition criterion. This higher-exterior-degree family is qualitatively broader than a single Grassmannian or low-rank spectral case while retaining a complete, falsifiable formula.

The natural next problem is phase-diagram rigidity: identify highest weights for which the scalar radial envelope has a unique positive branch and no reentrance. That question is not needed for the exact frontier proved here, and it remains open outside special dimension sequences.

Data and code availability

The source package contains the LaTeX manuscript, bibliography, the bounded Python replay, its JSON output, and the figure-generation script. No external data set is used. All floating-point output is diagnostic; the theorems do not depend on numerical optimization.

Use of AI tools

AI tools assisted with literature discovery, algebraic replay, drafting, and adversarial checking. The named author takes responsibility for the claims, proofs, citations, and release artifacts. No peer review or independent experimental validation is claimed.

References

- [1] Ayumu Sugita. “Proof of the Generalized Lieb–Wehrl Conjecture for Integer Indices Larger than One”. In: *Journal of Physics A: Mathematical and General* 35.42 (2002), pp. L621–L626. DOI: [10.1088/0305-4470/35/42/105](https://doi.org/10.1088/0305-4470/35/42/105). arXiv: [nlin/0208007](https://arxiv.org/abs/nlin/0208007).
- [2] Woody Lichtenstein. “A System of Quadrics Describing the Orbit of the Highest Weight Vector”. In: *Proceedings of the American Mathematical Society* 84.4 (1982), pp. 605–608. DOI: [10.1090/S0002-9939-1982-0643758-8](https://doi.org/10.1090/S0002-9939-1982-0643758-8).

- [3] Karin Baur. “Cartan Components and Decomposable Tensors”. In: *Transformation Groups* 8.4 (2003), pp. 309–319. DOI: [10.1007/s00031-003-1203-2](https://doi.org/10.1007/s00031-003-1203-2).
- [4] Peter Harremoës. “Maximum Entropy on Compact Groups”. In: *Entropy* 11.2 (2009), pp. 222–237. DOI: [10.3390/e11020222](https://doi.org/10.3390/e11020222). arXiv: [0901.0015](https://arxiv.org/abs/0901.0015) [cs.IT].
- [5] Wei Dai, Youjian Liu, and Brian Rider. “Quantization Bounds on Grassmann Manifolds and Applications to MIMO Communications”. In: *IEEE Transactions on Information Theory* 54.3 (2008), pp. 1108–1123. DOI: [10.1109/TIT.2007.915691](https://doi.org/10.1109/TIT.2007.915691).
- [6] Erwin Riegler, Günther Koliander, and Helmut Bölcskei. “Lossy Compression of General Random Variables”. In: *Information and Inference: A Journal of the IMA* 12.3 (2023), pp. 1759–1829. DOI: [10.1093/imaiai/iaac035](https://doi.org/10.1093/imaiai/iaac035). arXiv: [2111.12312](https://arxiv.org/abs/2111.12312) [cs.IT].
- [7] Alex Dytso and Martina Cardone. “Uniform Distribution on the $(n - 1)$ -Sphere: Rate-Distortion under Squared Error Distortion”. In: *2024 IEEE International Symposium on Information Theory (ISIT)*. 2024, pp. 873–878. DOI: [10.1109/ISIT57864.2024.10619427](https://doi.org/10.1109/ISIT57864.2024.10619427). arXiv: [2401.04248](https://arxiv.org/abs/2401.04248) [cs.IT].
- [8] Howard Barnum. “Quantum Rate-Distortion Coding”. In: *Physical Review A* 62 (2000), p. 042309. DOI: [10.1103/PhysRevA.62.042309](https://doi.org/10.1103/PhysRevA.62.042309). arXiv: [quant-ph/9806065](https://arxiv.org/abs/quant-ph/9806065).
- [9] Zahra Baghali Khanian and Andreas Winter. “A Rate-Distortion Perspective on Quantum State Redistribution”. In: *IEEE Transactions on Information Theory* 72.4 (2026), pp. 2307–2318. DOI: [10.1109/TIT.2024.3516505](https://doi.org/10.1109/TIT.2024.3516505). arXiv: [2112.11952](https://arxiv.org/abs/2112.11952) [quant-ph].
- [10] Toby Berger. *Rate Distortion Theory: A Mathematical Basis for Data Compression*. Englewood Cliffs, NJ: Prentice-Hall, 1971.
- [11] Imre Csiszár. “On an Extremum Problem of Information Theory”. In: *Studia Scientiarum Mathematicarum Hungarica* 9 (1974), pp. 57–61.
- [12] Per-Olov Löwdin. “Quantum Theory of Many-Particle Systems. I. Physical Interpretations by Means of Density Matrices, Natural Spin-Orbitals, and Convergence Problems in the Method of Configurational Interaction”. In: *Physical Review* 97 (1955), pp. 1474–1489. DOI: [10.1103/PhysRev.97.1474](https://doi.org/10.1103/PhysRev.97.1474).