Algebraic Query Support for Unitary Oracles

Exact Hilbert Laws, Harmonic Spectra, and General-Tester Bounds

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Abstract

The usual dimension bound for discrimination with k calls to a d-level unitary oracle is the ambient symmetric-tensor count $\binom{k+d^2-1}{k}$. It ignores polynomial relations satisfied by a restricted oracle family. We show that the exact linear support exposed by k forward calls is instead the degree-k Hilbert function of the projective Zariski closure of that family. Every causally ordered k-query protocol consequently obeys a Hilbert-rank discrimination bound.

For the noncentral, nontraceless fixed-angle qubit class $U(\boldsymbol{n}) = qI - ir\boldsymbol{n}\cdot\boldsymbol{\sigma}$, $\boldsymbol{n} \in S^2$, $qr \neq 0$, this gives the exact quadratic law $H_X(k) = (k+1)^2$, rather than the unrestricted cubic count $\binom{k+3}{3}$. At the traceless endpoint the rank is $\binom{k+2}{2}$, while a central class has rank one. We use the transitive orbit average and its constant leverage score to prove the stronger operational statement

$$P_{\text{succ}}^{\text{GEN}} \leq \min\{1, D_k/M\}$$

for every uniform M-hypothesis subset and every general tester. Thus the fixed-angle promise refines the unrestricted general-tester numerator $\binom{k+3}{3}$ to $D_k = (k+1)^2$. Beyond qubits, a selective phase oracle on an unknown ray in $\mathbb{C}^d$ has projective closure equal to a linearly transformed Segre variety. Its exact support is $\binom{k+d-1}{d-1}^2$: the ambient degree d^2-1 collapses to $2d-2$, and the two-query defect is $d^2(d-1)^2/4$. We then diagonalize the continuous k-copy Bell frame exactly: its eigenvalues are finite Funk–Hecke sums indexed by spherical harmonics of ranks $0 \leq \ell \leq k$. This yields an exact purity and a participation dimension that grows only linearly in k, despite quadratic algebraic support. We prove that the continuous frame is tight in the interior only for one query at $q^2 = 1/4$; it is never tight there for $k \geq 2$. Spherical $2k$-designs inherit the same spectrum. Great-circle oracles form a conic subfamily with exact rank $2k+1$ and a closed Fourier spectrum. The results separate support rank, effective rank, and operational attainability. They do not assert that every full-rank ensemble attains the dimension bound or that every general process matrix has a laboratory realization.

1 Introduction

A k-query circuit can only create amplitudes that are homogeneous polynomials of degree k in the entries of its oracle. The polynomial method is foundational in quantum query complexity [1, 2]. In unitary channel discrimination, an analogous ambient count gives the general bound $\binom{k+d^2-1}{k}/M$ for M equiprobable d-dimensional unitary hypotheses [3]. That count is optimal without structural information about the oracle set. It is not intrinsic to a constrained family.

The missing datum is the homogeneous ideal of the oracle family. If the allowed projective matrices lie on an algebraic variety X, every equation in $I(X)$ removes a symmetric-tensor direction. The surviving number of directions is exactly the Hilbert function $H_X(k)$ [4]. This observation turns the ambient polynomial method into a geometry-sensitive query bound.

Our main application is physically elementary but structurally nontrivial. Fix the eigenphases of an $SU(2)$ gate and vary only its rotation axis:

$$U(\boldsymbol{n}) = e^{-i\theta \boldsymbol{n}\cdot\boldsymbol{\sigma}/2} = qI - ir\,\boldsymbol{n}\cdot\boldsymbol{\sigma}, \quad q = \cos(\theta/2), \quad r = \sin(\theta/2), \quad \boldsymbol{n} \in S^2. \tag{1}$$

Here $q, r \in \mathbb{R}$, $q^2 + r^2 = 1$, and $\operatorname{Tr} U = 2q$. Although channel representatives have an arbitrary global phase, the promise itself is the phase-invariant quantity $|\operatorname{Tr} U|^2/4 = q^2$; the displayed representatives fix the $SU(2)$ convention. Physically, the sphere describes a pulse with known rotation angle and unknown field axis. This is a conjugacy class of fixed-angle qubit rotations, a family also appearing in quantum learning of rotations about an unknown direction [5]. Its nondegenerate projective closure is a quadric surface, so its k-query support grows as $(k+1)^2$. The result sharpens the unrestricted qubit count by

$$\binom{k+3}{3} - (k+1)^2 = \binom{k+1}{3}. \tag{2}$$

At two queries this is a one-dimensional defect; at general k it is a cubic cumulative deficit.

The contributions are:

1. an oracle-variety theorem identifying exact causal query support with a Hilbert function, together with uniform and nonuniform causal bounds;
2. the complete three-branch rank law for fixed-angle qubit rotations and the exact rank law for a great-circle restriction;
3. an arbitrary-dimension law for rank-one selective-phase oracles, whose Segre geometry changes the ambient support exponent from $d^2 - 1$ to $2d - 2$;
4. a transitive-orbit leverage theorem refining the ultimate general-tester bound of Bavaresco–Murao–Quintino from $\binom{k+3}{3}/M$ to D_k/M on the fixed-angle class;
5. a finite closed formula for every eigenvalue of the continuous k-copy frame, including endpoint cascades, exact purity, and effective rank;
6. a sharp obstruction showing that one-query tightness does not persist at any interior angle for $k \geq 2$;
7. an exact positive-scalability criterion that distinguishes a dimension upper bound from its attainability by a fixed effective state family.

The harmonic analysis, Hilbert functions, leverage inequality, and spherical designs used below are classical [6, 7]. The claimed contribution is their exact synthesis with algebraic quantum-query support and general-tester discrimination. We do not claim new general theory of spherical harmonics, algebraic varieties, or process matrices.

Dimensional and linear-independence bounds for repeated gate identification are direct antecedents [8, 9]. Likewise, the orbit-average domination mechanism is classical in covariant quantum estimation [10] and appears explicitly in the ultimate general-tester proof of [3, Appendix F]. Our new claim is therefore the exact promise-class numerator, its singular branch diagram, and the accompanying all-k harmonic spectra—not a new tester formalism or a first orbit-domination lemma.

She and Yuen's generalized polynomial method shows that a T-query unitary property test has an acceptance polynomial of degree at most $2T$ and uses invariant theory to derive property-testing lower bounds [2]. Our statement is complementary: it concerns the exact pure-state span of k forward calls, uses the full homogeneous ideal rather than an invariant acceptance polynomial, and feeds that span into finite-ensemble channel discrimination. We do not claim to originate the unitary polynomial method.

Table 1: Priority boundary: classical mechanisms versus the consequences derived here.

Source	Established mechanism	Present use or additional conclusion
Beals et al.; She–Yuen	Query amplitudes/acceptance probabilities are low-degree polynomials; invariant theory yields property-testing bounds.	The full oracle ideal gives the exact pure-state support $H_X(k)$ and a finite-ensemble rank converse.
Bavaresco et al.	Ambient symmetric-tensor domination for general testers.	The orbit numerator is replaced by the exact promise rank D_k, including all singular branches.
Funk–Hecke and spherical designs	Harmonic diagonalization and exact cubature for polynomial kernels.	Exact fixed-angle spectra, purity, endpoint cascades, and the all-k tightness obstruction.
Segre geometry	Rank-one matrices have coordinate space $H^0(\mathcal{O}(k,k))$.	Selective-phase qudit oracles acquire support $\binom{k+d-1}{d-1}^2$ and a GEN discrimination cap.

The $k = 2$ identity $10 \to 9$ was previously isolated for a specific 24-echo tetrahedral ensemble, together with a causal saturation certificate, in [11]. Here it is one instance of an all-k law for arbitrary fixed-angle subsets. The present work additionally treats the three singular branches, compact-orbit general testers, exact continuous spectra, and the planar-axis restriction; it does not reuse the special tetrahedral attainment certificate.

2 Forward-query model and projective oracle families

Let V be a finite-dimensional complex vector space containing the matrix entries of an oracle. A unitary channel is unchanged by multiplying its implementing matrix by a global phase, so oracle matrices naturally determine points of $\mathbb{P}(V)$. Let $\mathcal{U} \subset \mathbb{P}(V)$ be a family of projective unitary matrices and let $X = \overline{\mathcal{U}}^{Z}$ denote its complex projective Zariski closure. Write

$$H_X(k) = \dim_{\mathbb{C}}\big(\mathbb{C}[V]/I(X)\big)_k \tag{3}$$

for its degree-k Hilbert function.

We use the standard forward-oracle model: each query inserts the same unknown unitary U, not $U^\dagger$ and not a controlled version of an otherwise uncontrolled black box. Arbitrary known isometries, ancillas, coherent feedback, and intermediate measurements are allowed. Intermediate measurements can be deferred after adding a record system.

We also use the general-tester model of Bavaresco–Murao–Quintino [3]. For a d-level unitary let

$$|U\rangle\rangle = (I \otimes U) \sum_{a=0}^{d-1} |a\rangle|a\rangle, \qquad J_U = |U\rangle\rangle\langle\langle U|. \tag{4}$$

Thus $\operatorname{Tr} J_U = d$ and $\operatorname{Tr}_O J_U = I$. A general k-copy tester is a family $T_i \geq 0$ whose deterministic sum $W = \sum_i T_i$ obeys

$$\operatorname{Tr}[W(C_1 \otimes \cdots \otimes C_k)] = 1 \tag{5}$$

for every tuple of channel Choi operators $C_1, \ldots, C_k$. This feasible class contains parallel and causally ordered testers and is the mathematical general-process class used in that reference. We do not assume that every element has a laboratory realization.

Lemma 2.1 (Multilinear normal form)**.** *For every purified causally ordered circuit making k forward calls to U, there is a fixed linear map L, independent of U, such that its final pure state satisfies*

$$|\Psi_U\rangle = L\, U^{\otimes k}. \tag{6}$$

Tensor-factor permutations, fixed input vectors, ancillary identities, and all intermediate isometries are absorbed into L.

Proof. Expand the circuit in fixed bases. Every path amplitude contains exactly one matrix entry from each oracle call and hence is a homogeneous degree-k polynomial in the entries of U. Collecting its coefficients defines L. Purifying every known channel and coherently storing every measurement outcome reduces the general causal circuit to this case without changing the best final discrimination probability. □

3 The oracle-variety query law

Theorem 3.1 (Hilbert query-support law)**.** *For every $k \geq 0$,*

$$\dim\Big(\operatorname{span}\{U^{\otimes k} : [U] \in \mathcal{U}\}\Big) = H_X(k). \tag{7}$$

Consequently, the final states of every purified causal k-query circuit have linear span of dimension at most $H_X(k)$.

Proof. The Veronese vector $U^{\otimes k}$ belongs to $\mathrm{Sym}^k(V)$. Its annihilator inside $\mathrm{Sym}^k(V)^*$ consists exactly of homogeneous degree-k polynomials that vanish on $\mathcal{U}$. Polynomial vanishing is unchanged by Zariski closure, so the annihilator is $I(X)_k$. Rank–nullity gives

$$\dim\Big(\operatorname{span}\{U^{\otimes k}\}\Big) = \dim \mathrm{Sym}^k(V) - \dim I(X)_k = H_X(k). \tag{8}$$

The circuit statement follows from theorem 2.1 because a fixed linear map cannot increase span dimension. □

The support value in theorem 3.1 is operationally exposed by a causal parallel architecture: feed the k calls with k Bell pairs. The resulting pure vector is $|U\rangle\!\rangle^{\otimes k}$, and vectorization is injective, so its hypothesis span has dimension exactly $H_X(k)$. Thus $H_X(k)$ is the maximum causal support, not merely an upper bound on it.

Corollary 3.2 (Causal discrimination bounds)**.** *Let $U_1, \dots, U_M \in \mathcal{U}$ have priors p_i. Every causal k-query protocol satisfies*

$$P_{\text{succ}} \leq \min\{1, p_{\max} H_X(k)\}, \qquad p_{\max} = \max_i p_i. \tag{9}$$

For uniform priors this becomes $P_{\text{succ}} \leq \min\{1, H_X(k)/M\}$.

Proof. Let P project onto the span of the purified final hypothesis vectors and let $\{M_i\}$ be the final POVM. Since $|\Psi_i\rangle\langle\Psi_i| \leq P$,

$$P_{\text{succ}} = \sum_i p_i \operatorname{Tr}(M_i |\Psi_i\rangle\langle\Psi_i|) \tag{10}$$

$$\leq p_{\max} \sum_i \operatorname{Tr}(P M_i P) = p_{\max} \operatorname{Tr} P \leq p_{\max} H_X(k). \tag{11}$$

□

If X has positive projective dimension s and degree Δ, its Hilbert polynomial gives

$$H_X(k) = \frac{\Delta}{s!} k^s + O(k^{s-1}). \tag{12}$$

Thus the geometry of the oracle family fixes the exponent of accessible support. A necessary asymptotic condition for perfect identification of M generic hypotheses is $k \gtrsim (s! M/\Delta)^{1/s}$. It is only necessary: positivity and the geometry of the finite ensemble can impose further obstructions.

Remark 3.3 (Relation to the polynomial method). The fact that query amplitudes are low-degree polynomials is classical in Boolean-oracle complexity [1]. Theorem 3.1 identifies the exact quotient dimension after imposing the homogeneous ideal of a continuous matrix-oracle family. For $X = \mathbb{P}^{d^2-1}$ it reduces to the ambient count $\binom{k+d^2-1}{k}$ used in the general unitary discrimination bound [3].

4 Transitive orbits: an ultimate general-tester bound

The Hilbert law above applies to arbitrary projective oracle families but is proved through causal circuit amplitudes. Transitive unitary orbits admit a stronger statement because their leverage score is constant.

Lemma 4.1 (Compact-orbit leverage)**.** *Let a compact group act transitively on a unitary orbit $\{U_x : x \in \Omega\}$, with invariant probability measure μ. Set*

$$v_x = |U_x\rangle\!\rangle^{\otimes k}, \qquad \overline{J}_k = \int_\Omega |v_x\rangle\langle v_x|\, d\mu(x), \qquad D_k = \operatorname{rank} \overline{J}_k. \tag{13}$$

Then, for every x,

$$|v_x\rangle\langle v_x| \le D_k\, \overline{J}_k. \tag{14}$$

The constant D_k is the smallest orbit-independent constant for which this operator inequality holds.

Proof. The orbit representation is unitary, so $\overline{J}_k$ and its Moore–Penrose inverse commute with it. Hence $L(x) = \langle v_x|\overline{J}_k^+|v_x\rangle$ is constant. Averaging gives

$$L(x) = \operatorname{Tr}(\overline{J}_k^+ \overline{J}_k) = D_k. \tag{15}$$

For $A \ge 0$ and $v \in \operatorname{ran} A$, Cauchy–Schwarz applied to $A^{+1/2}v$ gives $|v\rangle\langle v| \le \langle v|A^+|v\rangle A$. This proves equation (14). Applying $\langle v_x|\overline{J}_k^+(\cdot)\overline{J}_k^+|v_x\rangle$ to any uniform domination shows that its constant cannot be smaller than D_k. □

Theorem 4.2 (Ultimate bound on a transitive unitary orbit)**.** *Let $U_1, \dots, U_M$ be any finite subset of a compact transitive unitary orbit and suppose the labels are equiprobable. Every general k-copy tester satisfies*

$$P_{\text{succ}}^{\text{GEN}} \le \min\left\{1, \frac{D_k}{M}\right\}. \tag{16}$$

For arbitrary priors, the safe variant is $P_{\text{succ}}^{\text{GEN}} \le \min\{1, p_{\max} D_k\}$.

Proof. Write $J_i = J_{U_i}$ and $W = \sum_i T_i$. By theorem 4.1, $J_i^{\otimes k} \le D_k \overline{J}_k$. Positivity and equation (5) give

$$P_{\text{succ}} = \frac{1}{M} \sum_i \operatorname{Tr}(T_i J_i^{\otimes k}) \tag{17}$$

$$\le \frac{D_k}{M} \operatorname{Tr}(W \overline{J}_k) = \frac{D_k}{M}, \tag{18}$$

because $\operatorname{Tr}(W J_{U_x}^{\otimes k}) = 1$ for every x and hence also after averaging. Replacing each prior by $p_{\max}$ proves the nonuniform bound. □

When $D_k \le M$, equality at D_k/M is not automatic. With $A_i = D_k \overline{J}_k - J_i^{\otimes k} \ge 0$, it holds for a given tester if and only if $\operatorname{Tr}(T_i A_i) = 0$ for every i. If P projects onto $\operatorname{ran} \overline{J}_k$, this forces

$$P T_i P = c_i |\overline{J}_k^+ v_i\rangle\langle \overline{J}_k^+ v_i|, \qquad c_i \ge 0, \tag{19}$$

together with the deterministic-tester normalization. Thus full orbit span is necessary but not sufficient for attaining the cap.

5 Fixed-angle qubit rotations: three exact branches

In the Pauli basis, equation (1) has projective coordinates $[x_0 : x_1 : x_2 : x_3] = [q : -irn_1 : -irn_2 : -irn_3]$.

Theorem 5.1 (Fixed-angle rank law)**.** *For $k \geq 1$, define*

$$\mathcal{V}_k(q, r) = \operatorname{span}_{\mathbb{C}}\{U(\boldsymbol{n})^{\otimes k} : \boldsymbol{n} \in S^2\}. \tag{20}$$

Then

$$\dim \mathcal{V}_k(q, r) = \begin{cases} (k+1)^2, & qr \neq 0, \\ \binom{k+2}{2}, & q = 0,\ r \neq 0, \\ 1, & r = 0. \end{cases} \tag{21}$$

For $qr \neq 0$, generic M axes have finite-ensemble rank $\min\{M, (k+1)^2\}$.

Proof. When $qr \neq 0$, the projective closure is the smooth quadric

$$x_1^2 + x_2^2 + x_3^2 = -\frac{r^2}{q^2} x_0^2 \subset \mathbb{P}^3. \tag{22}$$

Its Hilbert series is $(1 - t^2)/(1 - t)^4$, whose degree-k coefficient is

$$\binom{k+3}{3} - \binom{k+1}{3} = (k+1)^2. \tag{23}$$

Equivalently, the feature blocks are all symmetric monomials in $\boldsymbol{n}$ of degrees $0, \ldots, k$. Fischer decomposition identifies their restrictions to the sphere with $\bigoplus_{\ell=0}^{k} \mathcal{H}_\ell$, of total dimension $\sum_{\ell=0}^{k}(2\ell + 1) = (k+1)^2$.

If $q = 0$, only the homogeneous degree-k feature remains. The tensors $\boldsymbol{n}^{\otimes k}$ span $\mathrm{Sym}^k(\mathbb{C}^3)$: a homogeneous polynomial vanishing on the real unit sphere vanishes on every nonzero real vector by scaling and therefore vanishes identically. The dimension is $\binom{k+2}{2}$. If $r = 0$, the class contains one projective oracle. Finally, because the full evaluation span has the displayed finite dimension, some set of that many axes is independent; nonvanishing of an evaluation minor is Zariski open, which proves the generic finite-set statement. □

At the traceless endpoint $U(-\boldsymbol{n}) = -U(\boldsymbol{n})$, so antipodal axes implement the same unitary channel. The physical promise space is therefore $\mathbb{RP}^2$, not a sphere of distinct channel labels. Counting both $\boldsymbol{n}$ and $-\boldsymbol{n}$ as different hypotheses would introduce an unavoidable label ambiguity and cannot lead to perfect channel identification.

Corollary 5.2 (Ultimate fixed-angle query bound)**.** *For M equiprobable fixed-angle qubit hypotheses with $qr \neq 0$,*

$$P_{\mathrm{succ}}^{\mathrm{GEN}}(k) \leq \min\left\{1, \frac{(k+1)^2}{M}\right\}. \tag{24}$$

Perfect identification requires $k \geq \lceil \sqrt{M} \rceil - 1$. At $q = 0$ it instead requires $\binom{k+2}{2} \geq M$.

Proof. The $SU(2)$ conjugation action is transitive on the fixed-angle sphere, so theorem 4.2 applies with the ranks in theorem 5.1. The query conditions follow by requiring the relevant numerator to be at least M. They are necessary, not sufficient. □

Table 2: Exact k-query support dimensions for the indicated qubit oracle families and singular branches.

Oracle geometry	defining structure	support dimension
unrestricted projective matrices	$\mathbb{P}^3$	$\binom{k+3}{3}$
fixed angle, $qr \neq 0$	quadric surface	$(k+1)^2$
traceless fixed angle	homogeneous sphere features	$\binom{k+2}{2}$
fixed-angle great circle	plane conic	$2k+1$
traceless great circle	homogeneous binary features	$k+1$
central class	one point	1

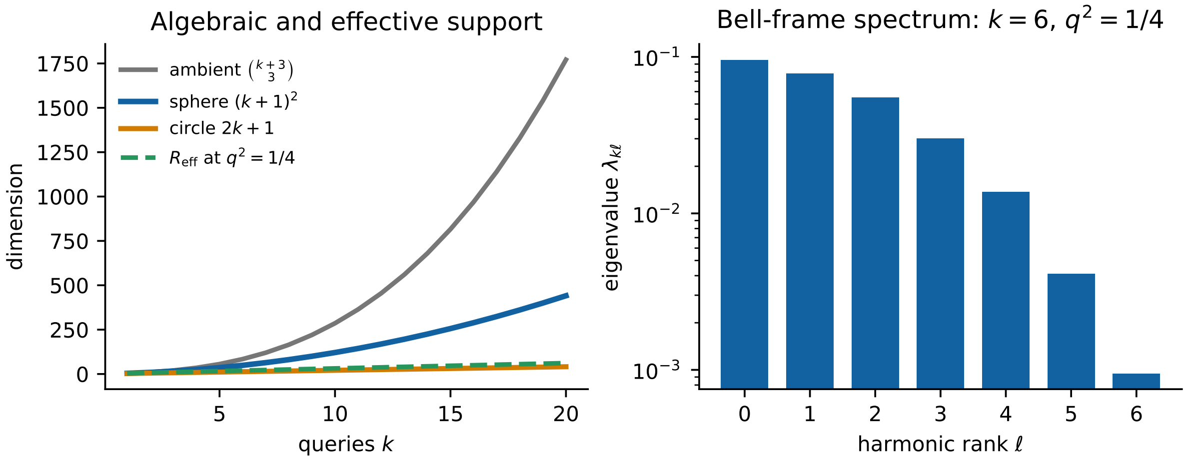

Figure 1: Left: unrestricted, fixed-sphere, and fixed-circle query dimensions, together with the Bell-frame effective rank at $q^2 = 1/4$. The fixed-angle promise changes the general-tester numerator from cubic to quadratic, while a planar-axis promise makes it linear. Right: the exact harmonic eigenvalues at $k = 6$ illustrate why effective and algebraic ranks differ.

6 Rank-one phase oracles in arbitrary dimension

The qubit quadric is the first member of a higher-dimensional family with a more dramatic support reduction. Let $d \geq 2$, let $|\psi\rangle \in \mathbb{C}^d$ be an unknown unit vector, and consider the selective phase oracle

$$U_\psi = I + cP_\psi, \qquad P_\psi = |\psi\rangle\langle\psi|, \qquad c = e^{i\vartheta} - 1. \tag{25}$$

It applies the known phase $e^{i\vartheta}$ to one unknown ray and acts as the identity on its orthogonal complement. The promise is transitive under unitary conjugation.

Theorem 6.1 (Segre query-support law)**.** *For $k \geq 1$, set*

$$D_{d,k}(c) = \dim \operatorname{span}\{U_\psi^{\otimes k} : |\psi\rangle \in \mathbb{C}^d,\ \|\psi\| = 1\}. \tag{26}$$

Then

$$D_{d,k}(c) = \begin{cases} 1, & c = 0, \\ \binom{k+2}{2}, & d = 2,\ c = -2, \\ \binom{k+d-1}{d-1}^2, & \textit{otherwise.} \end{cases} \tag{27}$$

On the generic branch the support exponent is $2d-2$, rather than the unrestricted matrix exponent $d^2 - 1$.

Proof. Projective rank-one matrices form the Segre variety

$$\Sigma_d = \mathbb{P}^{d-1} \times (\mathbb{P}^{d-1})^* \hookrightarrow \mathbb{P}(\operatorname{End}\mathbb{C}^d). \tag{28}$$

The physical projectors are Zariski dense in Σ_d: equivalently, the $U(d)$ orbit complexifies to the dense GL_d orbit of rank-one matrices with nonzero trace. Define the linear map

$$L_c(X) = \operatorname{Tr}(X)I + cX. \tag{29}$$

For $\operatorname{Tr} P_\psi = 1$, one has $L_c(P_\psi) = U_\psi$. On the scalar and traceless summands of $\operatorname{End}\mathbb{C}^d$, the eigenvalues of L_c are respectively $c+d$ and c. Hence L_c is invertible unless $c = 0$ or $c = -d$.

When it is invertible, the projective oracle closure is a linear image of Σ_d. Its degree-k coordinate space is

$$H^0(\Sigma_d, \mathcal{O}(k,k)) \cong \operatorname{Sym}^k(\mathbb{C}^d)^* \otimes \operatorname{Sym}^k(\mathbb{C}^d), \tag{30}$$

of dimension $\binom{k+d-1}{d-1}^2$. The Hilbert query-support law now gives the last line of equation (27).

For a unit-modulus phase, $c = -d$ can occur only when $d = 2$ and $c = -2$. Then L_c kills the scalar coordinate and sends the Bloch projectors onto the projective traceless space $\mathbb{P}^2$, whose Hilbert function is $\binom{k+2}{2}$. Finally, $c = 0$ makes every oracle equal to I. $\square$

Corollary 6.2 (Qudit phase-oracle discrimination)**.** *For M equiprobable hypotheses from the orbit in equation* (25)*, every general k-copy tester obeys*

$$P_{\text{succ}}^{\text{GEN}} \leq \min\{1, D_{d,k}(c)/M\}. \tag{31}$$

On the generic branch, perfect identification therefore requires

$$\binom{k+d-1}{d-1}^2 \geq M. \tag{32}$$

At two queries the exact reduction from the unrestricted support is

$$\binom{d^2+1}{2} - \binom{d+1}{2}^2 = \frac{d^2(d-1)^2}{4}. \tag{33}$$

Thus the qutrit support is 36 *rather than* 45*, and the four-level support is* 100 *rather than* 136*.*

Proof. The orbit is compact and transitive, so theorem 4.2 applies with theorem 6.1. Equation (33) is an elementary simplification of the two displayed dimensions. $\square$

For fixed d and large k, the generic support satisfies

$$D_{d,k}(c) = \frac{k^{2d-2}}{((d-1)!)^2} + O(k^{2d-3}), \tag{34}$$

whereas the unrestricted support is $k^{d^2-1}/(d^2-1)! + O(k^{d^2-2})$. The promise therefore changes not only a constant but the polynomial growth exponent. When $d = 2$, the generic Segre law reduces to $(k+1)^2$, and its exceptional phase $c = -2$ is exactly the traceless branch of theorem 5.1.

7 When is a support bound attained?

Rank alone does not construct a tester. The following statement concerns a fixed family of effective pure states, after the query architecture has been chosen; it is not an existence theorem for a causal architecture.

Theorem 7.1 (Fixed-state scalability criterion)**.** *Let $|\phi_i\rangle$ be normalized vectors with Gram matrix K, common span projector P, and rank D. For uniform priors, their dimension bound D/M is attained if and only if there are weights $c_i \geq 0$ such that*

$$\sum_i c_i |\phi_i\rangle\langle\phi_i| = P. \tag{35}$$

Equivalently, for $C = \mathrm{diag}(c_i)$,

$$KCK = K. \tag{36}$$

In that case $\sum_i c_i = D$ and $M_i = c_i|\phi_i\rangle\langle\phi_i|$ is an attaining POVM on the support.

Proof. The estimate $\mathrm{Tr}(M_i|\phi_i\rangle\langle\phi_i|) \leq \mathrm{Tr}(PM_iP)$ is an equality only if the compressed positive operator PM_iP is supported on $\mathbb{C}\phi_i$. Equality in the sum therefore forces $PM_iP = c_i|\phi_i\rangle\langle\phi_i|$ and POVM completeness gives equation (35). Conversely, equation (35) supplies a POVM with success $M^{-1}\sum_i c_i = D/M$. Multiplying the frame identity by the analysis and synthesis maps gives equation (36); the reverse implication holds on the frame range. Taking the trace gives $\sum_i c_i = D$. □

This criterion isolates three logically different resources: algebraic support D, the ability of a circuit to realize useful vectors inside that support, and positive scalability of those vectors. A full-rank evaluation table need not satisfy the last condition.

8 Exact harmonic spectrum of the continuous class

Use normalized one-query Bell/Choi vectors

$$|u_{\boldsymbol{n}}\rangle = q|0\rangle - ir\sum_{a=1}^{3} n_a|a\rangle, \qquad \langle u_{\boldsymbol{n}}|u_{\boldsymbol{m}}\rangle = a + b\,\boldsymbol{n}\cdot\boldsymbol{m}, \tag{37}$$

where $a = q^2$, $b = r^2$, and $a + b = 1$. Define the continuous k-copy frame operator

$$F_k = \int_{S^2} |u_{\boldsymbol{n}}\rangle\langle u_{\boldsymbol{n}}|^{\otimes k}\, d\omega(\boldsymbol{n}), \tag{38}$$

with normalized area measure.

Theorem 8.1 (Funk–Hecke spectrum)**.** *If $ab > 0$, the nonzero eigenvalues of F_k are*

$$\lambda_{k\ell}^{\times(2\ell+1)}, \qquad 0 \leq \ell \leq k, \tag{39}$$

where

$$\lambda_{k\ell} = \frac{1}{2}\int_{-1}^{1} (a + bx)^k P_\ell(x)\, dx \tag{40}$$

$$= \sum_{\substack{\ell \leq j \leq k \\ j-\ell \text{ even}}} \binom{k}{j} a^{k-j} b^j \frac{j!}{(j-\ell)!!(j+\ell+1)!!}. \tag{41}$$

Every displayed eigenvalue is positive and $\sum_{\ell=0}^{k}(2\ell+1)\lambda_{k\ell} = 1$. At $a = 0$, exactly the sectors $\ell \equiv k \pmod 2$ survive.

Proof. The integral operator with the same nonzero spectrum as F_k has zonal kernel $(a+b\boldsymbol{n}\cdot\boldsymbol{m})^k$. Funk–Hecke diagonalizes it on each spherical harmonic space $\mathcal{H}_\ell$ and gives equation (40). Expanding the binomial and using

$$\frac{1}{2}\int_{-1}^{1} x^j P_\ell(x)\,dx = \frac{j!}{(j-\ell)!!(j+\ell+1)!!} \tag{42}$$

when $j \geq \ell$ has the same parity as ℓ, and zero otherwise, proves equation (41). The $j = \ell$ term proves positivity for $ab > 0$. The trace identity follows by evaluating the kernel on the diagonal. ☐

Proposition 8.2 (Exact purity and effective rank)**.** *For* $b > 0$,

$$\operatorname{Tr} F_k^2 = \frac{1-(a-b)^{2k+1}}{2b(2k+1)}, \qquad R_{\text{eff}}(F_k) = \frac{2b(2k+1)}{1-(a-b)^{2k+1}}. \tag{43}$$

For every fixed interior angle, $R_{\text{eff}}(F_k) = 2b(2k+1)+o(1)$ *as* $k \to \infty$*, whereas* $\operatorname{rank} F_k = (k+1)^2$.

Proof. Using two independent axes and rotational invariance,

$$\begin{aligned}\operatorname{Tr} F_k^2 &= \iint |\langle u_{\boldsymbol{n}}|u_{\boldsymbol{m}}\rangle|^{2k} d\omega(\boldsymbol{n})d\omega(\boldsymbol{m}) && (44)\\ &= \frac{1}{2}\int_{-1}^{1} (a+bx)^{2k}\,dx, && (45)\end{aligned}$$

which integrates to equation (43), since $a + b = 1$. For $0 < a, b < 1$ one has $|a - b| < 1$, giving the asymptotic statement. ☐

Thus algebraic support and participation dimension have different scaling: the former is quadratic and the latter linear. The exact frame entropy is also immediately available as

$$S(F_k) = -\sum_{\ell=0}^{k}(2\ell+1)\lambda_{k\ell}\log\lambda_{k\ell}, \tag{46}$$

but no entropy optimality is asserted here.

8.1 Endpoint cascades

The rank jumps in equation (21) are resolved spectrally. As $b \downarrow 0$,

$$\lambda_{k\ell} = \frac{k!}{(k-\ell)!(2\ell+1)!!}a^{k-\ell}b^\ell + O(b^{\ell+1}), \tag{47}$$

so angular sectors disappear successively with harmonic rank. As $a \downarrow 0$, sectors with $k - \ell$ even have nonzero limits

$$\lambda_{k\ell} \longrightarrow \frac{k!}{(k-\ell)!!(k+\ell+1)!!}, \tag{48}$$

whereas opposite-parity sectors vanish at least linearly in $a = q^2$. More precisely, when $k - \ell$ is odd,

$$\lambda_{k\ell} = \frac{k!}{(k-\ell-1)!!(k+\ell)!!}\,a + O(a^2). \tag{49}$$

Exactly $k(k+1)/2$ dimensions disappear at the traceless endpoint.

9 Interior tightness is a one-query phenomenon

Theorem 9.1 (Sharp interior tightness obstruction)**.** *For $ab > 0$, the continuous frame F_k is tight on its support if and only if $k = 1$ and $a = q^2 = 1/4$. In particular, it is never tight for any $k \geq 2$ at an interior fixed angle.*

Proof. For $k = 1$, equation (41) gives $\lambda_{10} = a$ and $\lambda_{11} = b/3$, so equality holds exactly at $a = 1/4$. Let $k \geq 2$. Equality of the top two harmonic eigenvalues would force

$$\frac{\lambda_{k,k-1}}{\lambda_{k,k}} = (2k+1)\frac{a}{b} = 1. \tag{50}$$

At the resulting ratio $a/b = 1/(2k+1)$, direct evaluation of the next sector gives

$$\frac{\lambda_{k,k-2}}{\lambda_{k,k}} = \frac{2k+1}{2}\left[1 + (2k-1)(a/b)^2\right] = \frac{k(2k+3)}{2k+1} > 1. \tag{51}$$

The three sectors therefore cannot share one eigenvalue. $\square$

At the traceless endpoint $a = 0$, the one-query frame is tight on its three-dimensional support, while for $k \geq 2$ its surviving sectors already have unequal top eigenvalues because $\lambda_{k,k-2}/\lambda_{k,k} = (2k+1)/2 \neq 1$. At the central endpoint $a = 1$ the frame has rank one and is trivially tight for every k. These endpoint statements are separate from the interior theorem.

Corollary 9.2 (No positive reweighting restores Bell tightness)**.** *For $ab > 0$ and $k \geq 2$, no finite nonzero positive measure on the full fixed-angle orbit can make the k-copy Bell vectors a tight frame on their complete $(k+1)^2$-dimensional span.*

Proof. Let P_ℓ project onto the harmonic sector $\mathcal{H}_\ell$. Transitivity makes $\|P_\ell u_{\boldsymbol{n}}^{\otimes k}\|^2$ independent of $\boldsymbol{n}$; its value is $(2\ell+1)\lambda_{k\ell}$. Therefore every positive weighted frame of total mass w has sector trace $w(2\ell+1)\lambda_{k\ell}$. A scalar multiple of the full support projector would instead have sector trace proportional to $2\ell+1$, forcing all $\lambda_{k\ell}$ to coincide. Theorem 9.1 excludes this for $k \geq 2$. $\square$

Corollary 9.3 (Transfer to spherical designs)**.** *If a finite weighted axis set is a spherical $2k$-design, its discrete k-copy Bell frame equals F_k and has exactly the spectrum in theorem 8.1. Hence no such equal-weight frame is tight for $k \geq 2$ at an interior angle.*

Proof. Every matrix entry of the frame integrand is a polynomial of total degree at most $2k$ in the axis coordinates. The design identity therefore replaces the integral exactly by the finite weighted sum. $\square$

The corollary does not say that every design is a transitive group orbit, nor that no alternative query input can attain the general-tester dimension cap.

10 Planar-axis oracles: conic rank and Fourier spectrum

Restrict the axis to $\boldsymbol{n}(\varphi) = (\cos\varphi, \sin\varphi, 0)$. The projective closure is a plane conic. This is a control promise in which the unknown field axis is known a priori to lie in a plane (two available quadratures), rather than a separate physical memory resource.

Theorem 10.1 (Great-circle law)**.** *For $ab > 0$,*

$$\dim\Big(\mathrm{span}\{U(\boldsymbol{n}(\varphi))^{\otimes k} : \varphi \in [0, 2\pi)\}\Big) = 2k + 1. \tag{52}$$

The continuous frame has Fourier eigenvalues

$$\mu_{km} = \sum_{\substack{|m|\le j\le k\\ j-m\ \text{even}}} \binom{k}{j} a^{k-j} b^j 2^{-j} \binom{j}{(j-m)/2}, \qquad -k \le m \le k. \tag{53}$$

All are positive. At $a = 0$, only modes $m \equiv k \pmod 2$ survive and the rank becomes $k+1$.

Proof. The kernel is $(a+b\cos(\varphi-\psi))^k$. Its Fourier support is contained in $-k,\ldots,k$, and the $j = |m|$ term is nonzero when $ab > 0$, proving exact rank $2k+1$. Expanding $\cos^j\theta = 2^{-j}(e^{i\theta}+e^{-i\theta})^j$ gives equation (53). If $a = 0$, only $j = k$ remains. □

At $a = 0$, antipodal points again differ only by global phase, so the physical channel family is $\mathbb{RP}^1$. Distinct-hypothesis counts must quotient this identification before applying a perfect-discrimination criterion.

Because rotations around the normal act transitively on the circle, theorem 4.2 also gives

$$P_{\text{succ}}^{\text{GEN,circ}} \le \begin{cases} \min\{1,(2k+1)/M\}, & ab > 0,\\ \min\{1,(k+1)/M\}, & a = 0,\\ 1/M, & b = 0. \end{cases} \tag{54}$$

The circle is an explicit warning against inferring support dimension from the number of continuous parameters alone: both the sphere and the circle are infinite families, but their Hilbert functions are respectively quadratic and linear.

11 Operational consequences and comparisons

For nondegenerate fixed-angle qubit oracles, the accessible support hierarchy is

$$\underbrace{R_{\text{eff}}(F_k)}_{\Theta(k)} \le \underbrace{\operatorname{rank} F_k}_{(k+1)^2} \le \underbrace{\binom{k+3}{3}}_{\text{unrestricted ambient}}. \tag{55}$$

Each quantity answers a different question. The first measures spectral participation of one canonical continuous Bell encoding; the second is the maximum algebraic support exposed by the oracle family; the third ignores the fixed-angle relation.

For example, $M = 24$ generic fixed-angle hypotheses cannot be perfectly identified with fewer than four forward queries, because $(k+1)^2 < 24$ for $k \le 3$. This is a necessary query lower bound, not a four-query construction. At $k = 2$, the family-specific support cap is $9/M$, whereas the unrestricted qubit cap is $10/M$. The all-k deficit in equation (2) shows that this is the first member of a growing sequence rather than an isolated accident. The earlier tetrahedral-echo paper [11] proves this $k = 2$ defect for one nontransitive 24-label ensemble and, crucially, certifies causal attainment there. The present result generalizes the support defect to all k and arbitrary fixed-angle subsets, extends the bound to the transitive-orbit GEN class, and adds the singular branch laws and exact sphere/circle spectra; it makes no new claim about that ensemble's adaptive certificate.

More sharply, the class-to-ambient ratio is

$$\frac{(k+1)^2}{\binom{k+3}{3}} = \frac{6(k+1)}{(k+2)(k+3)} \sim \frac{6}{k}. \tag{56}$$

Thus the promise removes an asymptotically dominant fraction of the ambient query space, not merely a fixed finite defect.

The contrast with the unrestricted result is operational, not merely a rank calculation. Bavaresco–Murao–Quintino prove the ambient $\binom{k+3}{3}/M$ cap for general qubit testers [3]; theorem 4.2 replaces its numerator by the exact coordinate-ring rank of the transitive fixed-angle promise. For the full sphere the reduction is $\binom{k+1}{3}$ dimensions, while the circle promise reduces the numerator further to $2k+1$.

The higher-dimensional selective-phase family shows that the mechanism is not peculiar to a quadric surface. Its Segre closure changes the support growth from $\Theta(k^{d^2-1})$ to $\Theta(k^{2d-2})$. For fixed d, the corresponding necessary query scale for M generic labels drops from the ambient degree count to $k \gtrsim ((d-1)!^2 M)^{1/(2d-2)}$; this remains a converse, not a universal attaining construction.

Fixed-angle rotation learning [5] studies a different task: a direction is encoded in a spin system and later used to implement a target rotation. The present task is minimum-error identification from repeated black-box gate calls. The common conjugacy-class geometry explains the shared use of $SU(2)$ covariance, but neither result implies the other.

12 Reproducibility and scope

The accompanying exact-arithmetic verifier checks, for a configurable range of k, the Hilbert-rank identities, every rational Funk–Hecke eigenvalue at rational a, trace and purity, endpoint ranks, great-circle Fourier coefficients, and the tightness obstruction. It is a finite symbolic audit, not a computer proof of the general theorems.

The scope is deliberately precise:

- the Hilbert-function support identity is a causal forward-query theorem for arbitrary projective families; the stronger general-tester cap is proved only for compact transitive unitary orbits by constant leverage;
- the general-tester class is the mathematical optimization class of [3]; no claim is made that every general process matrix is physically realizable;
- support rank is not automatically an achievable success probability; theorem 7.1 is a criterion for a fixed effective state family, not a universal circuit synthesis theorem;
- the continuous spectrum concerns the Bell/Choi parallel frame; it does not optimize all inputs or all adaptive circuits;
- spherical harmonics, Funk–Hecke theory, Hilbert functions, and spherical designs are classical ingredients, as is the rank-one leverage inequality;
- controlled-U, calls to $U^\dagger$, postselection, and different oracle models require new feature spaces and are not covered;
- no noise robustness, experimental resource count, or asymptotic statistical estimation theorem is inferred from exact algebraic rank.

AI assistance. OpenAI Codex assisted with symbolic exploration, literature triage, drafting, and reproducibility checks. The author is responsible for the definitions, proofs, claims, citations, and final manuscript.

13 Conclusion

Quantum query support is controlled by the algebraic geometry of the oracle family, not only by the size of its ambient matrix space. The degree-k Hilbert function gives the exact exposed dimension and therefore a direct causal discrimination bound. On a compact transitive orbit, constant leverage upgrades the same rank to an ultimate general-tester cap. Fixed-angle qubit rotations provide a complete case study: quadratic generic support, a separate traceless branch, exact harmonic and Fourier spectra, linear participation rank, and a sharp failure of multi-query Bell tightness. Rank-one phase oracles show in arbitrary dimension that the same framework can reduce the ambient support exponent from $d^2 - 1$ to $2d - 2$. The resulting separation between ambient rank, promise-variety rank, spectral participation, and operational attainability supplies a portable framework for other constrained unitary orbits.

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