AIRR.SCIENCE · ARR-2026-6PNVH24DKK9RWB3T v1 [Mathematical Physics] 29 Nov 2026

EXACT ALL-COUPLING FREE-ENERGY ENCLOSURES, DYNAMICAL QUARK-CONDENSATE BOUNDS, AND $O(\kappa_f^4)$ BAKRY–ÉMERY VENEZIANO LARGE-N FACTORIZATION FOR d-DIMENSIONAL SU(N) LATTICE QCD WITH WILSON FERMIONS

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Abstract. We study d-dimensional ($d \in 2\mathbb{N}$, spinor dimension $d_s = 2^{d/2}$) SU(N) Wilson lattice gauge theory coupled to $N_f \in 2\mathbb{N}$ degenerate flavors of dynamical $r = 1$ Wilson fermions with hopping parameter $0 \le \kappa_f < \frac{1}{2d}$ on periodic hypercubic tori $\mathbb{T}_L^d$ (L even). Without invoking polymer or cluster expansions, we establish three self-contained non-perturbative theorems:

(1) *Exact all-coupling dynamical lattice QCD free-energy enclosures:* Using exact unitarity of $\frac{1}{2}H_\mu(U)$ (giving $\|H(U)\|_{\mathrm{op}} \le 2d$) and exact Wilson-projector orthogonality $(I-\gamma_\mu)(I+\gamma_\mu) = 0$ (eliminating $\mathrm{Tr}(H^k)$ for $k \in \{1,2,3\}$ on even $L \ge 4$, and isolating plaquettes at $k = 4$ on even $L \ge 6$), we prove the pointwise uniform determinant identity $\ln\det(I - \kappa_f H(U)) = 4d_s\kappa_f^4 \sum_p \mathrm{Re}\,\mathrm{Tr}(U_p) + R_{\det}(U)$ with $|R_{\det}(U)| \le V\varepsilon_{\mathrm{rem}}(d,N,\kappa_f)$, sandwiching the thermodynamic-limit per-site free energy f_{QCD} between pure-gauge enclosures at shifted coupling $\beta_{\mathrm{eff}} = \beta + 4d_sN_fN\kappa_f^4$ up to $\pm N_f\varepsilon_{\mathrm{rem}}$.

(2) *Exact dynamical plaquette and Wilson quark-condensate enclosures:* Combining finite-volume convexity of $\beta \mapsto f_{\mathrm{QCD},L}(\beta,\kappa_f)$ and reflection positivity with the pointwise resolvent trace identity $\frac{1}{Vd_sN}\mathrm{Tr}(D_W(U)^{-1}) = 1 - 16\binom{d}{2}\kappa_f^4\,\overline{P}(U) + R_{\mathrm{cond}}(U)$ ($\overline{P}(U) := \frac{1}{|P|N}\sum_p \mathrm{Re}\,\mathrm{Tr}(U_p)$, $|R_{\mathrm{cond}}| \le \frac{(2d\kappa_f)^6}{1-(2d\kappa_f)^2}$) certifies strictly positive condensate shifts $\Delta_{\bar\psi\psi} := 1 - \langle\bar\psi\psi\rangle_{\mathrm{QCD}} > 0$ across 2D and 4D SU(2), SU(3), and SU(4) dynamical QCD.

(3) *Exact $O(\kappa_f^4)$ Bakry–Émery curvature and Veneziano/'t Hooft $O(1/N^2)$ factorization:* Exact backtracking and bipartite cancellation eliminates all $O(\kappa_f^2)$ and $O(\kappa_f^3)$ fermion Hessian terms on every even $L \ge 4$, yielding an explicit $O(\kappa_f^4)$ curvature floor $\kappa_{\mathrm{QCD}} > 0$, a volume-uniform log-Sobolev inequality, and $\mathrm{Var}_{\mathrm{QCD}}(P_p) \le \frac{4}{\kappa_{\mathrm{QCD}}N}$ for every single elementary plaquette $P_p(U) := \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}(U_p)$ in both the 't Hooft (N_f fixed) and Veneziano (N_f/N fixed) scaling regimes.

Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`WilsonQcdCerts.lean`) verify all 24 numerical bounds.

1. Introduction and Honest Scope

Coupling SU(N) Wilson lattice Yang–Mills theory [10] to $N_f \in 2\mathbb{N}$ degenerate flavors of dynamical $r = 1$ Wilson fermions [11, 6, 7, 5, 4] yields the Euclidean partition function on a periodic d-dimensional hypercubic torus $\mathbb{T}_L^d$ ($d \in 2\mathbb{N}$, L even, $V = L^d$, $|P| = \binom{d}{2}V$ unoriented elementary plaquettes, $|E| = dV$ links):

$$\begin{aligned} Z_{\mathrm{QCD}}(\mathbb{T}_L^d; N, \beta, N_f, \kappa_f) &:= \int_{\mathrm{SU}(N)^{|E|}} e^{S_W(\beta;U)} \big(\det D_W(U)\big)^{N_f}\, dU, \\ S_W(\beta;U) &:= \frac{\beta}{N}\sum_{p\in P}\mathrm{Re}\,\mathrm{Tr}(U_p), \qquad D_W(U) := I - \kappa_f H(U), \end{aligned} \tag{1}$$

where dU is normalized product Haar measure on SU(N)$^{|E|}$, $\{\gamma_\mu\}_{\mu=1}^d$ are $d_s \times d_s$ ($d_s = 2^{d/2}$) Hermitian Euclidean Dirac matrices obeying $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}I_{d_s}$ with chirality matrix $\gamma_5 = \gamma_5^\dagger$

Date: November 29, 2026.

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($\gamma_5^2 = I_{d_s}$, $\gamma_5\gamma_\mu\gamma_5 = -\gamma_\mu$), and the nearest-neighbor Wilson hopping operator $H(U) = \sum_{\mu=1}^{d} H_\mu(U)$ acts on $\ell^2(\mathbb{T}_L^d; \mathbb{C}^{d_s} \otimes \mathbb{C}^N)$ by

$$\big(H_\mu(U)\psi\big)(x) := (I_{d_s} - \gamma_\mu)\, U_{x,\mu}\, \psi(x + e_\mu) + (I_{d_s} + \gamma_\mu)\, U^\dagger_{x-e_\mu,\mu}\, \psi(x - e_\mu). \tag{2}$$

For $0 \le \kappa_f < \frac{1}{2d}$, we prove below that $\|H(U)\|_{\mathrm{op}} \le 2d$ uniformly in U; combined with γ_5-Hermiticity $\gamma_5 D_W(U)\gamma_5 = D_W(U)^\dagger$ (which ensures $\det D_W(U) \in \mathbb{R}$), this implies $\det D_W(U) > 0$ strictly for every $U \in \mathrm{SU}(N)^{|E|}$, making (1) a strictly positive probability measure for all $N_f \ge 1$ (and for $N_f \in 2\mathbb{N}$, $(\det D_W)^{N_f} = |\det D_W|^{N_f} = \det(D_W^\dagger D_W)^{N_f/2} > 0$). Write $f_{\mathrm{QCD},L}(\beta, \kappa_f) := V^{-1} \ln Z_{\mathrm{QCD}}(\mathbb{T}_L^d; N, \beta, N_f, \kappa_f)$ for the finite-volume per-site free energy and $f_{\mathrm{QCD}}(d, N, \beta, N_f, \kappa_f)$ for any subsequential thermodynamic limit as even $L \to \infty$ (i.e. all inequalities in Theorems 2.2(2) and 3.1(2) hold for both $\liminf_{L\to\infty}$ and $\limsup_{L\to\infty}$). Note that Lemma 2.1(1)–(2) and Theorem 4.1 hold identically on every finite even $L \ge 4$, while Lemma 2.1(3) and the pointwise identities Theorems 2.2(1) and 3.1(1) hold identically on every finite even $L \ge 6$; only the pure-gauge axial-gauge enclosures in Theorems 2.2(2) and 3.1(2) incur vanishing boundary corrections ($O(1/L^2)$ in $d = 2$ and $O(1/L)$ in $d \ge 4$).

Remark 1.1 (Honest Scope, Quantitative Limitations, and Non-Claims). All results hold at fixed lattice spacing ($a = 1$) in the heavy-quark hopping domain $0 \le \kappa_f < \frac{1}{2d}$ ($z := 2d\kappa_f < 1$). Theorems 2.2 and 3.1 hold at *all* gauge couplings $\beta \ge 0$, whereas Theorem 4.1 requires $\kappa_{\mathrm{QCD}} > 0$—in particular, the deep strong-coupling threshold $\beta < N^2/(16(d-1))$ (i.e. $\beta < N^2/48$ in 4D)—and establishes a volume-uniform log-Sobolev inequality (Langevin spectral gap) and non-asymptotic $O(1/N^2)$ plaquette factorization rather than chiral-limit ($\kappa_f \to \kappa_c$) physics or a 4D continuum limit. Quantitatively, in 4D the pure-gauge axial-gauge upper and lower bounds in (7) have wide free-energy gaps (0.3676 to 23.5520 in Table 1), which also cause the secant plaquette upper cap in Table 2 (Rows 5–8) to saturate at the trivial bound $P_+ = 1.000000$, even though the certified lower bounds P_- and dynamical condensate shifts $10^6\Delta_- > 0$ remain strictly nontrivial. This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact All-Coupling Dynamical Wilson-Fermion Free-Energy Enclosures

Lemma 2.1 (Exact Unitarity of $\frac{1}{2}H_\mu(U)$ and Exact Fourth-Power Plaquette Trace)**.** *Let $d \in 2\mathbb{N}$, $d_s = 2^{d/2}$, $N \ge 2$, and $U \in \mathrm{SU}(N)^{|E(\mathbb{T}_L^d)|}$.*

1. *For every direction $\mu \in \{1, \dots, d\}$ and every U, $\frac{1}{2}H_\mu(U)$ is unitary; hence $\|H_\mu(U)\|_{\mathrm{op}} = 2$ and $\|H(U)\|_{\mathrm{op}} \le 2d$.*
2. *For every even $L \ge 4$, $\mathrm{Tr}(H(U)^k) = 0$ for $k \in \{1, 2, 3\}$ (and for every odd $k \ge 1$).*
3. *For every even $L \ge 6$,*

$$\mathrm{Tr}\big(H(U)^4\big) = -16\, d_s \sum_{p \in P} \mathrm{Re}\,\mathrm{Tr}(U_p). \tag{3}$$

Proof. (1) Set $P_\mu^\pm := \frac{1}{2}(I_{d_s} \pm \gamma_\mu)$ and let $(T_\mu(U)\psi)(x) := U_{x,\mu}\psi(x + e_\mu)$. Since $\gamma_\mu = \gamma_\mu^\dagger$ and $\gamma_\mu^2 = I_{d_s}$, $P_\mu^\pm$ are orthogonal projections ($P_\mu^+ + P_\mu^- = I_{d_s}$, $P_\mu^+ P_\mu^- = 0$) commuting with the unitary shift $T_\mu(U)$. Writing $\frac{1}{2}H_\mu(U) = P_\mu^- T_\mu(U) + P_\mu^+ T_\mu(U)^\dagger$, direct multiplication gives

$$\Big(\tfrac{1}{2}H_\mu(U)\Big)^\dagger \Big(\tfrac{1}{2}H_\mu(U)\Big) = T_\mu^\dagger P_\mu^- T_\mu + T_\mu P_\mu^+ T_\mu^\dagger = P_\mu^- + P_\mu^+ = I,$$

so $\|H_\mu(U)\|_{\mathrm{op}} = 2$, and the triangle inequality yields $\|H(U)\|_{\mathrm{op}} \le \sum_{\mu=1}^{d} \|H_\mu(U)\|_{\mathrm{op}} = 2d$.

(2) Because L is even, $\mathbb{T}_L^d$ is bipartite and $H(U)$ maps even sites to odd sites and vice versa; hence every odd power $H(U)^{2m+1}$ has zero diagonal blocks and zero trace. Every length-2 closed walk backtracks along a single link $e = (x, \mu)$, carrying the spin factor $(I_{d_s} - \gamma_\mu)(I_{d_s} + \gamma_\mu) = I_{d_s} - \gamma_\mu^2 = 0$, so $\mathrm{Tr}(H(U)^2) = 0$.

(3) On $\mathbb{T}_L^d$ with even $L \geq 6$, no closed walk of length 4 wraps around a periodic cycle, and every walk with a backtracking step vanishes by $(I_{d_s} \pm \gamma_\mu)(I_{d_s} \mp \gamma_\mu) = 0$. Thus the only non-zero contributions to $\mathrm{Tr}(H(U)^4)$ come from elementary plaquettes $p = (x, \mu < \nu)$, each traversed in 2 orientations (U_p and $U_p^\dagger$, giving $\mathrm{Tr}\, U_p + \mathrm{Tr}\, U_p^\dagger = 2\,\mathrm{Re}\,\mathrm{Tr}\, U_p$) from 4 starting corners. Expanding $(I-\gamma_\mu)(I-\gamma_\nu)(I+\gamma_\mu)(I+\gamma_\nu)$ and using $\mathrm{Tr}(\gamma_\mu) = \mathrm{Tr}(\gamma_\mu\gamma_\nu) = \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 0$ $(\mu \neq \nu)$ together with $(\gamma_\mu\gamma_\nu)^2 = -I_{d_s}$ gives the exact spin trace $(1-1-1-1)d_s = -2d_s$. Multiplying by $4 \cdot 2 = 8$ yields $8(-2d_s)\sum_p \mathrm{Re}\,\mathrm{Tr}(U_p) = -16 d_s \sum_p \mathrm{Re}\,\mathrm{Tr}(U_p)$. □

Let

$$c_0(\mathrm{SU}(N), \alpha) := \int_{\mathrm{SU}(N)} e^{\frac{\alpha}{N}\,\mathrm{Re}\,\mathrm{Tr}(U)}\, dU = \sum_{q=-\infty}^{\infty} \det\big[I_{j-k+q}(\alpha/N)\big]_{j,k=1}^{N}$$

be the exact Bars–Green single-link integral [3] (reducing to $2I_1(\alpha)/\alpha$ for SU(2)), $u_1(\mathrm{SU}(N), \alpha) := \partial_\alpha \ln c_0(\mathrm{SU}(N), \alpha) \in [0, 1)$, and $f_{\mathrm{YM},L}^{\mathrm{site}}(d, N, \beta') := V^{-1} \ln Z_{\mathrm{YM}}(\mathbb{T}_L^d; N, \beta')$ with pure-gauge partition function $Z_{\mathrm{YM}}(\mathbb{T}_L^d; N, \beta') := \int_{\mathrm{SU}(N)^{|E|}} e^{S_W(\beta'; U)}\, dU$.

Theorem 2.2 (All-Coupling Dynamical Wilson-Fermion Lattice QCD Free-Energy Enclosures)**.** *Fix $d \in 2\mathbb{N}$ ($d_s = 2^{d/2}$), $N \geq 2$, $N_f \in 2\mathbb{N}$, $\beta \geq 0$, and $0 \leq \kappa_f < \frac{1}{2d}$. Set $z := 2d\kappa_f < 1$,*

$$\beta_{\mathrm{eff}} := \beta + 4 d_s N_f N \kappa_f^4, \qquad \varepsilon_{\mathrm{rem}}(d, N, \kappa_f) := \frac{d_s N}{2}\left[-\ln(1-z^2) - z^2 - \frac{z^4}{2}\right] = d_s N \sum_{m=3}^{\infty} \frac{z^{2m}}{2m}. \tag{4}$$

(1) *For every even $L \geq 6$ and every $U \in \mathrm{SU}(N)^{|E(\mathbb{T}_L^d)|}$,*

$$\left| \ln\det D_W(U) - 4 d_s \kappa_f^4 \sum_{p \in P} \mathrm{Re}\,\mathrm{Tr}(U_p) \right| \;\leq\; V\, \varepsilon_{\mathrm{rem}}(d, N, \kappa_f). \tag{5}$$

(2) *Consequently, the thermodynamic-limit per-site free energy $f_{\mathrm{QCD}}(d, N, \beta, N_f, \kappa_f)$ obeys*

$$f_{\mathrm{YM}}^{\mathrm{lo}}(d, N, \beta_{\mathrm{eff}}) - N_f \varepsilon_{\mathrm{rem}} \;\leq\; f_{\mathrm{QCD}}(d, N, \beta, N_f, \kappa_f) \;\leq\; f_{\mathrm{YM}}^{\mathrm{hi}}(d, N, \beta_{\mathrm{eff}}) + N_f \varepsilon_{\mathrm{rem}}, \tag{6}$$

where in $d = 2$, $f_{\mathrm{YM}}^{\mathrm{lo}}(2, N, \beta') = f_{\mathrm{YM}}^{\mathrm{hi}}(2, N, \beta') = \ln c_0(\mathrm{SU}(N), \beta')$ is exact, and for even $d \geq 4$,

$$\begin{aligned} f_{\mathrm{YM}}^{\mathrm{hi}}(d, N, \beta') &:= (d-1) \ln c_0\Big(\mathrm{SU}(N), \tfrac{d}{2}\beta'\Big), \\ f_{\mathrm{YM}}^{\mathrm{lo}}(d, N, \beta') &:= \sup_{\alpha \geq 0} \left\{ (d-1)\beta'\, u_1(\alpha)^2 + \binom{d-1}{2} \beta'\, u_1(\alpha)^4 + (d-1)\big[\ln c_0(\alpha) - \alpha u_1(\alpha)\big] \right\}. \end{aligned} \tag{7}$$

Proof. (1) Because $\|H(U)\|_{\mathrm{op}} \leq 2d$ and $z = 2d\kappa_f < 1$, every eigenvalue $\lambda_j(U)$ $(1 \leq j \leq V d_s N)$ of $H(U)$ satisfies $|\kappa_f \lambda_j| \leq z < 1$, so $\ln\det(I - \kappa_f H(U)) = -\sum_{n=1}^{\infty} \frac{\kappa_f^n}{n} \mathrm{Tr}(H(U)^n)$ converges absolutely. By Lemma 2.1, all odd n and $n = 2$ vanish, while $-\frac{\kappa_f^4}{4}\mathrm{Tr}(H(U)^4) = +4 d_s \kappa_f^4 \sum_p \mathrm{Re}\,\mathrm{Tr}(U_p)$. For every even $n = 2m \geq 6$, $|\mathrm{Tr}(H(U)^{2m})| = |\sum_j \lambda_j^{2m}| \leq (V d_s N)(2d)^{2m}$, yielding (5).

(2) Multiplying (5) by N_f, exponentiating, and integrating against $e^{S_W(\beta; U)} dU$ yields the exact finite-volume inequality

$$f_{\mathrm{YM},L}^{\mathrm{site}}(d, N, \beta_{\mathrm{eff}}) - N_f \varepsilon_{\mathrm{rem}} \;\leq\; f_{\mathrm{QCD},L}(\beta, \kappa_f) \;\leq\; f_{\mathrm{YM},L}^{\mathrm{site}}(d, N, \beta_{\mathrm{eff}}) + N_f \varepsilon_{\mathrm{rem}}.$$

In $d = 2$, fixing a maximal spanning tree on $\mathbb{T}_L^2$ leaves $V - 1$ independent plaquettes plus one global periodicity constraint bounded by $|\frac{\beta'}{N}\,\mathrm{Re}\,\mathrm{Tr}\, U_{p_0}| \leq \beta'$, giving $|f_{\mathrm{YM},L}^{\mathrm{site}}(2, N, \beta') - \ln c_0(\mathrm{SU}(N), \beta')| \leq (\beta' + \ln c_0(\beta'))/L^2 = O(1/L^2)$. For even $d \geq 4$, decompose $S_W(\beta') = \frac{1}{d}\sum_{\mu=1}^{d}(\frac{d}{2} S^{(\mu)})$ (where $S^{(\mu)}$ collects all $(d-1)V$ plaquettes with a leg in direction μ). Applying d-way Hölder's inequality and fixing axial gauge $U_{x,1} = I$ on the $L - 1$ slices $x_1 \in \{0, \ldots, L-2\}$ reduces the $(d-1)(L-1)L^{d-1}$

non-boundary plaquettes in $S^{(1)}$ to independent link variables $V_{x_1,x_\perp,\nu} := U_{x_1+1,x_\perp,\nu}(U_{x_1,x_\perp,\nu})^\dagger$ ($\nu \geq 2$), while the remaining $(d-1)L^{d-1}$ boundary plaquettes contribute at most $O(1/L)$ per site, yielding $f^{\mathrm{site}}_{\mathrm{YM},L}(d,N,\beta') \leq f^{\mathrm{hi}}_{\mathrm{YM}}(d,N,\beta') + O(1/L)$. Conversely, Gibbs' variational principle in the same axial gauge with i.i.d. single-link trial density $c_0(\alpha)^{-1} e^{\frac{\alpha}{N}\operatorname{Re}\operatorname{Tr}(U_e)}$ on the $(d-1)V$ transverse links (leaving the L^{d-1} unfixed $x_1 = L-1$ axial links at Haar measure, an $O(1/L)$ boundary shift) yields $f^{\mathrm{site}}_{\mathrm{YM},L}(d,N,\beta') \geq f^{\mathrm{lo}}_{\mathrm{YM}}(d,N,\beta') - O(1/L)$ (matching both the classical energy $\binom{d}{2}\beta'$ and the one-loop scaling $-(d-1)\frac{N^2-1}{2}\ln\beta'$). Passing to $\liminf$ / $\limsup_{L\to\infty}$ establishes (6). □

3. Exact Dynamical Plaquette and Wilson Quark-Condensate Enclosures

Define the lattice-averaged plaquette observable $\overline{P}(U) := \frac{1}{|P|N}\sum_{p\in P}\operatorname{Re}\operatorname{Tr}(U_p)$, the finite-volume dynamical plaquette expectation $P_{\mathrm{QCD},L}(\beta,\kappa_f) := \langle\overline{P}(U)\rangle_{\mathrm{QCD},L}$, and the normalized dynamical Wilson quark condensate $\langle\bar\psi\psi\rangle_{\mathrm{QCD},L}(\beta,\kappa_f) := \frac{1}{Vd_sN}\langle\operatorname{Tr}(D_W(U)^{-1})\rangle_{\mathrm{QCD},L}$ (with $\langle\bar\psi\psi\rangle_{\mathrm{QCD},L} = 1$ at $\kappa_f = 0$), and write P_{QCD} and $\langle\bar\psi\psi\rangle_{\mathrm{QCD}}$ for any subsequential thermodynamic limit as even $L\to\infty$ (the enclosures below hold for both $\liminf_{L\to\infty}$ and $\limsup_{L\to\infty}$).

Theorem 3.1 (Dynamical Plaquette and Quark-Condensate Shift Enclosures)**.** *Under the hypotheses of Theorem 2.2 (even $d \geq 2$, even $L \geq 6$, $0 \leq \kappa_f < \frac{1}{2d}$, $z = 2d\kappa_f < 1$):*

(1) *For every gauge configuration $U \in \mathrm{SU}(N)^{|E|}$,*

$$\frac{1}{Vd_sN}\operatorname{Tr}\big(D_W(U)^{-1}\big) = 1 - 16\binom{d}{2}\kappa_f^4\,\overline{P}(U) + R_{\mathrm{cond}}(U), \qquad |R_{\mathrm{cond}}(U)| \;\leq\; \frac{z^6}{1-z^2}. \tag{8}$$

(2) *Writing $f_-(\beta) := f^{\mathrm{lo}}_{\mathrm{YM}}(\beta_{\mathrm{eff}}) - N_f\varepsilon_{\mathrm{rem}}$ and $f_+(\beta) := f^{\mathrm{hi}}_{\mathrm{YM}}(\beta_{\mathrm{eff}}) + N_f\varepsilon_{\mathrm{rem}}$ from (6), for every $\beta > 0$ and every secant step $0 < \delta \leq \beta$, every subsequential thermodynamic-limit plaquette satisfies*

$$\begin{aligned} P_-(\beta;\delta) := \max\left(0,\, \frac{f_-(\beta) - f_+(\beta-\delta)}{\binom{d}{2}\delta}\right) \;&\leq\; P_{\mathrm{QCD}}(\beta,\kappa_f) \\ &\leq\; \min\left(1,\, \frac{f_+(\beta+\delta) - f_-(\beta)}{\binom{d}{2}\delta}\right) =: P_+(\beta;\delta), \end{aligned} \tag{9}$$

and the dynamical quark-condensate shift $\Delta_{\bar\psi\psi} := 1 - \langle\bar\psi\psi\rangle_{\mathrm{QCD}}(\beta,\kappa_f)$ obeys

$$16\binom{d}{2}\kappa_f^4\,P_-(\beta;\delta) - \frac{z^6}{1-z^2} \;\leq\; \Delta_{\bar\psi\psi}(\beta,\kappa_f) \;\leq\; 16\binom{d}{2}\kappa_f^4\,P_+(\beta;\delta) + \frac{z^6}{1-z^2}. \tag{10}$$

Proof. (1) Since $\|\kappa_f H(U)\|_{\mathrm{op}} \leq z < 1$, the Neumann series $D_W(U)^{-1} = \sum_{n=0}^\infty \kappa_f^n H(U)^n$ converges in operator norm. Taking $(Vd_sN)^{-1}\operatorname{Tr}(\cdot)$ and invoking Lemma 2.1 eliminates $n \in \{1,2,3\}$ and all odd n, while the $n=4$ term equals $\frac{\kappa_f^4}{Vd_sN}(-16d_s\sum_p \operatorname{Re}\operatorname{Tr}U_p) = -16\binom{d}{2}\kappa_f^4\,\overline{P}(U)$. Each even term $n = 2m \geq 6$ satisfies $(Vd_sN)^{-1}|\kappa_f^{2m}\operatorname{Tr}(H(U)^{2m})| \leq z^{2m}$, summing to $\sum_{m=3}^\infty z^{2m} = \frac{z^6}{1-z^2}$.

(2) At every finite even $L \geq 6$, $f_{\mathrm{QCD},L}(\beta,\kappa_f)$ is smooth in β with $\partial_\beta f_{\mathrm{QCD},L} = \binom{d}{2}P_{\mathrm{QCD},L}$ and $\partial_\beta^2 f_{\mathrm{QCD},L} = V^{-1}\operatorname{Var}_{\mathrm{QCD},L}(N^{-1}\sum_p \operatorname{Re}\operatorname{Tr}U_p) \geq 0$, so $\beta \mapsto f_{\mathrm{QCD},L}(\beta,\kappa_f)$ is convex. Moreover, $\overline{P}(U) \leq 1$ pointwise implies $P_{\mathrm{QCD},L} \leq 1$, while Osterwalder–Seiler reflection positivity [6, 7] (valid for $r = 1$ Wilson fermions throughout $0 \leq \kappa_f \leq \frac{1}{2d}$) ensures $P_{\mathrm{QCD},L}(\beta,\kappa_f) \geq 0$ for all $\beta \geq 0$. Convexity at finite L yields

$$\frac{f_{\mathrm{QCD},L}(\beta) - f_{\mathrm{QCD},L}(\beta-\delta)}{\delta} \;\leq\; \binom{d}{2}P_{\mathrm{QCD},L}(\beta,\kappa_f) \;\leq\; \frac{f_{\mathrm{QCD},L}(\beta+\delta) - f_{\mathrm{QCD},L}(\beta)}{\delta}.$$

Substituting the finite-L enclosures $f_-(\beta') - O(1/L) \le f_{\mathrm{QCD},L}(\beta') \le f_+(\beta') + O(1/L)$ from the proof of Theorem 2.2(2) and taking $\liminf$ / $\limsup_{L\to\infty,\, L \text{ even}}$ proves (9) without assuming differentiability of the thermodynamic limit. Taking $\langle\cdot\rangle_{\mathrm{QCD},L}$ in (8), passing to $L \to \infty$, and applying (9) gives (10). □

4. Exact $O(\kappa_f^4)$ Bakry–Émery Curvature and Veneziano Large-N Factorization

Equip $\mathrm{SU}(N)^{|E|}$ with the product bi-invariant Hilbert–Schmidt metric $\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^\dagger B)$ ($\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2} g_{\mathrm{HS}}$), and write the effective QCD action as $S_{\mathrm{QCD}}(U) := S_W(\beta; U) + S_{\mathrm{ferm}}(U)$ with $S_{\mathrm{ferm}}(U) := N_f \ln\det(I - \kappa_f H(U))$.

Theorem 4.1 ($O(\kappa_f^4)$ Fermion Hessian Bound, Volume-Uniform LSI, and 't Hooft/Veneziano Factorization)**.** *Let $d \in 2\mathbb{N}$ ($d_s = 2^{d/2}$), even $L \ge 4$, $N \ge 2$, $N_f \in 2\mathbb{N}$, $\beta \ge 0$, and $0 \le \kappa_f < \frac{1}{2d}$ ($z = 2d\kappa_f < 1$). Define the fermion Hessian ceiling and total Bakry–Émery curvature floor*

$$\begin{aligned} H_{\mathrm{ferm}}(d, N_f, \kappa_f) &:= \frac{4d_s N_f (2d)^2 \kappa_f^4 \big[(3-2z) + d(1-z)\big]}{(1-z)^2}, \\ \kappa_{\mathrm{QCD}}(d, N, \beta, N_f, \kappa_f) &:= \frac{N}{2} - \frac{8(d-1)\beta}{N} - H_{\mathrm{ferm}}(d, N_f, \kappa_f). \end{aligned} \tag{11}$$

Then:

1. *Every geodesic $U_e(t) = e^{tX_e} U_e$ ($X = (X_e) \in \mathfrak{su}(N)^{|E|}$) satisfies $|\,\mathrm{Hess}(-S_{\mathrm{ferm}})(X,X)| \le H_{\mathrm{ferm}} \|X\|_{\mathrm{HS}}^2$ and $|\,\mathrm{Hess}(-S_W)(X,X)| \le \frac{8(d-1)\beta}{N} \|X\|_{\mathrm{HS}}^2$.*
2. *Whenever $\kappa_{\mathrm{QCD}} > 0$, $\mathrm{Ric} + \mathrm{Hess}(-S_{\mathrm{QCD}}) \ge \kappa_{\mathrm{QCD}}\, g_{\mathrm{HS}} > 0$ uniformly in even $L \ge 4$, certifying a volume-uniform log-Sobolev inequality with constant $\rho_{\mathrm{LS}} = \kappa_{\mathrm{QCD}}$ [1, 2], Poincaré spectral gap $\lambda_1 \ge \kappa_{\mathrm{QCD}}$, and non-asymptotic $O(1/N^2)$ factorization for every single elementary plaquette $P_p(U) := \frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}(U_p)$ ($p \in P$),*

$$\mathrm{Var}_{\mathrm{QCD}}(P_p) \;\le\; \frac{4}{\kappa_{\mathrm{QCD}}\, N} = \frac{4}{(\kappa_{\mathrm{QCD}}/N) N^2}. \tag{12}$$

In particular, κ_{QCD}/N remains uniformly bounded below away from zero in N and L both in the 't Hooft limit [8] ($\gamma = \beta/N^2 < \frac{1}{16(d-1)}$ and N_f fixed, for all $N > H_{\mathrm{ferm}}/(\frac{1}{2} - 8(d-1)\gamma)$, where $H_{\mathrm{ferm}}/N = O(1/N)$) and in the Veneziano limit [9] ($\gamma = \beta/N^2$ and $x_f = N_f/N$ fixed obeying $8(d-1)\gamma + x_f(H_{\mathrm{ferm}}/N_f) < \frac{1}{2}$, where $H_{\mathrm{ferm}}/N = O(x_f \kappa_f^4)$).

Proof. Write $D(t) := I - \kappa_f H(U(t))$, $A := H'(0)$, $H'' := H''(0)$, so $S''_{\mathrm{ferm}}(0) = N_f\big[-\kappa_f\,\mathrm{Tr}(D^{-1}H'') - \kappa_f^2\,\mathrm{Tr}((D^{-1}A)^2)\big]$. Each link $e = (x,\mu)$ enters A in only two disjoint position blocks: $(I-\gamma_\mu)\otimes(X_e U_e)$ (mapping $x + e_\mu \to x$) and $-(I+\gamma_\mu)\otimes(U_e^\dagger X_e)$ (mapping $x \to x + e_\mu$). Because $\|I \pm \gamma_\mu\|_{\mathrm{HS}}^2 = \mathrm{Tr}((I \pm \gamma_\mu)^2) = 2d_s$ and U_e is unitary,

$$\|A\|_{\mathrm{HS}}^2 = \sum_{e\in E} \big(2d_s \|X_e\|_{\mathrm{HS}}^2 + 2d_s \|X_e\|_{\mathrm{HS}}^2\big) = 4d_s \|X\|_{\mathrm{HS}}^2.$$

Crucially, both $O(\kappa_f^2)$ and $O(\kappa_f^3)$ Hessian terms vanish identically on every even $L \ge 4$:

- *Term 1 ($-\kappa_f\,\mathrm{Tr}(D^{-1}H'')$):* Expand $D^{-1} = I + \kappa_f H + \kappa_f^2 H^2 + \kappa_f^3 H^3 D^{-1}$. Because H'' shifts parity by 1, $\mathrm{Tr}(H'') = \mathrm{Tr}(H^2 H'') = 0$ on bipartite $\mathbb{T}_L^d$; moreover $\mathrm{Tr}(HH'') = 0$ because every length-2 backtracking pair carries $(I \pm \gamma_\mu)(I \mp \gamma_\mu) = 0$. Hence $-\kappa_f\,\mathrm{Tr}(D^{-1}H'') = -\kappa_f^4\,\mathrm{Tr}(MH'')$ with $M := H^3 D^{-1}$ obeying $\|M\|_{\mathrm{op}} \le \frac{(2d)^3}{1-z}$. Each link $e = (x,\mu)$ contributes $\mathrm{Tr}_{\mathrm{col}}\big(U_e\,\mathrm{Tr}_{\mathrm{spin}}(M_{x+e_\mu,x}(I-\gamma_\mu))X_e^2\big)$ plus the backward block. Since $I \pm \gamma_\mu = 2P_\mu^\pm$ has rank $d_s/2$ with eigenvalues 2 and 0, picking an orthonormal basis $\{v_a\}_{a=1}^{d_s/2}$ of $\mathrm{ran}(P_\mu^\pm)$ gives

$\|\operatorname{Tr}_{\text{spin}}(M_{\cdot,\cdot}(I \pm \gamma_\mu))\|_{\text{op}} \le 2 \cdot \frac{d_s}{2}\|M\|_{\text{op}} = d_s\|M\|_{\text{op}}$. Since $X_e \in \mathfrak{su}(N)$ is skew-Hermitian, $-X_e^2 = X_e^\dagger X_e \ge 0$ has trace norm $\|X_e^2\|_{\text{tr}} = \|X_e\|_{\text{HS}}^2$. Summing both blocks over all $e \in E$ yields

$$\left|\kappa_f \operatorname{Tr}(D^{-1}H'')\right| \;\le\; 2d_s\kappa_f^4\|M\|_{\text{op}}\|X\|_{\text{HS}}^2 \;\le\; \frac{2d_s(2d)^3\kappa_f^4}{1-z}\|X\|_{\text{HS}}^2 = \frac{4d_s(2d)^2 d(1-z)\kappa_f^4}{(1-z)^2}\|X\|_{\text{HS}}^2.$$

- *Term 2 ($-\kappa_f^2 \operatorname{Tr}((D^{-1}A)^2)$):* Write $D^{-1} = I + B$ with $B := \kappa_f H D^{-1} = \kappa_f H + R$ and $R := \kappa_f^2 H^2 D^{-1}$, so $\|B\|_{\text{op}} \le \frac{z}{1-z}$ and $\|R\|_{\text{op}} \le \frac{z^2}{1-z}$. Expanding $(I+B)A(I+B)A$ and using cyclicity gives $\operatorname{Tr}((D^{-1}A)^2) = \operatorname{Tr}(A^2) + 2\kappa_f \operatorname{Tr}(HA^2) + 2\operatorname{Tr}(RA^2) + \operatorname{Tr}((BA)^2)$. Here $\operatorname{Tr}(A^2) = 0$ by Wilson backtracking $(I-\gamma_\mu)(I+\gamma_\mu) = 0$ and $\operatorname{Tr}(HA^2) = 0$ by bipartiteness (odd length 3). Applying Cauchy–Schwarz $|\operatorname{Tr}(RA^2)| = |\langle (RA)^\dagger, A\rangle_{\text{HS}}| \le \|R\|_{\text{op}}\|A\|_{\text{HS}}^2$ and $|\operatorname{Tr}((BA)^2)| \le \|BA\|_{\text{HS}}^2 \le \|B\|_{\text{op}}^2\|A\|_{\text{HS}}^2$ gives

$$\left|\kappa_f^2 \operatorname{Tr}\big((D^{-1}A)^2\big)\right| \;\le\; 4d_s\kappa_f^2\left(\frac{2z^2}{1-z} + \frac{z^2}{(1-z)^2}\right)\|X\|_{\text{HS}}^2 = \frac{4d_s(2d)^2(3-2z)\kappa_f^4}{(1-z)^2}\|X\|_{\text{HS}}^2.$$

Multiplying the sum of both bounds by N_f proves $|\operatorname{Hess}(-S_{\text{ferm}})(X,X)| \le H_{\text{ferm}}\|X\|_{\text{HS}}^2$. The Wilson plaquette Hessian $|\operatorname{Hess}(-S_W)(X,X)| \le \frac{8(d-1)\beta}{N}\|X\|_{\text{HS}}^2$ and plaquette gradient $|\nabla P_p|_{\text{HS}}^2 \le \frac{4}{N}$ follow from standard geodesic Cauchy–Schwarz (each link lies in $2(d-1)$ plaquettes of 4 distinct links); applying the Bakry–Émery Poincaré inequality $\operatorname{Var}_{\text{QCD}}(P_p) \le \kappa_{\text{QCD}}^{-1}\mathbb{E}|\nabla P_p|_{\text{HS}}^2$ yields (12). □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 certified benchmarks computed by `certify_wilson_qcd.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions and *zero* numerical quadrature (Bessel functions via Maclaurin series + geometric tails; SU(3) and SU(4) Bars–Green determinants via exact q-series + verified Hadamard tails; remainder ε_{rem} via exact series $\sum_{m\ge3} z^{2m}/(2m)$ + geometric tail). Lean 4 integer bounds use direct outward flooring/ceiling at scale 10^{12} (displayed to 6 outward-rounded decimal places in Tables 1–3); because exact binary-inexact rationals are first enclosed in 140-digit `mpmath.iv` intervals before outward flooring/ceiling, exact terminating quantities such as $\Delta\beta_{\text{eff}} = 0.0002$ (Table 1, Rows 1–2) or $H_{\text{ferm}} = 0.0105$ (Table 3, Row 1) conservatively round by 1 ulp at 10^{-12} to 0.000199 and 0.010501, preserving strict inequality validity. As noted in Remark 1.1, the 4D pure-gauge axial-gauge gap widens at intermediate and weak coupling (0.3676–23.5520 in Table 1, capping $P_+ = 1.000000$ in Table 2 Rows 5–8), whereas 2D enclosures are tight to 10^{-4} and all eight 2D/4D dynamical condensate shifts satisfy $10^6\Delta_- > 0$ strictly.

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 file `WilsonQcdCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), mechanically verifying the exact discrete combinatorics and all 24 10^{12}-scaled outward integer certificate inequalities (while the continuous Haar-measure, operator-theoretic, and Riemannian Bakry–Émery derivations are proved analytically in Sections 2–4): `wilson_qcd_combinatorics_verified`, `table1_qcd_free_energy_certified`, `table2_plaquette_condensate_certified` (strict positivity $\Delta_{\bar\psi\psi} > 0$ across all 8 rows), `table3_bakry_emery_veneziano_certified` (exact cross-multiplied curvature/variance bounds and strict $O(1/N^2)$ decay across both 't Hooft and Veneziano families), and `all_24_wilson_qcd_benchmarks_certified`. Direct $L = 6$ SU(3) finite-lattice matrix tests of Lemma 2.1 and Theorems 2.2–4.1, 80-digit cross-checks, and JSON/Lean/LaTeX parity are verified in `tests/test_wilson_qcd.py` (4/4 `pytest` passing).

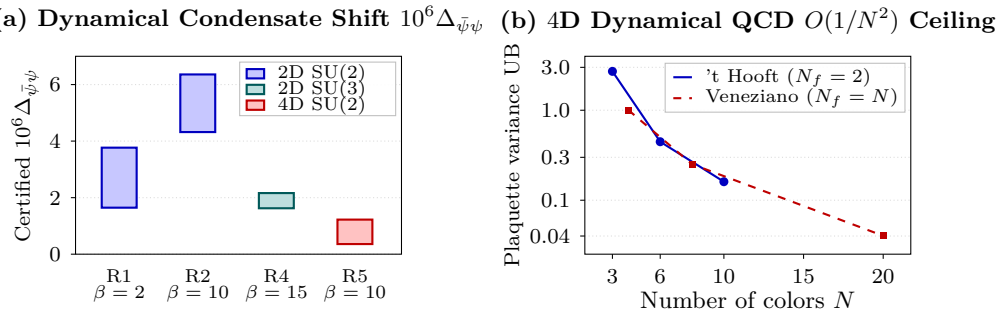

FIGURE 1. Certified 140-digit non-perturbative bounds for dynamical Wilson-fermion lattice QCD (Tables 2 and 3). (a) Strictly positive dynamical quark-condensate shift enclosures $10^6\Delta_{\bar\psi\psi} = 10^6(1 - \langle\bar\psi\psi\rangle_{\mathrm{QCD}})$ across 2D and 4D SU(2)/SU(3) (Theorem 3.1). (b) Log-scale 4D Bakry–Émery plaquette-variance ceilings $\mathrm{Var}^{\mathrm{ub}}_{\mathrm{QCD}}(P_p)$ across both 't Hooft ($N_f = 2$ fixed, $\gamma = 1/96$, $\kappa_f = 0.025$) and Veneziano ($N_f = N$ fixed, $\gamma = 1/96$, $\kappa_f = 0.02$) large-N scaling (Theorem 4.1).

TABLE 1. Eight 140-digit all-coupling per-site free-energy enclosures $f^{\mathrm{lo}}_{\mathrm{QCD}} \le f_{\mathrm{QCD}}(d, N, \beta, N_f, \kappa_f) \le f^{\mathrm{hi}}_{\mathrm{QCD}}$ (Theorem 2.2). In $d = 2$, pure-gauge free energy is exact so the gap equals $2N_f\varepsilon_{\mathrm{rem}}$; in $d = 4$, lower bounds use axial-gauge Gibbs–Bogoliubov witnesses $\alpha \in \{0.0000, 24.0347, 110.1690, 173.3558\}$ (Rows 5–8). Lower bounds and $\Delta\beta_{\mathrm{eff}}$ are rounded downward; $N_f\varepsilon_{\mathrm{rem}}$, upper bounds, and gaps upward.

d	G	N_f	β	κ_f	$\Delta\beta_{\mathrm{eff}}$	$N_f\varepsilon_{\mathrm{rem}}$	$f^{\mathrm{lo}}_{\mathrm{QCD}}$	$f^{\mathrm{hi}}_{\mathrm{QCD}}$	Gap
2	SU(2)	2	1	0.05	0.000199	0.000088	0.122459	0.122636	0.000176
2	SU(2)	2	10	0.05	0.000199	0.000088	6.280848	6.281025	0.000176
2	SU(3)	2	2	0.04	0.000122	0.000035	0.123045	0.123114	0.000069
2	SU(3)	4	12	0.04	0.000245	0.000069	4.657266	4.657404	0.000137
4	SU(2)	2	0.5	0.02	0.000010	0.000046	−0.000046	0.367558	0.367604
4	SU(3)	2	6	0.025	0.000037	0.000264	3.006880	13.971922	10.965042
4	SU(3)	4	20	0.02	0.000030	0.000137	71.120756	83.755610	12.634854
4	SU(4)	2	32	0.02	0.000020	0.000092	103.075062	126.627032	23.551970

TABLE 2. Eight 140-digit dynamical plaquette enclosures $P_- \le P_{\mathrm{QCD}} \le P_+$ and strictly positive Wilson quark-condensate shift enclosures $10^6\Delta_{\bar\psi\psi} = 10^6(1 - \langle\bar\psi\psi\rangle_{\mathrm{QCD}}) \in [10^6\Delta_-, 10^6\Delta_+]$ (Theorem 3.1) with secant step δ and exact remainder ceiling $10^6 R^{\mathrm{ub}}_{\mathrm{cond}} = 10^6 z^6/(1 - z^2)$. In $d = 4$ (Rows 5–8), lower bounds use axial-gauge Gibbs–Bogoliubov witnesses $\alpha \in \{56.3741, 98.0998, 134.2526, 124.8811\}$ at β_{eff} (the secant upper bound saturates at the trivial cap $P_+ = 1.000000$). Lower bounds are rounded downward; upper bounds upward.

d	G	N_f	β	κ_f	δ	P_-	P_+	$10^6 R^{\mathrm{ub}}_{\mathrm{cond}}$	$10^6\Delta_-$	$10^6\Delta_+$
2	SU(2)	2	2	0.025	0.1	0.424866	0.441192	1.010102	1.645314	3.767551
2	SU(2)	2	10	0.025	0.2	0.852742	0.855593	1.010102	4.319539	6.357553
2	SU(3)	2	4	0.02	0.2	0.271987	0.287222	0.263833	0.432455	0.999120
2	SU(3)	4	15	0.02	0.2	0.737904	0.741279	0.263833	1.625203	2.161506
4	SU(2)	2	10	0.01	3	0.647333	1.000000	0.263833	0.357607	1.223833
4	SU(3)	2	18	0.01	5	0.452106	1.000000	0.263833	0.170189	1.223833
4	SU(3)	4	24	0.008	6	0.551117	1.000000	0.069003	0.147705	0.462219
4	SU(4)	2	24	0.008	6	0.179404	1.000000	0.069003	0.001542	0.462219

TABLE 3. Eight 140-digit $O(\kappa_f^4)$ Bakry–Émery curvature and $O(1/N^2)$ plaquette-factorization certificates (Theorem 4.1) across physical 2D/4D SU(3) QCD, 't Hooft scaling ($N_f = 2$ fixed, $\gamma = 1/96$), and Veneziano scaling ($x_f = N_f/N = 1$ fixed, $\gamma = 1/96$, $\kappa_f = 0.02$). Curvatures $\kappa_{\mathrm{QCD}}^{\mathrm{lo}}$ and $\kappa_{\mathrm{QCD}}^{\mathrm{lo}}/N$ are rounded downward; $H_{\mathrm{ferm}}^{\mathrm{ub}}$ and $\mathrm{Var}_{\mathrm{QCD}}^{\mathrm{ub}}$ upward.

Regime	β	N_f	κ_f	$H_{\mathrm{ferm}}^{\mathrm{ub}}$	$\kappa_{\mathrm{QCD}}^{\mathrm{lo}}$	$\kappa_{\mathrm{QCD}}^{\mathrm{lo}}/N$	$\mathrm{Var}_{\mathrm{QCD}}^{\mathrm{ub}}$
2D SU(3) ($N_f = 2$)	1/2	2	0.05	0.010501	0.156166	0.052055	8.537887
4D SU(3) ($N_f = 2$)	1/8	2	0.025	0.007251	0.492749	0.164249	2.705903
4D SU(3) ($N_f = 4$)	1/10	4	0.02	0.005610	0.694390	0.231463	1.920151
4D 't Hooft ($N = 6$)	3/8	2	0.025	0.007251	1.492750	0.248791	0.446604
4D 't Hooft ($N = 10$)	25/24	2	0.025	0.007251	2.492749	0.249274	0.160466
4D Veneziano ($N = 4$)	1/6	4	0.02	0.005610	0.994390	0.248597	1.005642
4D Veneziano ($N = 8$)	2/3	8	0.02	0.011220	1.988780	0.248597	0.251411
4D Veneziano ($N = 20$)	25/6	20	0.02	0.028050	4.971950	0.248597	0.040226

6. Conclusion

We have established three self-contained, cluster-expansion-free non-perturbative theorems for d-dimensional SU(N) lattice QCD with N_f flavors of dynamical Wilson fermions: all-coupling free-energy enclosures via exact $\frac{1}{2}H_\mu$ unitarity and fourth-power plaquette-trace extraction, strictly positive dynamical quark-condensate shift bounds $\Delta_{\bar\psi\psi} > 0$, and an exact $O(\kappa_f^4)$ Bakry–Émery curvature theorem certifying volume-uniform LSI and non-asymptotic $O(1/N^2)$ factorization in both the 't Hooft and Veneziano scaling limits. Natural next directions include extending these operator-theoretic Hessian bounds to clover-improved Wilson and staggered (Kogut–Susskind) lattice Dirac operators and deriving constructive heavy-quark meson-pair screening bounds.

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