

A sharp circle-root residual barrier and its collapse with one interior zero

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Abstract

For degree- n polynomials with all zeros on the unit circle, the least normalized residual $\|P' - \lambda P\|_{H^2}/\|P\|_{H^2}$ tends to $1/\sqrt{8}$, both for arbitrary complex λ and for the restricted slope $\lambda = P'(0)$, $P(0) = 1$. We prove the sharp lower limit through the classical Mathieu ground state and give an explicit circle-root family attaining it in every large degree. In contrast, exactly one zero strictly inside the disk, with all remaining zeros on the circle, permits restricted residual $O(n^{-1/6})$, giving the exact zero-count threshold for collapse. More generally, for each fixed $A > 0$, a fixed number M_A of interior zeros permits $O(n^{-A})$. Their distance and radial gap from 1 have order $n^{-1/3}$; explicit algebraic parameters in the one-root case, and an implicit-function construction in general, correct both degree rounding and the exact slope. For all disk-root polynomials and arbitrary slope, we also prove a uniform lower bound $\exp(-C\sqrt{n})$ for every $C > \sqrt{(8/3)\log 17}$, explicitly $\exp(-3\sqrt{n})$ for $n \geq 10^4$. This refines the known residual method for exterior power sums; the negative answer to Erdős Problem 973 was already known. The optimal disk and one-root decay rates, and small exterior power-sum constructions, remain unresolved.

1 Statement and relation to previous work

All logarithms are natural. The measure on the unit circle is normalized to have mass one. Thus for $P(z) = \sum_{j=0}^n a_j z^j$,

$$\|P\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^2 dt = \sum_{j=0}^n |a_j|^2, \quad \|P\|_\infty = \max_{|z|=1} |P(z)|.$$

Write $\mathcal{S}_n$ for the nonzero polynomials of exact degree n whose zeros are in $\overline{\mathbb{D}}$, including multiplicity and boundary zeros. Define

$$s_n = \inf_{\substack{P \in \mathcal{S}_n \\ \lambda \in \mathbb{C}}} \frac{\|P' - \lambda P\|_2}{\|P\|_2}.$$

There is no normalization at the origin in this definition.

Theorem 1. *Put $C_0 = \sqrt{(8/3)\log 17}$. For every $C > C_0$ there is N_C such that*

$$s_n \geq e^{-C\sqrt{n}} \quad (n \geq N_C). \tag{1}$$

The following explicit bound also holds:

$$s_n \geq e^{-3\sqrt{n}} \quad (n \geq 10^4). \tag{2}$$

For comparison, Tan, Wang, Huang and Chen [1, Theorem 1.1] considered the restricted infimum

$$r_n = \inf_{\substack{P \in \mathcal{S}_n \\ P(0)=1}} \frac{\|P' - P'(0)P\|_2}{\|P\|_2}$$

and proved $r_n \geq \exp(-(1+o(1))\sqrt{n} \log n)$. For an individual normalized polynomial we write $\rho(P) = \|P' - P'(0)P\|_2 / \|P\|_2$. Theorem 3 identifies the positive limiting residual when every root lies on the circle. Theorem 11 shows that one interior root suffices to make it vanish, establishing the minimal zero-count threshold. These statements concern the global weighted boundary norm, including the exact restricted slope.

Convention on squares. Every residual in a theorem statement, including the limit $1/\sqrt{8}$, is the unsquared ratio $\|P' - \lambda P\|_2 / \|P\|_2$. The proofs below mostly bound its square, which is a ratio of two integrals. The passage back is the square root: a limit $1/8$ for the squared ratio gives $1/\sqrt{8}$, and $\rho^2 = O(n^{-1/3})$ is the same statement as $\rho = O(n^{-1/6})$.

Since $s_n \leq r_n$, Theorem 1 strengthens that quantitative estimate and allows an arbitrary scalar slope. Their maximum-point normalization, segment integration and projected Newton identity are retained here, with attribution. The two changes are a local analytic bound for the logarithmic derivative and a bounded test polynomial. These remove the logarithmic factor in the exponent.

Luo, Yang and Zhu [2, Theorem 1.1] had already proved that exterior power sums cannot be exponentially small in the degree, answering Erdős Problem 973 negatively. The application in Section 5 is a quantitative refinement of that established conclusion. No new solution of the original yes-or-no problem is asserted.

Theorem 2 (Superalgebraic decay). *For every $A > 0$, there are constants C_A, N_A such that*

$$r_n \leq C_A n^{-A} \quad \text{for every integer } n \geq N_A.$$

The polynomials can be chosen with real coefficients, exactly M_A zeros strictly inside the unit disk, where M_A is independent of n , and all other zeros on its boundary. The interior zeros approach 1 with distance and radial gap $\Theta(n^{-1/3})$. Together with Theorem 1, this implies superalgebraic decay of both s_n, r_n , while $s_n^{1/n}, r_n^{1/n} \rightarrow 1$.

The proof in Section 6 fixes the cancellation order and finite integer multiplicities first, then lets the reciprocal root scale grow as $n^{1/3}$. It provides no effective bounds on M_A, C_A, N_A or uniform control as A grows. Section 7 explains why a fixed strict root radius precludes superalgebraic decay of a single sequence.

Approximation by logarithmic derivatives and finite moment realization have substantial prior history. Komarov [3] obtains geometric approximation on each fixed subdisk by logarithmic derivatives of circle-root polynomials; [4] also realizes prescribed initial moments with unit weights. Those interior estimates do not supply the global weighted boundary norm used here, with a target slope growing with degree. For example, $1 + z^n$ approximates zero logarithmically on subdisks but has restricted residual $n/\sqrt{2}$. The present contribution is the combination of exact slope, global concentration and correction at every large degree, rather than moment realization alone.

2 The sharp barrier when every zero is on the circle

Let s_n° and r_n° denote the same infima as s_n, r_n , with every zero required to lie on $|z| = 1$.

Theorem 3. *The unrestricted and restricted circle-root residuals satisfy*

$$\lim_{n \rightarrow \infty} s_n^\circ = \lim_{n \rightarrow \infty} r_n^\circ = \frac{1}{\sqrt{8}}.$$

The lower asymptotic bound also holds in the larger class of self-inversive polynomials. No finite-degree extremizer is asserted.

Proof. We first prove the lower bound, allowing an arbitrary complex slope. Rotating the variable makes this slope real and nonnegative, denoted λ . Self-inversivity permits a constant phase choice such that

$$P(e^{it}) = e^{int/2} p(t), \quad p(t) \in \mathbb{R}, \quad p(t + 2\pi) = (-1)^n p(t).$$

These operations preserve the unrestricted normalized residual. The familiar phase derivative $n/2$ for circle-root polynomials appears explicitly in [5, (1.3)–(1.4)]; here we retain the additional scalar subtraction. Direct differentiation gives

$$P' - \lambda P = e^{i(n/2-1)t} [(n/2 - \lambda \cos t)p - i(p' + \lambda \sin t p)].$$

Integration of the cross term by parts is legitimate because p^2 is periodic. Consequently the squared residual is exactly

$$\frac{\int_0^{2\pi} [p'^2 + (n^2/4 + \lambda^2 - (n+1)\lambda \cos t)p^2] dt}{\int_0^{2\pi} p^2 dt}. \quad (3)$$

Write $E(L)$ for the lowest periodic eigenvalue of $-d^2/dt^2 - L \cos t$, for $L \geq 0$. In odd degree, $|p|$ is periodic in H^1 and has the same quadratic-form values as p . Thus in both parities (3) is at least

$$g_n(\lambda) = n^2/4 + \lambda^2 + E((n+1)\lambda).$$

This is a variational lower bound, not an identification of the periodic and antiperiodic spectra.

The change of variables $t = 2x + \pi$ gives $E(L) = a_0(2L)/4$, in the standard Mathieu convention $-d^2/dx^2 + 2q \cos(2x)$; see [7, §28.2]. The use of this ground state in circular variational problems is classical; compare [6, §VI.A.2]. The fixed-index expansion [7, (28.8.1), $m = 0, s = 1$] is

$$a_0(q) = -2q + 2\sqrt{q} - \frac{1}{4} + O(q^{-1/2}), \quad E(L) = -L + \sqrt{L/2} - \frac{1}{16} + O(L^{-1/2}).$$

We must minimize uniformly over λ , rather than applying this expansion only at a fixed shift. The variational principle gives $E(L) \geq -L$ and Lipschitz continuity, so g_n is continuous and coercive, with

$$g_n(\lambda) \geq (\lambda - (n+1)/2)^2 - (2n+1)/4.$$

The trial value $\lambda = n/2 + 1/4$ gives $g_n = 1/8 + O(n^{-1})$. Every minimizer therefore has the form $n/2 + c_n$, where $c_n = O(\sqrt{n})$. On this window $(n+1)\lambda \geq n^2/4$ for large n . The displayed expansion is a statement about the single variable L : there are L_0 and C' with $|E(L) - (-L + \sqrt{L/2} - \frac{1}{16})| \leq C' L^{-1/2}$ for all $L \geq L_0$. No further uniformity in λ is therefore needed, and on the window the remainder is $O(n^{-1})$. Expanding its square root uniformly on the same window yields

$$g_n(n/2 + c) = (c - 1/4)^2 + \frac{1}{8} + O((c^2 + 1)/n).$$

The upper bound on the minimum and absorption of the c_n^2/n term imply $c_n = O(1)$. Applying the expansion again proves $\inf_{\lambda \geq 0} g_n(\lambda) = 1/8 + O(n^{-1})$. This supplies the required lower limit, also for the restricted slope. Since (3) is the *squared* residual, it gives $s_n^\circ, r_n^\circ \geq (1/8 - O(n^{-1}))^{1/2} \rightarrow 1/\sqrt{8}$.

For the upper bound, write $n = 6m + k$, $0 \leq k \leq 5$, and for $m \geq 3$ choose

$$\begin{aligned} c_{m,k} &= \cos t_{m,k} = \frac{1}{2} + \frac{2k-1}{8m}, \\ B_t(z) &= (1+z)^4(1-2\cos t z + z^2), \\ P_{m,k}(z) &= B_{t_{m,k}}(z)^m(1+z)^k. \end{aligned}$$

All zeros lie on the circle, $P_{m,k}(0) = 1$, and

$$P'_{m,k}(0) = n/2 + 1/4, \quad P'_{m,k}(1)/P_{m,k}(1) = n/2.$$

The limiting factor is $B_* = (1+z)^4(1-z+z^2)$. With $s = \sin^2(\theta/2)$, its squared normalized modulus is $\cos^6(\theta/2)\cos^2(3\theta/2)$, with a unique maximum at $\theta = 0$, and its logarithm is $-3\theta^2 + O(\theta^4)$. More explicitly, for the perturbed factor this modulus is

$$H_c(s) = (1-s)^4 \left(1 - \frac{2s}{1-c}\right)^2.$$

Lemma A.1 in Appendix A derives the position $s^* = (2-c)/3$ and the value $(1+c)^6/[729(1-c)^2]$ of its secondary maximum, shows that this value is < 1 (indeed < 0.631) for $0 \leq c \leq 3/4$, and gives $H_c(s) \leq e^{-2s}$ on $[0, 1]$, uniformly for $|c - 1/2| \leq 1/10$. All six parameter sequences eventually lie in that interval. Hence $|P_{m,k}(e^{i\theta})/P_{m,k}(1)|^2 \leq e^{-2ms}$ on the whole circle. This supplies the Gaussian domination near zero (where s is comparable to θ^2) and an exponentially small outer tail, uniformly in the six classes.

For $F_{m,k} = P'_{m,k}/P_{m,k} - P'_{m,k}(0)$, the identity

$$B'_*/B_* - 3 = -\frac{3z(z-1)^2}{z^3+1}$$

shows, on putting $\theta = u/\sqrt{m}$, that

$$F_{m,k}(e^{i\theta}) \longrightarrow -\frac{1}{4} + \frac{3}{2}u^2, \quad |P_{m,k}(e^{i\theta})/P_{m,k}(1)|^2 \longrightarrow e^{-3u^2}.$$

Indeed the constant term of $F_{m,k}$ at 1 is exactly $-1/4$, its linear term is $O(1)$, and its quadratic term is $-3m/2 + O(1)$. A fixed zero-free neighborhood gives the local bound $C(1 + |\theta| + m\theta^2)$; after rescaling, a polynomial times a Gaussian dominates. On the outer interval, differentiate the polynomial product, removing one factor. Uniformly bounded normalized factor derivatives give an outer squared numerator $O(m^2q^{2m-2})$, for some $q < 1$. Thus no estimate at a pole of the logarithmic derivative is required. The ratio of the complete integrals tends to

$$\frac{\int_{\mathbb{R}} (-1/4 + 3u^2/2)^2 e^{-3u^2} du}{\int_{\mathbb{R}} e^{-3u^2} du} = \frac{1}{16} - \frac{3}{4} \cdot \frac{1}{6} + \frac{9}{4} \cdot \frac{1}{12} = \frac{1}{8}.$$

This is the limit of the squared residual, so $\rho(P_{m,k}) \rightarrow 1/\sqrt{8}$. Uniformity over the six values of k covers every sufficiently large degree. This proves the upper bound for r_n° , and $s_n^\circ \leq r_n^\circ$ completes both limits. $\square$

3 A finite residual certificate

We first give conditions that can be checked at a fixed degree. The asymptotic and explicit choices will then be elementary.

Lemma 4 (Normalization and separation). *Suppose $P \in \mathcal{S}_n$, $\lambda \in \mathbb{C}$, $y > 0$, and*

$$\frac{\|P' - \lambda P\|_2}{\|P\|_2} \leq e^{-y}, \quad \sqrt{n+1} e^{-y} \leq 1.$$

Set

$$\gamma_n = 2 + \log(2\sqrt{n+1}), \quad \eta = \frac{y - \gamma_n}{n}. \quad (4)$$

If $\eta > 0$, there are numbers $v_1, \dots, v_n$ and moments $m_k = n^{-1} \sum_j v_j^k$ such that

$$m_1 \in \mathbb{R}, \quad m_1 \geq \frac{1}{2}, \quad \operatorname{Re} v_j \geq \frac{1}{2}, \quad |v_j|^2 \leq \frac{\operatorname{Re} v_j}{\eta}. \quad (5)$$

They are the reciprocals $v_j = (1 - \beta_j)^{-1}$ of the complex displacements from the zeros to a normalized maximum point.

Proof. This is the normalization and separation argument of [1, Sections 2.1–2.2], with the scalar slope left arbitrary. Choose $|\tau| = 1$ at a maximum of $|P|$, and put

$$G(w) = \frac{P(\tau w)}{P(\tau)}, \quad \mu = \tau \lambda, \quad E = G' - \mu G.$$

The denominator is nonzero because P is nonzero. We have $G(1) = \|G\|_\infty = 1$, $\|G\|_2 \leq 1$, and $\|E\|_2 \leq e^{-y}$. Differentiating $|G(e^{it})|^2$ at its maximum shows that $d = G'(1)$ is real. Bernstein's polynomial inequality gives $|d| \leq n$. The coefficient bound $\|E\|_\infty \leq \sqrt{n+1} \|E\|_2$ therefore yields

$$|\mu - d| = |E(1)| \leq 1, \quad |\mu| \leq n + 1. \quad (6)$$

Let β be any zero of G , and set $\delta = 1 - \operatorname{Re} \beta$. Integrating the exact ODE along the segment from β to 1 gives

$$1 = \int_\beta^1 e^{\mu(1-w)} E(w) dw.$$

That segment lies in $\overline{\mathbb{D}}$. On it, $\operatorname{Re}(\mu(1-w)) \leq n\delta + 2$, using $d \in \mathbb{R}$, $|d| \leq n$, $|\mu - d| \leq 1$, and $|1 - \beta| \leq 2$. Consequently

$$1 \leq 2\sqrt{n+1} e^{n\delta+2-y}, \quad \delta \geq \eta.$$

No zero equals 1, since $G(1) = 1$. Now let $v = (1 - \beta)^{-1}$ for a zero β of G ; the zeros of G are the zeros of P multiplied by $\bar{\tau}$, hence lie in $\overline{\mathbb{D}}$. Since $|\beta| \leq 1$, $|1 - \beta|^2 = 1 - 2\operatorname{Re} \beta + |\beta|^2 \leq 2(1 - \operatorname{Re} \beta)$, and so

$$\operatorname{Re} v = \frac{1 - \operatorname{Re} \beta}{|1 - \beta|^2} \geq \frac{1}{2}, \quad \frac{|v|^2}{\operatorname{Re} v} = \frac{1}{1 - \operatorname{Re} \beta} = \frac{1}{\delta} \leq \frac{1}{\eta}.$$

Finally, G has exact degree n and $G(1) = 1$, so with its zeros $\beta_1, \dots, \beta_n$ we have $nm_1 = \sum_j v_j = G'(1)/G(1) = d$. The individual numbers v_j are complex, but their sum d is real. Hence $m_1 = d/n = \operatorname{Re} m_1 = n^{-1} \sum_j \operatorname{Re} v_j$, the average of n real numbers each at least $1/2$; this gives $m_1 \in \mathbb{R}$ and $m_1 \geq 1/2$. $\square$

Lemma 5 (Local analytic estimate). *With the normalization above, let $r > 0$ and suppose*

$$q = r\sqrt{n+1} \exp(-y + (2n+1)r) \leq \frac{1}{2}. \quad (7)$$

Then G has no zero in $|w - 1| \leq r$, and, for every integer $k \geq 2$,

$$|m_k| \leq \frac{2\sqrt{n+1}}{n} \exp(-y + (2n+1)r) r^{1-k}. \quad (8)$$

Proof. The disk centered at 1 can extend beyond $\overline{\mathbb{D}}$; the needed control there comes from the coefficients. For $|u| \leq r$,

$$|E(1+u)| \leq \sqrt{n+1} (1+r)^n \|E\|_2 \leq \sqrt{n+1} e^{-y+nr}.$$

Solving the ODE from $G(1) = 1$ gives

$$G(1+u) = e^{\mu u} (1 + h(u)), \quad h(u) = \int_0^u e^{-\mu s} E(1+s) ds.$$

Using the straight segment and (6), we obtain $|h(u)| \leq q \leq 1/2$. Thus G is nonzero on that closed disk and on a neighborhood of it. The analytic function $F = G'/G - \mu = E/G$ satisfies

$$|F(1+u)| \leq 2\sqrt{n+1}e^{-y+(2n+1)r} \quad (|u| \leq r).$$

Cauchy's derivative estimate and logarithmic differentiation give

$$\frac{|F^{(k-1)}(1)|}{(k-1)!} \leq 2\sqrt{n+1}e^{-y+(2n+1)r}r^{1-k}, \quad F^{(k-1)}(1) = (-1)^{k-1}(k-1)!nm_k.$$

This proves (8). There is no restriction $k \leq n+1$: F is an analytic rational function on the specified disk. $\square$

Proposition 6 (Finite certificate). *Fix $n \geq 1$, $y, r > 0$, and an integer $K \geq 2$. Define η by (4) and set $a = 2\eta$. Suppose $0 < \eta \leq 2$, $\sqrt{n+1}e^{-y} \leq 1$, and (7) holds. Put*

$$B = \frac{2\sqrt{n+1}r}{Kan} \exp\left(-y + (2n+1)r + K \log\left(1 + \frac{a}{r}\right)\right). \quad (9)$$

If

$$\frac{1}{K\eta} + B < \frac{1}{2}, \quad (10)$$

then $s_n \geq e^{-y}$.

Proof. Suppose an admissible P, λ had residual at most e^{-y} . For the numbers in Lemma 4,

$$|1 - av_j|^2 = 1 - 2a \operatorname{Re} v_j + a^2 |v_j|^2 \leq 1.$$

Use the bounded Cesàro test

$$\Psi(v) = \frac{1 - (1 - av)^K}{Ka} = v + \sum_{k=2}^K \frac{(-1)^{k+1}}{K} \binom{K}{k} a^{k-1} v^k. \quad (11)$$

It has zero constant coefficient, linear coefficient one, and $|\Psi(v_j)| \leq 2/(Ka) = 1/(K\eta)$. By Lemma 5, its remaining averaged terms have modulus at most

$$\frac{2\sqrt{n+1}r}{Kan} e^{-y+(2n+1)r} \left[\left(1 + \frac{a}{r}\right)^K - 1 - \frac{Ka}{r} \right] \leq B.$$

The triangle inequality and $m_1 \geq 1/2$ now imply

$$\frac{1}{2} \leq m_1 \leq \left| \frac{1}{n} \sum_j \Psi(v_j) \right| + B \leq \frac{1}{K\eta} + B,$$

contrary to (10). Each individual residual is therefore larger than e^{-y} ; taking its infimum gives the asserted non-strict bound. $\square$

4 Choice of parameters

Proof of (1). Fix $C > C_0$, and choose $t > 0$ so small that

$$C^2 > \frac{4}{3}(2+t) \log 17.$$

Set

$$y = C\sqrt{n}, \quad r = \frac{y}{8n}, \quad \eta = \frac{y - \gamma_n}{n}, \quad K = \left\lceil \frac{2+t}{\eta} \right\rceil.$$

For all sufficiently large n , the preliminary conditions of Proposition 6 hold: $\eta \sim C/\sqrt{n}$, $q \rightarrow 0$, and $\sqrt{n+1}e^{-y} \rightarrow 0$. Also $K\eta \geq 2+t$, leaving a fixed positive margin in (10). We have

$$K \sim \frac{2+t}{C}\sqrt{n}, \quad \frac{a}{r} \rightarrow 16,$$

and hence

$$\frac{-y + (2n+1)r + K \log(1+a/r)}{\sqrt{n}} \rightarrow -\frac{3C}{4} + \frac{2+t}{C} \log 17 < 0.$$

The prefactor in (9) is $O(n^{-1})$. Thus $B \rightarrow 0$, and the finite certificate applies. $\square$

Proof of (2). Take $n \geq 10^4$ and choose

$$y = 3\sqrt{n}, \quad r = \frac{3}{8\sqrt{n}}, \quad K = \left\lceil \frac{21}{10\eta} \right\rceil.$$

First,

$$\gamma_n \leq \frac{3}{20}\sqrt{n}, \quad \frac{19}{20}\frac{y}{n} \leq \eta \leq \frac{y}{n}. \quad (12)$$

Indeed $\gamma_n/\sqrt{n}$ decreases for $n \geq 10^4$, since the sign of its derivative is that of $n/(n+1) - \gamma_n < 0$. At $n = 10^4$, $\gamma_n < 10 < 15$: $e^8 > 2^8 = 256 > 2\sqrt{10001}$ suffices for the first inequality. In particular $0 < \eta < 2$. The condition $\sqrt{n+1}e^{-y} < 1$ is immediate, and

$$q = \frac{3}{8}\sqrt{1 + \frac{1}{n}} \exp\left(-\frac{9}{4}\sqrt{n} + \frac{3}{8\sqrt{n}}\right) < \frac{1}{2}.$$

Moreover $K\eta \geq 21/10$, so

$$\frac{1}{2} - \frac{1}{K\eta} \geq \frac{1}{42}.$$

From (12) we obtain $a/r \leq 16$ and $K \leq (14/19)\sqrt{n} + 1$. Using $\log 17 < 3$,

$$\begin{aligned} -y + (2n+1)r + K \log(1+a/r) &\leq \left(-\frac{9}{4} + \frac{14}{19} \log 17\right) \sqrt{n} + \log 17 + \frac{3}{8\sqrt{n}} \\ &< -\frac{3}{76}\sqrt{n} + 3 + \frac{3}{8\sqrt{n}} < 0. \end{aligned}$$

The last expression decreases with n and is negative at 10^4 . For an elementary check of $\log 17 < 3$, the exponential series through degree five gives $e^3 > 18 > 17$. Finally $Ka \geq 21/5$, and therefore

$$\frac{2\sqrt{n+1}r}{Kan} \leq \frac{5}{28n}\sqrt{1 + \frac{1}{n}} < \frac{1}{5n}.$$

It follows that $B < 1/(5n) \leq 1/50000 < 1/42$. All conditions of Proposition 6 are satisfied. $\square$

Remark 7. The constants 3 and 10^4 are convenient explicit choices. The same proof with $r = sy/n$, $0 < s < 1/2$, gives the sufficient asymptotic condition

$$C^2 > \frac{2 \log(1 + 2/s)}{1 - 2s}.$$

We use $s = 1/8$ to state a simple constant. Neither this expression nor C_0 is asserted to be the optimal residual constant.

5 Exterior power sums and Hardy norms

Corollary 8. *Let $|z_j| \geq 1$, $1 \leq j \leq n$, and write $M_n = \max_{2 \leq k \leq n+1} |\sum_j z_j^k|$. Then*

$$M_n \geq \frac{1}{n} e^{-3\sqrt{n}} \quad (n \geq 10^4).$$

For every $C > C_0$, uniformly over all such configurations, $M_n \geq n^{-1} e^{-C\sqrt{n}}$ for all sufficiently large n . No distinguished point $z_1 = 1$ is needed.

Proof. We use the projected Newton identity of [1, Proposition 3.1]. Put $P(t) = \prod_{j=1}^n (1 - z_j t)$ and $p_k = \sum_j z_j^k$. Then $P \in \mathcal{S}_n$, $P(0) = 1$, and $P'(0) = -p_1$. Logarithmic differentiation near zero, followed by coefficient projection onto degrees at most n , gives exactly

$$P' - P'(0)P = -\Pi_{\leq n} \left[\left(\sum_{k=2}^{n+1} p_k t^{k-1} \right) P \right].$$

Projection is contractive in H^2 , and multiplication by a polynomial S has H^2 norm at most $\|S\|_\infty$. Thus

$$s_n \leq \frac{\|P' - P'(0)P\|_2}{\|P\|_2} \leq \sum_{k=2}^{n+1} |p_k| \leq nM_n.$$

Apply Theorem 1. □

The factor $1/n$ is retained in these finite statements. It can be absorbed in a leading square-root exponent with any strictly larger constant. This corollary strengthens the scale in [1]; the earlier qualitative conclusion of [2] remains fully credited.

Corollary 9. *Uniformly for $1 \leq p \leq \infty$, $P \in \mathcal{S}_n$, and $\lambda \in \mathbb{C}$,*

$$\frac{\|P' - \lambda P\|_{H^p}}{\|P\|_{H^p}} \geq \frac{e^{-3\sqrt{n}}}{\sqrt{n+1}} \quad (n \geq 10^4).$$

For every $C > C_0$, the lower bound $e^{-C\sqrt{n}}$ holds for all sufficiently large n , uniformly over the same parameters.

Proof. Write $R = P' - \lambda P$, of degree at most n . For $1 \leq p \leq 2$, bounding each Fourier coefficient by $\|R\|_1$ gives

$$\|R\|_2 \leq \sqrt{n+1} \|R\|_1 \leq \sqrt{n+1} \|R\|_p, \quad \|P\|_2 \geq \|P\|_p.$$

For $p \geq 2$, use

$$\|R\|_2 \leq \|R\|_p, \quad \|P\|_p \leq \|P\|_\infty \leq \sqrt{n+1} \|P\|_2.$$

In both cases $\|R\|_p / \|P\|_p \geq (\|R\|_2 / \|P\|_2) / \sqrt{n+1}$. For the asymptotic statement, first apply Theorem 1 with a constant strictly between C_0 and C , and absorb $\frac{1}{2} \log(n+1) = o(\sqrt{n})$. □

6 Superalgebraic upper bounds

We prove the upper statement by cancelling a prescribed finite number of logarithmic-derivative terms at a strict global maximum. All constants in this section may depend on the prescribed cancellation order K . They are not asserted to be uniform as $K \rightarrow \infty$.

Lemma 10 (A finite positive moment cone). *Fix $K \geq 3$. For a rational point $\zeta = x + iy$, $x > 0$, $y \geq 0$, define the real column*

$$V(\zeta) = \begin{cases} (\zeta^2, \dots, \zeta^K)^T, & y = 0, \\ (2\operatorname{Re}\zeta^2, \dots, 2\operatorname{Re}\zeta^K)^T, & y > 0. \end{cases}$$

There is a finite collection of these columns whose cone is $\mathbb{R}^{K-1}$. The collection can contain any prescribed K distinct positive rational real points. Its column matrix A_∞ admits a strictly positive rational vector α^0 satisfying

$$A_\infty \alpha^0 = (0, 3/2, 0, \dots, 0)^T.$$

Proof. All columns are real vectors in $\mathbb{R}^{K-1}$, and a rational point has rational columns. Consider a real linear functional $a = (a_2, \dots, a_K)$, $a_j \in \mathbb{R}$ (a vector of the dual space, so its coordinates are real by construction), that is nonnegative on every column. Put $h(z) = \operatorname{Re} \sum_{j=2}^K a_j z^j = \sum_j a_j \operatorname{Re} z^j$. For $y > 0$, $a \cdot V(\zeta) = 2h(\zeta)$, and for $y = 0$, $a \cdot V(\zeta) = h(\zeta)$; so $h \geq 0$ at every rational point of the open right half-plane. Since h is continuous and $h(\bar{z}) = h(z)$, it follows that $h \geq 0$ on $\operatorname{Re} z > 0$. Suppose $a \neq 0$ and let $l \geq 2$ be the largest index with $a_l \neq 0$. Because $l \geq 2$, the numbers $\cos(l\theta)$, $\theta \in (-\pi/2, \pi/2)$, take both signs, so we may choose θ in this interval with $a_l \cos(l\theta) < 0$. Then $h(te^{i\theta}) = a_l t^l \cos(l\theta) + O(t^{l-1}) < 0$ for large $t > 0$, a contradiction. Hence the dual cone is $\{0\}$, and the closure of the cone generated by all columns is $\mathbb{R}^{K-1}$.

For each coordinate vector $\pm e_i$ choose a finite conical combination $p_i^\pm$ of columns within distance δ of it. For small δ the $2(K-1)$ vectors $p_i^\pm$ positively span $\mathbb{R}^{K-1}$ (a small perturbation of the positive spanning set $\{\pm e_i\}$ keeps 0 in the interior of the convex hull). The finitely many columns used already generate the whole space; append the prescribed real points if necessary. Call the resulting collection $s_1, \dots, s_N$; the matrix $A_\infty = (s_1 | \dots | s_N)$ is rational with full row rank.

For each i the vector $-s_i$ is a nonnegative combination of the columns, and the set of such coefficient vectors is a nonempty rational polyhedron, so it contains a rational point. Adding the relations $s_i + \sum_j c_{ij} s_j = 0$ over all i gives a rational kernel vector $\kappa \geq 1$ (every coordinate is at least 1). Finally, since the cone is the whole space, the target $(0, 3/2, 0, \dots, 0)^T$ equals $A_\infty \alpha'$ for a rational $\alpha' \geq 0$; the same polyhedron argument gives rationality. Then $\alpha^0 = \alpha' + \Theta \kappa$, for a large rational Θ , is strictly positive and rational, and $A_\infty \alpha^0 = (0, 3/2, 0, \dots, 0)^T$. $\square$

6.1 A fixed number of interior zeros

For a real reciprocal variable $v > 1$, put $\beta = 1 - 1/v$ and $f_v(z) = 1 - z/\beta$. For $v = x + iy$, $y > 0$, use the conjugate-pair factor $f_v(z) = (1 - z/\beta)(1 - \bar{z}/\bar{\beta})$. Write $\nu_v = 1, 2$, respectively, and define

$$M_j(v) = \begin{cases} v^j, & v \text{ real}, \\ 2\operatorname{Re} v^j, & \Im v > 0, \end{cases} \quad d(v) = \begin{cases} -v^2/(v-1), & v \text{ real}, \\ 2\operatorname{Re}[-v^2/(v-1)], & \Im v > 0. \end{cases}$$

Thus $d(v) = f'_v(0) - f'_v(1)/f_v(1)$, and $\operatorname{Re} v > 1/2$ is equivalent to $|\beta| < 1$. Use the circle-root baseline

$$B(z) = (1+z)^4(1-z+z^2), \quad b_j = \sum_{\beta: B(\beta)=0} (1-\beta)^{-j}.$$

It has degree six, constant coefficient one, $B'(0) = B'(1)/B(1) = 3$, $b_2 = 0$, and $b_3 = -3/2$.

Proof of Theorem 2. Fix $K \geq 3$. Apply Lemma 10, with K designated distinct positive real nodes among the ζ_i . Clear the denominators of its strictly positive weights by an integer Q , and put $m_i = Q\alpha_i^0$. These are positive integer group multiplicities, fixed independently of n . Set

$$M = \sum_i \nu_i m_i, \quad S = \sum_i m_i M_1(\zeta_i) > 0.$$

The moments of orders $2, \dots, K$ are now $(0, 3Q/2, 0, \dots, 0)$. For every large integer n , define

$$R = (n/(6Q))^{1/3}, \quad \epsilon = R^{-1}, \quad c_n \equiv n - M \pmod{6}, \quad |c_n - 2SR| \leq 3, \quad b_n = (n - M - c_n)/6.$$

Choose a nearest integer in the specified congruence class; both b_n, c_n are positive eventually. Then $t_n = c_n/R = 2S + O(\epsilon)$ and $u_n = b_n/R^3 = Q + O(\epsilon^2)$.

Let ξ_i vary near ζ_i , varying only the K designated real nodes. Define $d_\epsilon(\xi) = -\xi^2/(\xi - \epsilon)$ for a real node, and twice its real part for a pair. Consider the K equations

$$0 = t/2 + \sum_i m_i d_\epsilon(\xi_i), \tag{13}$$

$$0 = t\epsilon/4 + \sum_i m_i M_2(\xi_i), \tag{14}$$

$$0 = b_j u \epsilon^{j-3} + t 2^{-j} \epsilon^{j-1} + \sum_i m_i M_j(\xi_i), \quad 3 \leq j \leq K. \tag{15}$$

They vanish at $(\epsilon, t, u, \xi) = (0, 2S, Q, \zeta)$. The Jacobian in the designated real positions has columns

$$m_i(-1, 2\zeta_i, 3\zeta_i^2, \dots, K\zeta_i^{K-1})^T.$$

It equals $\text{diag}(-1, 2, 3, \dots, K) [\xi_i^{j-1}]_{j,i} \text{diag}(m_i)$, a Vandermonde matrix of distinct nodes with positive scalings, hence nonsingular. The left sides of (13)–(15) are polynomial in (ϵ, t, u, ξ) except for the terms d_ϵ , which are real-analytic for $\xi_i > 0$ and $|\epsilon|$ small; so they are C^1 near the base point. The real implicit-function theorem there gives a C^1 solution $\xi(\epsilon, t, u)$ with $\xi(0, 2S, Q) = \zeta$. Since $|t_n - 2S| \leq 3\epsilon$ and $u_n - Q = -Q(M + c_n)/n = O(\epsilon^2)$, we obtain $\xi_{i,n} = \xi(\epsilon, t_n, u_n)_i = \zeta_i + O(\epsilon)$. The size of the neighborhood is not quantified; this is the only non-effective step apart from Lemma 10. Put $v_{i,n} = R\xi_{i,n}$ and

$$P_n(z) = B(z)^{b_n} (1+z)^{c_n} \prod_i f_{v_{i,n}}(z)^{m_i}.$$

For large n , its degree is exactly n , its constant coefficient is one, and its coefficients are real. Exactly M zeros, counted with multiplicity, lie strictly inside the disk; the other $n - M$ lie on the circle. The interior zeros satisfy $|1 - \beta_{i,n}| \asymp 1 - |\beta_{i,n}| \asymp R^{-1}$. Multiplying (13)–(15) by $R, R^2, \dots, R^K$, respectively, gives the exact identities

$$\lambda_n := P'_n(0) = P'_n(1)/P_n(1), \quad \sum_{\beta: P_n(\beta)=0} (1-\beta)^{-j} = 0 \quad (2 \leq j \leq K).$$

In particular integer rounding has been corrected, and $\lambda_n = 3b_n + c_n/2 + R \sum_i m_i M_1(\xi_{i,n}) = O(n)$.

We now verify the hypotheses of the master estimate, Proposition A.5 of Appendix A, with this K and $P = P_n$. Below $C, c > 0$ depend only on K and on the fixed data (ζ_i, m_i, Q) , never on n . Write $v_j = (1 - \beta_j)^{-1}$ for the reciprocals of the zeros of P_n .

(a) *Slope and moment identities.* These are the exact identities displayed above: $P'_n(0) = \sum_j v_j = \lambda_n$ and $\sum_j v_j^k = 0$ for $2 \leq k \leq K$. The polynomial has real coefficients, exact degree n , $P_n(0) = 1$, and no zero at 1.

(b) *Sizes of the reciprocals.* The $n - M$ circle zeros have $\text{Re } v = \frac{1}{2}$; for the baseline and for -1 one finds $|v| \leq 1$ (the values are $1/2$ and $e^{\pm i\pi/3}$). The interior reciprocals are $v_{i,n} = R\xi_{i,n}$ with $\xi_{i,n} \rightarrow \zeta_i$, so $c_1 R \leq |v_{i,n}| \leq c_2 R$. Consequently $V := \max_j |v_j| \leq c_2 R$, $\Sigma_2 := \sum_j |v_j|^2 \leq n + M c_2^2 R^2 \leq 2n$, and $\Sigma_{K+1} := \sum_j |v_j|^{K+1} \leq n + M c_2^{K+1} R^{K+1} \leq C R^{K+1}$, because $K \geq 3$ gives $R^{K+1} \gtrsim n$.

(c) *Group bounds.* Take the groups B (multiplicity b_n), $1+z$ (multiplicity c_n) and the interior groups of Lemma A.2 (multiplicities m_i). By Lemma A.1(c), $\rho_B^2 \leq e^{-3s}$; and $\rho_{1+z}^2 = 1 - s \leq e^{-s}$.

For B one has $L_B \leq 3$, because the coefficients a_k of B satisfy $\sum_k k|a_k| = 48$ and $B(1) = 16$; for $1+z$, $L = 1/2$. By Lemma A.2 the interior groups have $\tau_i \leq 20|v_{i,n}|^2 \leq CR^2$ and $L_i \leq 6|v_{i,n}|^2 \leq CR^2$. Hence

$$T = -3b_n - c_n + \sum_i \tau_i \mu_i \leq -\frac{1}{2}(n - M) + CR^2 \leq -\frac{n}{3}, \quad \Lambda \leq 3b_n e^{3/2} + c_n + CR^2 \leq Cn,$$

for large n . Here $6b_n + c_n = n - M$ by the definition of b_n , so $3b_n + c_n = \frac{1}{2}(n - M + c_n) \geq \frac{1}{2}(n - M)$; and $R^2 \asymp n^{2/3}$. Also $\lambda_n = O(n)$.

(d) *Conclusion.* Here $\tau_0 = 1/(2V) \geq c/R$, which is much larger than $\Sigma_2^{-1/2} \geq (2n)^{-1/2}$; so $\tau_1 = \Sigma_2^{-1/2} \geq (2n)^{-1/2}$ for large n . Proposition A.5 gives

$$\rho(P_n)^2 \leq C\sqrt{n} \left[R^{2K+2} n^{-K-1/2} + n^2 e^{-cn/R^2} \right] \leq C_K R^{2K+2} n^{-K} \leq C_K n^{(2-K)/3},$$

since $n/R^2 \asymp n^{1/3}$ makes the second term smaller than any power of n . Hence $\rho(P_n) \leq C_K n^{(2-K)/6}$, and $r_n \leq \rho(P_n)$ because $P_n(0) = 1$.

Choose $K \geq 6A + 2$. Then $M_A = M$ is independent of n , as required. The proof does not give effective bounds on M_A , C_A , or the starting degree as functions of A . $\square$

Theorem 11 (One interior zero suffices). *For every sufficiently large integer n , there is a real degree- n polynomial P_n , with $P_n(0) = 1$, exactly one zero strictly inside the unit disk and all remaining zeros on the circle, such that $\rho(P_n) = O(n^{-1/6})$. The interior zero is real and positive, with distance and radial gap from 1 of order $n^{-1/3}$. In conjunction with Theorem 3, the minimum number of interior zeros permitting a sequence with vanishing residual is exactly one. No optimal rate for one interior zero is asserted.*

Proof. We use two adjustable circle angles in the degree-34 baseline

$$B_{a,c}(z) = (1 - 2az + z^2)^{15} (1 - 2cz + z^2)^2, \quad -1 < a, c < 1.$$

Its slope difference and second reciprocal moment are, respectively,

$$D(a, c) = -30a - 4c - 17, \quad T(a, c) = -\frac{15a}{1-a} - \frac{2c}{1-c}.$$

Both vanish at $(a_*, c_*) = (-2/3, 3/4)$. Write $n - 1 = 34m + k$, $0 \leq k \leq 33$, and let $v = v_n > 3/2$ be the unique solution of

$$v^3 - \frac{3}{2}v^2 = (n - 1)/4. \quad (16)$$

The left side is strictly increasing on this interval, and $v \sim (n/4)^{1/3}$. Set $\beta = 1 - 1/v \in (0, 1)$. For large n , $m > 0$. Define the real targets

$$d_n = \frac{v^2/(v-1) - k/2}{m} = O(n^{-2/3}), \quad t_n = \frac{-v^2 - k/4}{m} = O(n^{-1/3}).$$

The following explicit formulas select circle parameters solving $D(a_n, c_n) = d_n$, $T(a_n, c_n) = t_n$:

$$\begin{aligned} L &= (51 + d_n)/2, & U &= 17 - t_n, \\ X &= \frac{UL + 221 + \sqrt{(UL - 289)(UL - 169)}}{30U}, \\ Y &= (L - 15X)/2, & a_n &= 1 - X, & c_n &= 1 - Y. \end{aligned}$$

Indeed these equations are equivalent to $15X + 2Y = L$, $15/X + 2/Y = U$; elimination gives $15UX^2 - (UL + 221)X + 15L = 0$. The positive-square-root branch above tends to $X = 5/3, Y = 1/4$, since $UL \rightarrow 867/2$ and the square root tends to $391/2$. *Uniformity over the 34 classes.* The

map $(d, t) \mapsto (X, Y)$ is real-analytic near $(0, 0)$: at that point $U = 17$, $L = 51/2$, $UL = 867/2$, and $(UL - 289)(UL - 169) = \frac{289}{2} \cdot \frac{529}{2} > 0$ (the square root is $391/2$), while $U > 0$. Hence there are δ_0, C_1 with $|X - 5/3| + |Y - 1/4| \leq C_1(|d| + |t|)$ whenever $|d| + |t| \leq \delta_0$; δ_0, C_1 do not involve k . Since $0 \leq k \leq 33$, $m \geq (n - 34)/34$ and $v^2/(v - 1) \leq 2v$ for $v \geq 2$, we get $|d_n| \leq C(v + 1)/n$ and $|t_n| \leq C(v^2 + 1)/n$, uniformly in k ; with $v \asymp n^{1/3}$ both are $O(n^{-1/3})$, indeed $O(n^{-2/3})$ and $O(n^{-1/3})$. So for n beyond a threshold independent of k the discriminant and denominators are positive, $a_n, c_n \in (-1, 1)$, and $|a_n + 2/3| + |c_n - 3/4| \leq Cn^{-1/3}$. The “34 residue classes” are therefore a finite family of sequences governed by one analytic map, and every “eventually” below is uniform over them.

Put

$$P_n(z) = B_{a_n, c_n}(z)^m (1 + z)^k (1 - z/\beta).$$

It has exactly the claimed degree, constant term and zero locations. The two target equations give the exact identities

$$P'_n(0) = P'_n(1)/P_n(1) =: \lambda_n, \quad \sum_{j=1}^n v_j^2 = 0, \quad v_j = (1 - \beta_j)^{-1}.$$

Here the extra roots at -1 contribute $k/2$ to the slope difference and $k/4$ to the second moment; the interior root contributes $-v^2/(v - 1)$ and v^2 , respectively. For a conjugation-symmetric multiset of N circle roots avoiding 1, its reciprocal moments satisfy

$$M_3^o = \frac{3}{2}M_2^o - N/4. \quad (17)$$

This follows for each pair $1/2 \pm iy$, and also for the real reciprocal $1/2$. Conjugation symmetry is essential and holds here. Taking $N = n - 1$ and $M_2^o = -v^2$, (16) gives $\sum_j v_j^3 = 0$ exactly. Moreover $\lambda_n = (n - 1)/2 + v = O(n)$.

We now verify the hypotheses of Proposition A.5 of Appendix A with $K = 3$ and $P = P_n$. All constants below are absolute. Let $v_j = (1 - \beta_j)^{-1}$ denote the reciprocals of the zeros of P_n .

(a) *Identities.* P_n has real coefficients, exact degree n , $P_n(0) = 1$, no zero at 1, and (shown above) $P'_n(0) = \sum_j v_j = \lambda_n$, $\sum_j v_j^2 = 0 = \sum_j v_j^3$.

(b) *Sizes.* A circle zero $e^{\pm i\varphi} \neq 1$ has $v = \frac{1}{2} \pm \frac{i}{2} \cot(\varphi/2)$. For the baseline, $\cos \varphi = a_n \rightarrow -2/3$ and $\cos \varphi = c_n \rightarrow 3/4$, so $|v| \leq 2$ eventually; the zeros at -1 have $v = \frac{1}{2}$, and the interior zero has $v_n = v > 3/2$ real. Hence $V = v$ eventually, $\Sigma_2 \leq 4n + v^2 \leq 5n$, and $\Sigma_4 \leq 16n + v^4 \leq Cn^{4/3}$, since $v^3 - \frac{3}{2}v^2 = (n - 1)/4$ gives $v \asymp n^{1/3}$.

(c) *Group bounds.* The groups are B_{a_n, c_n} (multiplicity m), $1 + z$ (multiplicity k) and $f = 1 - z/\beta$ (multiplicity 1). By Lemma A.1(e), for large n the baseline has $\rho^2 \leq e^{-6s}$, i.e. $\tau = -6$; for $1 + z$, $\tau = -1$ and $L = 1/2$; for f Lemma A.2 gives $\tau = 4v(v - 1)$ and $L = v$. For the baseline, writing $g_a = 1 - 2az + z^2$, $g_c = 1 - 2cz + z^2$, $B' = 15g'_a g_a^{14} g_c^2 + 2g'_c g_a^{15} g_c$, and on the circle $|g'_a| \leq 4$, $|g'_c| \leq 4$, $g_a(1) = 2(1 - a) \geq 3$, $g_c(1) = 2(1 - c) \geq 0.4$, $|g_a/g_a(1)| = |1 - \alpha s| \leq 1$, $|g_c/g_c(1)| = |1 - \gamma s| \leq 9$ (with $\alpha = 2/(1 - a) \leq 2$, $\gamma = 2/(1 - c) \leq 9$). Hence $L_B \leq 15 \cdot \frac{4}{3} \cdot 81 + 2 \cdot 10 \cdot 9 < 2000$. Therefore

$$T = -6m - k + 4v(v - 1) \leq -\frac{6}{34}(n - 34) + 4v^2 \leq -\frac{n}{7}, \quad \Lambda \leq 2000 e^3 m + k + v \leq Cn,$$

for large n , and $\lambda_n = (n - 1)/2 + v = O(n)$.

(d) *Conclusion.* Here $\tau_0 = 1/(2v) \gg \Sigma_2^{-1/2}$ and $\tau_1 = \Sigma_2^{-1/2} \geq (5n)^{-1/2}$. Proposition A.5 with $K = 3$ gives

$$\rho(P_n)^2 \leq C\sqrt{n} \left[n^{8/3} n^{-7/2} + n^2 e^{-n\tau_0^2/(7\pi^2)} \right],$$

and $n\tau_0^2 = n/(4v^2) \asymp n^{1/3}$. The second term is smaller than every power of n , and the first is $O(n^{-1/3})$. Hence $\rho(P_n)^2 = O(n^{-1/3})$, as required.

Finally, Theorem 3 excludes vanishing residuals with zero interior roots; this construction permits them with one. This proves the zero-count threshold, not a sharp rate or the superalgebraic conclusion with one root. No numerical starting degree or norm constant is claimed. $\square$

7 A fixed radial gap forces a polynomial barrier

The upper construction uses the full closed disk. A fixed strict root radius gives a complementary obstruction, by the same local estimate and bounded test as the lower-bound proof.

Proposition 12. *For $0 \leq q < 1$, let $s_n(q)$ denote the infimum defining s_n with all zeros restricted to $|z| \leq q$. Set*

$$a = 1 - q^2, \quad K = \lceil 4/a \rceil, \quad \Delta = \frac{1}{1+q} - \frac{1+q^K}{Ka} > 0, \quad c_q = \min \left\{ \frac{1}{\sqrt{2}}, \frac{\Delta Ka}{2\sqrt{2}e^3(1+a)^K} \right\}.$$

Then $s_n(q) \geq c_q n^{3/2-K}$ for every $n \geq 41$.

Proof. Write ρ for an admissible residual. If $\rho > 1/\sqrt{n+1}$, the conclusion is immediate. Otherwise the maximum normalization gives $|\mu| \leq n+1$. Take $r = 1/n$. The smallness parameter in the proof of Lemma 5 is at most $e^3/n < 1/2$. That local proof does not need the separate zero-separation parameter η , and gives

$$|m_j| \leq \frac{2\sqrt{n+1}}{n} \rho e^3 r^{1-j} \quad (j \geq 2).$$

The exact image of $|\beta| \leq q$ under $v = (1-\beta)^{-1}$ is $|v - 1/a| \leq q/a$. Thus $|1 - av| \leq q$ and $\operatorname{Re} v \geq 1/(1+q)$. Use $\Psi(v) = [1 - (1 - av)^K]/(Ka)$, whose modulus is at most $(1 + q^K)/(Ka)$. Estimating its higher averaged terms gives

$$\Delta \leq \frac{2e^3\sqrt{n+1}\rho}{Kan^2} [(1+an)^K - 1 - Kan] \leq \frac{2e^3\sqrt{n+1}\rho}{Kan^2} (1+an)^K.$$

Now $\sqrt{n+1} \leq \sqrt{2n}$ and $1+an \leq (1+a)n$. The resulting lower bound is the stated one. Finally $Ka \geq 4$, $q^K < 1$, and $1/(1+q) > 1/2$ ensure $\Delta > 0$. $\square$

In particular, any single sequence of admissible degree- n polynomials with residual $o(n^{-A})$ for every fixed $A > 0$ must have its maximum root modulus tending to one. Otherwise an infinite subsequence would satisfy a fixed polynomial lower bound. No sharp radial exponent is asserted.

8 Scope and verification

The uniform slope formulation bounds approximate eigenvectors of polynomial differentiation under a zero-location constraint. It does not describe the full unconstrained pseudospectrum. For example, for $u = n/4$ and $T_n(z) = \sum_{k=0}^n (uz)^k/k!$,

$$T'_n - uT_n = -\frac{u^{n+1}}{n!} z^n, \quad \frac{\|T'_n - uT_n\|_2}{\|T_n\|_2} \leq \frac{n}{4} \left(\frac{e}{4}\right)^n,$$

using $\|T_n\|_2 \geq 1$ and $n! \geq (n/e)^n$. Thus a root constraint is essential for a bound of our form.

The lower and upper disk bounds do not match in order. Whether the restricted exterior power-sum minima stay bounded away from zero remains unresolved here, as does the optimal order of s_n and r_n . The superalgebraic upper bound is not a stretched-exponential upper bound, and its constants are not optimized. The minimal interior-root count permitting collapse is one, but the optimal rate with one such root is not established. We do not prove superalgebraic decay with one interior root. The elementary radial barrier is a consequence of the same local method. We do not assert a theorem for arbitrary starting exponents, weighted power sums, or H^p with $p < 1$. The elementary norm comparison in Corollary 9 is a consequence of the main estimate.

The analytic proof of the explicit threshold uses only the displayed elementary inequalities. It does not rely on optimization or sampling. The supplementary diagnostics check exact projected Newton identities over Gaussian rationals, the bounded test on admissible reciprocal zeros, binomial identities, and numerical instances of the finite certificate. Rational checks of constants and high-precision evaluations are included for reproducibility. The circle-root family is checked with complete exact coefficient Parseval ratios. The one-interior-root family has symbolic and high-precision checks of the cubic, slope, moments and circle parameters, without quadrature. The independent audit covers all 34 degree classes at two scales. These finite checks do not certify the full norm asymptotics. Symbolic identities support the constructions. Finite moment searches motivated the general construction but are not premises of its arbitrary-order proof. These checks do not establish historical priority or replace the proof.

Version 2. Appendix A writes out the uniform estimates that version 1 stated without proof: the secondary maximum and the Gaussian gaps of the baseline factors, the interior-factor inequality, the local series bound, the lower bound for the denominator, the outer-arc bound by the polynomial product rule, and the resulting master estimate (Proposition A.5) with explicit constants. Lemma 10 now makes explicit that its linear functionals are real. What remains non-effective or unproved: the implicit-function neighborhood in Theorem 2 and the number ε_0 in Lemma A.1(e) are existence statements; M_A , C_A and N_A are still not computed; the constants of Proposition A.5 are far from optimal and are non-trivial (a bound below 1 for the one-interior-zero family) only for $n \gtrsim 10^{21}$; the odd-degree comparison in Section 2 and the external Mathieu expansion are used as before. The additional numerical checks of version 2 (exact identities in 50-digit arithmetic, and quadrature of the residual for the one-interior-zero family up to $n = 10^{12}$, where $\rho^2 n^{1/3}$ stays between 0.37 and 0.46; and, for Theorem 2 with $K = 3$, an explicit rational cone solution with $M = 114$ interior zeros, for which $\rho^2 n^{1/3} \approx 771$ for $10^9 \leq n \leq 10^{21}$) illustrate the statements; they are not part of any proof.

The comparison uses the exact primary-source versions listed below. A targeted search did not identify the bounds in this exact setting; exhaustive priority certification is not claimed. The accessible Komarov versions listed below were used for the stated comparison. A related 2020 journal text was not available in full in this review; no exclusion of every possible bibliographic overlap is claimed. AI tools assisted exploration, drafting and computational checking. Model review is distinct from independent human peer review and formal proof verification.

A Uniform estimates behind Theorems 2 and 11

This appendix is new in version 2. Version 1 stated the estimates below in a line or two; here each is derived. Throughout, $z = e^{i\theta}$, $|\theta| \leq \pi$, $s = \sin^2(\theta/2)$, so that $|z - 1|^2 = 4s$, $|z - 1| \leq |\theta|$, and $s \geq \theta^2/\pi^2$. For a polynomial g with $g(1) \neq 0$ put

$$\rho_g(\theta) = \left| \frac{g(e^{i\theta})}{g(1)} \right|, \quad L_g = \sup_{\theta} \frac{|g'(e^{i\theta})|}{|g(1)|}.$$

For a zero $\beta \neq 1$ put $v = (1 - \beta)^{-1}$; then $z - \beta = (1 + v(z - 1))/v$ and, for $g(z) = 1 - z/\beta$, $\rho_g(\theta) = |1 + v(z - 1)|$.

Lemma A.1 (Baseline moduli). *For $0 \leq c < 1$ let $H_c(s) = (1 - s)^4(1 - \frac{2s}{1-c})^2$, $0 \leq s \leq 1$.*

(a) $H_c(s) = \rho_g(\theta)^2$ for $g = (1 + z)^4(1 - 2cz + z^2)$. It decreases on $[0, (1 - c)/2]$, vanishes at $(1 - c)/2$ and at 1, and has exactly one critical point $s^* = (2 - c)/3$ in $((1 - c)/2, 1)$, a maximum, with value $(1 + c)^6/[729(1 - c)^2]$.

(b) For $0 \leq c \leq 3/4$ this value is at most $117649/186624 < 0.631$.

(c) $H_{1/2}(s) \leq e^{-3s}$ on $[0, 1]$. In particular $\rho_B^2 \leq e^{-3s}$ for $B = (1 + z)^4(1 - z + z^2)$.

(d) If $|c - 1/2| \leq 1/10$, then $H_c(s) \leq e^{-2s}$ on $[0, 1]$.

(e) There is $\varepsilon_0 > 0$ such that, for $|a + 2/3| \leq \varepsilon_0$, $|c - 3/4| \leq \varepsilon_0$, the polynomial $B_{a,c} = (1 - 2az + z^2)^{15}(1 - 2cz + z^2)^2$ satisfies $\rho_{B_{a,c}}^2 = (1 - \alpha s)^{30}(1 - \gamma s)^4 \leq e^{-6s}$, where $\alpha = 2/(1 - a)$, $\gamma = 2/(1 - c)$.

Proof. (a) On the circle $|1 + z|^2 = 4(1 - s)$, and since $\cos \theta = 1 - 2s$, $|1 - 2cz + z^2| = |z + z^{-1} - 2c| = 2|1 - c - 2s|$. Dividing by the values at $z = 1$ gives H_c . Put $w = 2/(1 - c)$; then $(\log H_c)' = -4/(1 - s) - 2w/(1 - ws)$. On $[0, 1/w)$ both terms are negative, so H_c decreases from 1 to its zero at $1/w = (1 - c)/2$. On $(1/w, 1)$ the equation $(\log H_c)' = 0$ reads $-4(1 - ws) - 2w(1 - s) = 0$, i.e. $6ws = 4 + 2w$, whose only solution is $s^* = (2 + w)/(3w) = (2 - c)/3$, which lies in $(1/w, 1)$. As H_c is positive there and vanishes at both ends, s^* is its maximum. There $1 - s^* = (1 + c)/3$ and $1 - ws^* = -(1 + c)/(3(1 - c))$, so $H_c(s^*) = ((1 + c)/3)^4(1 + c)^2/(9(1 - c)^2)$, as claimed.

(b) $(1 + c)^6/(1 - c)^2$ increases on $[0, 1]$; at $c = 3/4$ it equals $(7/4)^6 \cdot 16$, so the value is $7^6 \cdot 16/(4^6 \cdot 729) = 117649/186624$.

(c) Here $s^* = 1/2$ and the secondary maximum is $1/16$. On $[0, 1/4]$: $0 \leq 1 - 4s \leq 1$, so $H_{1/2} \leq e^{-4s}e^{-8s} \leq e^{-3s}$. On $[1/4, 7/10]$: $H_{1/2} \leq 1/16 < e^{-21/10} \leq e^{-3s}$, since $e^{2.1} < 8.2 < 16$. On $[7/10, 1]$, $H_{1/2}$ is decreasing (as $s > s^*$), so $H_{1/2} \leq H_{1/2}(7/10) = (3/10)^4(9/5)^2 < 0.0263 < e^{-3} \leq e^{-3s}$.

(d) Now $w \in [10/3, 5]$. On $[0, 1/w]$, $0 \leq 1 - ws \leq e^{-ws}$ and $1 - s \leq e^{-s}$, so $H_c \leq e^{-(4+2w)s} \leq e^{-2s}$. On $[1/w, 7/10]$, by (a) and the monotonicity in (b), $H_c \leq (1.6)^6/(729 \cdot 0.4^2) < 0.144 < e^{-7/5} \leq e^{-2s}$. On $[7/10, 1]$ we have $s > s^*$ (as $s^* \leq 0.54$), hence $H_c \leq H_c(7/10) = (0.3)^4(0.7w - 1)^2 \leq 0.0081 \cdot 6.25 < 0.0507 < e^{-2}$.

(e) Let $\varepsilon_0 \leq 1/50$. Then $\alpha \in [1.18, 1.22]$ and $\gamma \in [7.4, 8.7]$. On $[0, 1/10]$ both $1 - \alpha s$ and $1 - \gamma s$ are positive (note $1/\gamma > 0.11$), and $\log H$ is a sum of concave functions, hence lies below its tangent at 0: $\log H \leq -(30\alpha + 4\gamma)s \leq -65s$. On $[1/10, 1]$, for the center $(a, c) = (-2/3, 3/4)$ we have $H_*(s) = (1 - 6s/5)^{30}(1 - 8s)^4$. It decreases on $[1/10, 1/8]$, has the single critical point $5/24$ on $(1/8, 5/6)$ (from $-36/(1 - 6s/5) = 32/(1 - 8s)$), and increases on $[5/6, 1]$. Hence

$$\sup_{[1/10, 1]} H_* = \max\{H_*(\frac{1}{10}), H_*(\frac{5}{24}), H_*(1)\} = \max\{3.46 \cdot 10^{-5}, 3.53 \cdot 10^{-5}, 2.6 \cdot 10^{-18}\} < 4 \cdot 10^{-5}.$$

Since $(a, c, s) \mapsto H_{a,c}(s)$ is continuous on a compact set, there is $\varepsilon_0 \in (0, 1/50]$ with $H_{a,c} \leq 10^{-3} < e^{-6} \leq e^{-6s}$ on $[1/10, 1]$ for the nearby parameters. Together with the bound $e^{-65s} \leq e^{-6s}$ on $[0, 1/10]$ this proves (e). (A grid evaluation gives $\sup H_{a,c} < 1.1 \cdot 10^{-4}$ on $[1/10, 1]$ for $\varepsilon_0 = 1/50$; only the existence of ε_0 is used.) $\square$

Lemma A.2 (Interior factors). *Let $v = x + iy$ with $x > 1/2$, $\beta = 1 - 1/v$ (so $|\beta| < 1$).*

(i) *If $y = 0$ and $g = 1 - z/\beta$, then $\rho_g^2 = 1 + 4x(x - 1)s \leq e^{\tau s}$ with $\tau = 4x(x - 1)$, and $L_g = |v|$.*

(ii) *If $y > 0$ and $g = (1 - z/\beta)(1 - z/\bar{\beta})$, put $Y = y^2$, $U = |v|^2$, $V' = |v - 1|^2$. Then*

$$\rho_g^2 = 1 + 8(x^2 - x - Y)s + 16UV's^2 \leq e^{\tau s}, \quad \tau = 4(U + V'),$$

and for $|v| \geq 1$: $\tau \leq 20|v|^2$ and $L_g \leq 6|v|^2$.

Proof. We have $\rho = |1 + v(z - 1)|$ for each factor. Since $\operatorname{Re}(z - 1) = -2s$ and $|z - 1|^2 = 4s$, $|1 + v(z - 1)|^2 = 1 + 2\operatorname{Re}(v(z - 1)) + |v|^2|z - 1|^2 = 1 - 4xs - 2y\sin\theta + 4|v|^2s$. For $y = 0$ this is $1 + 4x(x - 1)s$; and $1 + u \leq e^u$ gives (i). Also $g' = -1/\beta$ and $g(1) = -1/(v\beta)$, so $L_g = |v|$.

For (ii), the conjugate factor gives the same expression with $\sin\theta$ replaced by $-\sin\theta$, so with $\sin^2\theta = 4s(1 - s)$,

$$\rho_g^2 = (1 + 4(U - x)s)^2 - 4Y\sin^2\theta = 1 + 8(U - x - 2Y)s + 16[(U - x)^2 + Y]s^2.$$

Here $U - x - 2Y = x^2 - x - Y$, and $(U - x)^2 + Y = UV'$ because $V' = U - 2x + 1$. Moreover $UV' - (x^2 - x - Y)^2 = (2x - 1)^2 Y \geq 0$: with $A = x^2 - x$ one has $UV' = (A + Y)^2 + Y$, and $(A + Y)^2 - (A - Y)^2 = 4AY$, so the difference is $Y + 4AY = (2x - 1)^2 Y$. Thus $x^2 - x - Y \leq \sqrt{UV'}$ and $\rho_g^2 \leq (1 + 4\sqrt{UV's})^2 \leq e^{8\sqrt{UV's}} \leq e^{4(U+V')s}$, using $\sqrt{UV'} \leq (U + V')/2$. If $|v| \geq 1$ then $V' \leq (|v| + 1)^2 \leq 4|v|^2$, so $\tau \leq 20|v|^2$. Finally $g' = f'_v f'_v + f_v f'_v$ for the two factors, each factor has $|f'|/|f(1)| = |v|$ and $\rho_f \leq 1 + |v||z - 1| \leq 1 + 2|v| \leq 3|v|$; hence $L_g \leq 2|v| \cdot 3|v| = 6|v|^2$. $\square$

Lemma A.3 (Window estimates). *Let P be a polynomial of exact degree n with real coefficients, $P(1) \neq 0$, and zeros $\beta_j \in \mathbb{D} \setminus \{1\}$, with $v_j = (1 - \beta_j)^{-1}$. Put $\lambda = \sum_j v_j = P'(1)/P(1)$, $V = \max_j |v_j|$, $\Sigma_m = \sum_j |v_j|^m$, $\tau_0 = 1/(2V)$.*

(i) *If $\sum_j v_j^k = 0$ for $2 \leq k \leq K$, then P has no zero in $|z - 1| \leq \tau_0$ and there $|P'(z)/P(z) - \lambda| \leq 2|z - 1|^K \Sigma_{K+1}$.*

(ii) *For $|\theta| \leq \tau_0$, $\rho_P(\theta)^2 \geq \exp(-4\Sigma_2\theta^2)$. Consequently, with $\tau_1 = \min\{\tau_0, \Sigma_2^{-1/2}\}$, $\int_{-\pi}^{\pi} \rho_P^2 d\theta \geq 2e^{-4}\tau_1$.*

Proof. (i) Let $w = z - 1$, $|w| \leq \tau_0$; then $|v_j w| \leq 1/2$, so $z - \beta_j = (1 + v_j w)/v_j \neq 0$ and $P'/P = \sum_j 1/(z - \beta_j) = \sum_j v_j/(1 + v_j w)$. The identity

$$\frac{v}{1 + vw} = \sum_{k=0}^{K-1} (-1)^k v^{k+1} w^k + \frac{(-1)^K v^{K+1} w^K}{1 + vw}$$

(multiply by $1 + vw$ and telescope) summed over j gives $P'/P = \lambda + \sum_{k=1}^{K-1} (-1)^k w^k \sum_j v_j^{k+1} + (-1)^K w^K \sum_j v_j^{K+1}/(1 + v_j w)$. The middle sums vanish by hypothesis, and $|1 + v_j w| \geq 1/2$.

(ii) Let $f(\theta) = \log \rho_P^2 = \sum_j [\log |z - \beta_j|^2 - \log |1 - \beta_j|^2]$. Then $f(0) = 0$, and $f'(0) = 0$ because the coefficients are real, so $\rho_P(\theta) = \rho_P(-\theta)$. A direct computation gives $\frac{d^2}{d\theta^2} \log |z - \beta|^2 = 2 \operatorname{Re}[z\beta/(z - \beta)^2]$, whose modulus is at most $2|\beta|/|z - \beta|^2$. For $|\theta| \leq \tau_0$, $|z - \beta_j| \geq |1 - \beta_j| - |z - 1| \geq 1/|v_j| - 1/(2V) \geq 1/(2|v_j|)$, so $f'' \geq -8\Sigma_2$, and Taylor's formula gives $f(\theta) \geq -4\Sigma_2\theta^2$. On $|\theta| \leq \tau_1$ this is at least -4 , and the integral bound follows. $\square$

Lemma A.4 (Outer arc). *Let $P = \prod_{i=1}^r g_i^{\mu_i}$ with integers $\mu_i \geq 1$, $g_i(1) \neq 0$, $\rho_{g_i}^2 \leq e^{\tau_i s}$ for all θ (with real τ_i , of either sign) and $L_i = L_{g_i} < \infty$. Put $T = \sum_i \mu_i \tau_i$, $\tau_i^- = \max\{0, -\tau_i\}$ and $\Lambda = \sum_i \mu_i L_i e^{\tau_i^-/2}$. Then for all θ*

$$\rho_P \leq e^{sT/2}, \quad \frac{|P'(e^{i\theta})|}{|P(1)|} \leq \Lambda e^{sT/2}.$$

Proof. The first bound is the product of the factor bounds. For the second,

$$P' = \sum_i \mu_i g_i' g_i^{\mu_i - 1} \prod_{h \neq i} g_h^{\mu_h}$$

is a sum of products of polynomials, so no quotient by a zero of P occurs. Dividing the i -th term by $|P(1)|$, it is at most

$$\mu_i L_i \prod_h \rho_{g_h}^{\mu_h - \delta_{ih}} \leq \mu_i L_i e^{s(T - \tau_i)/2} \leq \mu_i L_i e^{\tau_i^-/2} e^{sT/2},$$

using $0 \leq s \leq 1$. $\square$

Proposition A.5 (Master estimate). *Let $K \geq 2$, and let P be as in Lemma A.3 with $P(0) = 1$. Assume*

(H1) $P'(0) = \lambda = \sum_j v_j$ and $\sum_j v_j^k = 0$ for $2 \leq k \leq K$;

(H2) $P = \prod_i g_i^{\mu_i}$ as in Lemma A.4, with $T < 0$.

With the notation of Lemmas A.3 and A.4,

$$\rho(P)^2 = \frac{\|P' - P'(0)P\|_2^2}{\|P\|_2^2} \leq \frac{e^4}{\tau_1} \left[2\Sigma_{K+1}^2 \Gamma(K + \tfrac{1}{2}) \left(\frac{\pi^2}{|T|} \right)^{K+1/2} + \pi(\Lambda + |\lambda|)^2 e^{-|T|\tau_0^2/\pi^2} \right].$$

Proof. Both norms carry the factor $1/2\pi$, which cancels; we integrate over $[-\pi, \pi]$ and divide by $|P(1)|^2$. On $|\theta| \leq \tau_0$, Lemma A.3(i) gives $|P' - \lambda P| = |P'/P - \lambda||P| \leq 2|\theta|^K \Sigma_{K+1}|P|$, and $|P|^2/|P(1)|^2 \leq e^{sT} \leq e^{-|T|\theta^2/\pi^2}$. Hence the local part is at most $4\Sigma_{K+1}^2 \int_{\mathbb{R}} \theta^{2K} e^{-|T|\theta^2/\pi^2} d\theta = 4\Sigma_{K+1}^2 \Gamma(K + \tfrac{1}{2}) (\pi^2/|T|)^{K+1/2}$. On $\tau_0 \leq |\theta| \leq \pi$, Lemma A.4 gives $|P' - \lambda P|/|P(1)| \leq (\Lambda + |\lambda|)e^{sT/2}$ and $sT \leq -|T|\tau_0^2/\pi^2$, so this part is at most $2\pi(\Lambda + |\lambda|)^2 e^{-|T|\tau_0^2/\pi^2}$. Divide by the lower bound $2e^{-4}\tau_1$ of Lemma A.3(ii). $\square$

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