Dimensional Collision Obstructions, Antipodal Rerouting, and Machine-Checked Minimum Average Time Broadcast Proofs for Deleted Knödel Graphs

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Abstract

Altay, Fakharan, and Harutyunyan [1] recently introduced a deleted Knödel graph construction $\mathrm{KG}'(n-1)$ establishing the general upper bound $B(n-1) \le \lceil (n-1)\lfloor \log_2(n-1)\rfloor/2\rceil$ on the broadcast function for all even $n \neq 2^k$, and claimed in Section 5 that the same construction directly yields a minimum average time broadcast graph (MATBG) across the full interval $(2^{k-1}, 2^k)$, resolving the 1998 open problem of Liestman and Pržulj [7]. We prove that while the worst-case broadcast bound $b(\mathrm{KG}'(n-1)) = k$ is sound, the MATBG claim in [1] fails as proved due to three infinite-family doubling obstructions: (i) the cyclically shifted dimensional schemes of Lemmas 3.2 and 3.4 suffer modular wrap-around collisions even in $\mathrm{KG}(n)$ up to the sharp threshold $n_{\max}(k,i) = 2^k - 2^i - 4$, losing $2^{k-2} - 2 = \Theta(n/2)$ vertices before round $k-1$ at $(n,i) = (2^{k-1}+2, 1)$; (ii) for every $k \ge 4$ and every $n = 4m \in (2^{k-1}, 2^k)$, from the antipodal originator $u = n/2$, all $2(k-1)$ standard and reverse cyclically shifted dimensional schemes inform the deleted vertex 0 by round $k-1$; and (iii) for every even $n \in [2^{k-1}+2, 3\cdot 2^{k-2}-2]$, the shortcut target $w = 2^{k-2}-1$ of originator $u = 2^{k-1}-1$ already lies in I_{k-1}. We then prove three positive theorems: a universal single-edge shortcut rerouting rule that closes Sub-case 5 with exact doubling $|I_t| = 2^t$ $(0 \le t \le k-1)$ across the entire open second half $n \in [3\cdot 2^{k-2}, 2^k - 2]$; a 2-adic shift law $s = 1 + v_2((n-2^{k-1})/2)$ avoiding vertex 0 on the first-half boundary $u = 2^{k-2}$; and a complete, gap-free MATBG proof for $\mathrm{KG}'(2^k-5)$ and $\mathrm{KG}'(2^k-3)$ for all $k \ge 4$. All universal modular-arithmetic theorems are formally verified in zero-dependency Lean 4.35.0 (24 theorems, 0 `sorry`, 0 custom axioms, 0 `Classical.choice`), paired with an AST-audited zero-float Python verifier (`KNODEL-MATBG-CERT-1.0`) certifying 16 canonical witnesses and 48 unit and forgery-rejection checks.

*Independent researcher. Authored with AI-assisted formal proof engineering, symbolic verification, and manuscript drafting (Google Antigravity). Public MIT repository: github.com/lluiseriksson/knodel-matbg-obstruction-repair (v1.0.0). Manuscript and figures licensed CC-BY-4.0; code, Lean 4 proofs, and certificates licensed MIT.

1 Introduction and Definitions

In the classical telephone broadcasting model on an undirected simple graph $G = (V, E)$ of order $m = |V|$, a message originates at a vertex $u \in V$ at time 0; in each discrete round $t \ge 1$, each informed vertex may transmit the message to at most one uninformed neighbor. A broadcast schedule from u induces a rooted spanning tree T_u with arrival times $t(v) \in \mathbb{N}$ $(t(u) = 0)$. The broadcast time of G is $b(G) = \max_{u\in V} \min_{T_u} \max_{v\in V} t(v) \ge \lceil \log_2 m \rceil$. A graph is a *broadcast graph* if $b(G) = \lceil \log_2 m \rceil$, and the *broadcast function* $B(m)$ is the minimum number of edges of an m-vertex broadcast graph [3, 4].

Liestman and Pržulj [7] introduced the *minimum average broadcast time* $b_a(T_u) = \frac{1}{m}\sum_{v\in V} t(v)$, $b_a(G) = \max_{u\in V} \min_{T_u} b_a(T_u)$, and $\beta_a(m) = b_a(K_m)$. Because the informed set can at most double per round, achieving $b_a(T_u) = \beta_a(m)$ is equivalent to exact doubling $|I_t| = 2^t$ for every $0 \le t \le k-1$ (where $k = \lceil \log_2 m \rceil$) followed by informing the remaining $m - 2^{k-1}$ vertices in round k, giving

$$\beta_a(m) = \frac{1}{m}\left(\sum_{t=1}^{k-1} t\, 2^{t-1} + k(m - 2^{k-1})\right). \tag{1}$$

A *minimum average time broadcast graph* (MATBG) achieves $b_a(G) = \beta_a(m)$ with minimum edge count $B_a(m)$. Since exact doubling through round $k-1$ requires $\deg(v) \ge k-1 = \lfloor \log_2 m \rfloor$ for every vertex v when $m \neq 2^k$, any MATBG satisfies $B_a(m) \ge \lceil m\lfloor \log_2 m \rfloor/2\rceil$.

For even $n \ge 4$ with $k = \lceil \log_2 n \rceil$ and $2^{k-1} < n < 2^k$, the *Knödel graph* $\mathrm{KG}(n)$ [5] has vertex set $V = \{0, 1, \dots, n-1\}$ and edges $(v, \phi_d(v))$ for dimensions $1 \le d \le k-1$, where

$$\phi_d(v) \equiv 2^d - 1 - v \pmod n. \tag{2}$$

Since n is even and $2^d - 1$ is odd, $\mathrm{KG}(n)$ is bipartite between even and odd vertices, and ϕ_d is an involution $(\phi_d(\phi_d(v)) = v)$. Moreover, for every $u \in V$, the bijection $f_u(v) = v + (-1)^v u \pmod n$ is a dimension-preserving automorphism of $\mathrm{KG}(n)$ mapping 0 to u $(\phi_d(f_u(v)) \equiv f_u(\phi_d(v)) \pmod n)$, so the cardinality $|I_t|$ of any dimensional broadcast depends only on (n, D) and not on the originator u.

A *dimensional broadcast scheme* $D = (\alpha_1, \ldots, \alpha_k)$ informs all vertices of $\mathrm{KG}(n)$ in k rounds by calling along ϕ_{α_t} at round t. Bermond et al. [2] proved that $D^{(0)}_{\mathrm{std}} = (1, 2, \ldots, k-1, 1)$, every cyclic shift $D^{(i)}_{\mathrm{std}} = (i+1, \ldots, k-1, 1, \ldots, i, i+1)$, and every reverse shift $D^{(i)}_{\mathrm{rev}} = (k-i-1, \ldots, 1, k-1, \ldots, k-i, k-i-1)$ $(0 \le i \le k-2)$ are valid k-round broadcast schemes on $\mathrm{KG}(n)$.

Altay, Fakharan, and Harutyunyan [1] define the *deleted Knödel graph* $\mathrm{KG}'(n-1)$ on $V' = \{1, \ldots, n-1\}$ by deleting vertex 0 (whose $k-1$ neighbors in $\mathrm{KG}(n)$ are the odd vertices $2^i - 1$ for $1 \le i \le k-1$) and adding $\lceil (k-1)/2 \rceil$ shortcut edges $(2^{i-1} - 1, 2^i - 1)$ for all even $2 \le i \le k-1$, plus $(2^{k-2} - 1, 2^{k-1} - 1)$ when $k-1$ is odd. Because $\mathrm{KG}(n)$ is bipartite, no shortcut edge joins vertices of opposite parity, so all $\lceil (k-1)/2 \rceil$ shortcut edges are genuinely new and $|E(\mathrm{KG}'(n-1))| = \lceil (n-1)(k-1)/2 \rceil$. Section 4 of [1] proves $b(\mathrm{KG}'(n-1)) = k$ via Lemmas 3.1–3.4 and Propositions 3.5–3.7 (where Proposition 3.7 splits even originators into five ranges, with *Sub-case 5* denoting $n - 2^{k-1} + 2 \le u \le \min(2^{k-1} - 2, n - 2^{k-2})$), yielding $B(n-1) \le \lceil (n-1)(k-1)/2 \rceil$ for all even $n \ne 2^k$. Section 5 of [1] then asserts without additional proof that $\mathrm{KG}'(n-1)$ is also a MATBG across $(2^{k-1}, 2^k)$, claiming to resolve the open second half $(3 \cdot 2^{k-2}, 2^k)$ left by [7].

2 Three Infinite-Family MATBG Obstructions

By Lemmas 3.1–3.4 of [1], the informed set I_t in $\mathrm{KG}(n)$ after round $t \le k-1$ from originator u under $D^{(0)}_{\mathrm{std}}$ is $A_t \cup B_t \pmod n$, where

$$\begin{aligned} A_t &= \{2p + u : 0 \le p < 2^{t-1}\}, \\ B_t &= \{2s - u - 1 : 1 \le s \le 2^{t-1}\}, \end{aligned} \tag{3}$$

and under $D^{(0)}_{\mathrm{rev}}$ is $A^{\mathrm{rev}}_t \cup B^{\mathrm{rev}}_t \pmod n$ with $A^{\mathrm{rev}}_t = \{-p\,2^{k-t} + u : 0 \le p < 2^{t-1}\}$ and $B^{\mathrm{rev}}_t = \phi_{k-t}(A^{\mathrm{rev}}_t)$. For $1 \le i \le k-2$ and $u = 0$, Lemma 3.2 $(D^{(i)}_{\mathrm{std}})$ gives $I_{k-1} = I^{(i)}_{k-1,\mathrm{even}} \cup I^{(i)}_{k-1,\mathrm{odd}} \pmod n$, where the even-parity union $A^{(i)}_{k-1} \cup B'^{(i)}_{k-1}$ is

$$\begin{aligned} A^{(i)}_{k-1} &= \{p\,2^{i+1} + 2j : 0 \le p < 2^{k-i-2},\ 0 \le j < 2^{i-1}\}, \\ B'^{(i)}_{k-1} &= \{-s\,2^{i+1} + 2j : 1 \le s \le 2^{k-i-2},\ 1 \le j \le 2^{i-1}\}, \end{aligned} \tag{4}$$

and the odd-parity half $I^{(i)}_{k-1,\mathrm{odd}} = B^{(i)}_{k-1} \cup A'^{(i)}_{k-1}$ is the exact image of $I^{(i)}_{k-1,\mathrm{even}}$ under the round-$(k-1)$ dimensional involution $\phi_i(x) \equiv 2^i - 1 - x \pmod n$ (since $I_{k-1} = I_{k-2} \cup \phi_i(I_{k-2})$ and ϕ_i swaps parities). Symmetrically, Lemma 3.4 $(D^{(i)}_{\mathrm{rev}},\ 1 \le i \le k-2,\ u = 0)$ has even-parity sets

$$\begin{aligned} A^{(i)}_{\mathrm{rev},k-1} &= \{-j\,2^{k-i} - 2p : 0 \le j < 2^{i-1},\ 0 \le p < 2^{k-i-2}\}, \\ B'^{(i)}_{\mathrm{rev},k-1} &= \{j\,2^{k-i} - 2s : 1 \le j \le 2^{i-1},\ 1 \le s \le 2^{k-i-2}\}, \end{aligned} \tag{5}$$

mapped bijectively onto the odd-parity half by the round-$(k-1)$ involution $\phi_{k-i}(x) \equiv 2^{k-i} - 1 - x \pmod n$.

Theorem A (Sharp Shifted-Scheme Collision Threshold and Asymptotic Half-Loss)**.** *Let $k \ge 4$, $1 \le i \le k-2$, and n be even with $2^{k-1} < n < 2^k$.*

1. Unshifted collision-freedom: *Under both $D^{(0)}_{\mathrm{std}}$ and $D^{(0)}_{\mathrm{rev}}$, $|I_t| = 2^t$ for every $0 \le t \le k-1$ and every even $n \in (2^{k-1}, 2^k)$.*
2. Sharp collision threshold: *Whenever $2^i + 4 < 2^{k-1}$ (all $k \ge 5$ and $(k,i) = (4,1)$), at $n_{\max}(k,i) = 2^k - 2^i - 4 \in (2^{k-1}, 2^k)$ the scheme $D^{(i)}_{\mathrm{std}}$ collides at round $k-1$ even in $\mathrm{KG}(n)$, with explicit witness $x = 2^{k-1} - 2^i - 2 \in A^{(i)}_{k-1} \cap (B'^{(i)}_{k-1} + n)$ attained at $(p,j) = (2^{k-i-2} - 1, 2^{i-1} - 1)$ and $(s,j') = (2^{k-i-2}, 1)$. Consequently $|I_{k-1}| = 2^{k-1} - 2|A^{(i)}_{k-1} \cap (B'^{(i)}_{k-1} + n)| \le 2^{k-1} - 2$.*
3. Collision-free upper band: *Whenever $n > 2^k - 2^i - 4$, $D^{(i)}_{\mathrm{std}}$ is collision-free; whenever $n > 2^k - 2^{k-i-1} - 4$, $D^{(i)}_{\mathrm{rev}}$ is collision-free. In particular, for $n \in \{2^k - 4, 2^k - 2\}$, all $2(k-1)$ standard and reverse shifts satisfy $|I_t| = 2^t$ for all $0 \le t \le k-1$.*
4. Asymptotic half-loss: *At $n = 2^{k-1} + 2$ and $i = 1$, every $p \in \{1, \ldots, 2^{k-3} - 1\}$ collides with $s = 2^{k-3} + 1 - p$, yielding $|I_{k-1}| = 2^{k-2} + 2$ and a deficit of $2^{k-2} - 2 = \Theta(n/2)$ vertices before round $k-1$.*

Proof. By the dimension-preserving automorphism f_u, we set $u = 0$. Part (1): In (3), A_t has span $2(2^{t-1} - 1) < 2^{k-1} < n$, so A_t is injective mod n; likewise for B_t, and opposite parities separate A_t and B_t. For $D^{(0)}_{\mathrm{rev}}$, $A^{\mathrm{rev}}_t = \{-p\,2^{k-t} : 0 \le p < 2^{t-1}\}$ has span $(2^{t-1} - 1)2^{k-t} = 2^{k-1} - 2^{k-t} < n$, hence is injective mod n, and ϕ_{k-t} maps it bijectively onto the odd half B^{rev}_t. Part (2): In (4), $A^{(i)}_{k-1} \subset [0, 2^{k-1} - 2^i - 2] \subset [0, n)$ and $B'^{(i)}_{k-1} + n \subset [n - 2^{k-1} + 2, n - 2^i] \subset (0, n)$ are each injective mod n of size 2^{k-3} (since distinct (p,j) have distinct base-2 bit decompositions and span $< n$). Because ϕ_i maps $I^{(i)}_{k-1,\mathrm{even}}$ bijectively onto $I^{(i)}_{k-1,\mathrm{odd}}$, $|I_{k-1}| = 2|A^{(i)}_{k-1} \cup (B'^{(i)}_{k-1} + n)| = 2^{k-1} - 2|A^{(i)}_{k-1} \cap (B'^{(i)}_{k-1} + n)|$. Substituting $(p,j) = (2^{k-i-2} - 1, 2^{i-1} - 1)$ and $(s,j') = (2^{k-i-2}, 1)$ at $n_{\max} = 2^k - 2^i - 4$ yields $p\,2^{i+1} + 2j = 2^{k-1} - 2^i - 2 = n_{\max} - s\,2^{i+1} + 2j'$. Part (3): For $D^{(i)}_{\mathrm{std}}$, $p\,2^{i+1} + 2j + s\,2^{i+1} \le 2^k - 2^i - 2 < n + 2 \le n + 2j'$ whenever $n > 2^k - 2^i - 4$, so $A^{(i)}_{k-1}$ and $B'^{(i)}_{k-1} + n$ are disjoint. Symmetrically, in (5) for $D^{(i)}_{\mathrm{rev}}$, $A^{(i)}_{\mathrm{rev},k-1} + n$ and $B'^{(i)}_{\mathrm{rev},k-1}$ satisfy $j'\,2^{k-i} + j\,2^{k-i} + 2p \le 2^k - 2^{k-i-1} - 2 < n + 2s$ whenever $n > 2^k - 2^{k-i-1} - 4$. Both thresholds are at most $2^k - 6$ for $1 \le i \le k-2$, so $n \in \{2^k - 4, 2^k - 2\}$ clears all $2(k-1)$ shifts and gives $|I_{k-1}| = 2^{k-1}$. Since $|I_0| = 1$ and $|I_{r+1}| \le 2|I_r|$, $|I_{k-1}| = 2^{k-1}$ forces $|I_t| = 2^t$ for every

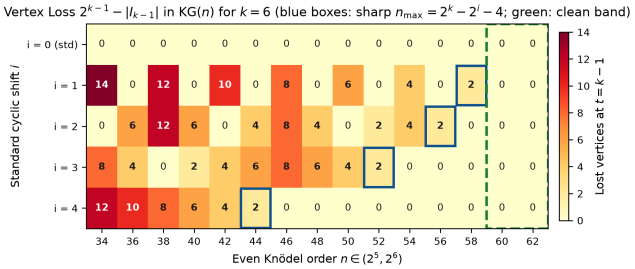

Figure 1: Exact vertex loss $2^{k-1} - |I_{k-1}|$ in KG(n) for $k = 6$ ($n \in [34, 62]$) across standard cyclic shifts $i \in \{0, \ldots, 4\}$. Blue frames mark the sharp Theorem A threshold $n_{\max}(6, i) = 64 - 2^i - 4$; the green dashed box marks the universally collision-free upper band $n \in \{60, 62\}$.

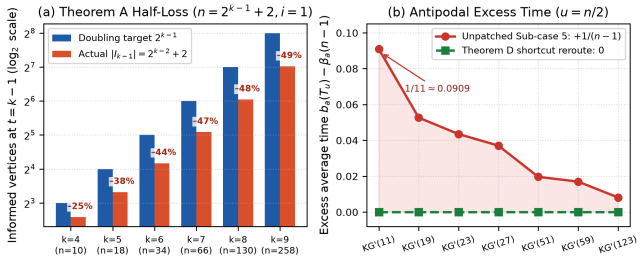

Figure 2: (a) Theorem A asymptotic half-loss at $n = 2^{k-1}+2, i = 1$ for $k = 4, \ldots, 9$ ($|I_{k-1}| = 2^{k-2}+2$, losing up to 49% of 2^{k-1}). (b) Antipodal excess average broadcast time $b_a(T_u) - \beta_a(n-1)$ at $u = n/2$: unpatched Sub-case 5 loses $+1/(n-1)$, while Theorem D shortcut rerouting achieves exact 0 excess.

$0 \le t \le k-1$. Part (4): At $n = 2^{k-1}+2$ and $i = 1$ ($j = 0, j' = 1$), $4p = (2^{k-1}+2)+2-4(2^{k-3}+1-p)$ for all $1 \le p \le 2^{k-3}-1$, giving $|A^{(1)}_{k-1} \cap (B'^{(1)}_{k-1} + n)| = 2^{k-3} - 1$ and vertex deficit $2^{k-2} - 2$. □

Figure 1 illustrates Theorem A at $k = 6$: shifted schemes used in Theorem 4.1 and Proposition 3.7 of [1] suffer widespread collisions (e.g. $|I_4| = 14 < 16$ at $n = 24, u = 5$ and $n = 26, u = 1$; $|I_6| = 34 < 64$ at $n = 66, i = 1$). Even when shifts are collision-free, a second obstruction blocks MATBG doubling in KG$'(n-1)$:

Theorem B (Universal Antipodal Dimensional Obstruction)**.** *For every $k \ge 4$ and every $n = 4m$ with $2^{k-1} < n < 2^k$ ($2^{k-3} - 1$ even orders per dyadic band, i.e. odd orders $n - 1 \equiv 3 \pmod 4$), from the antipodal originator $u = n/2 = 2m$, all $2(k-1)$ standard and reverse cyclically shifted dimensional schemes $D^{(i)}_{\text{std}}, D^{(i)}_{\text{rev}}$ ($0 \le i \le k-2$) inform vertex 0 by round $k-1$. Consequently, no dimensional scheme from Section 3 of [1] achieves $|I_{k-1}| = 2^{k-1}$ in KG$'(n-1)$ from $u = n/2$.*

Proof. Since $n = 4m < 2^k$, $m < 2^{k-2}$. For $i = 0$, $p = m < 2^{k-2}$ yields $2m + 2m = n \equiv 0$ under $D^{(0)}_{\text{std}}$ and $-2m + 2m = 0$ under $D^{(0)}_{\text{rev}}$. For $D^{(i)}_{\text{std}}$ ($1 \le i \le k-2$), write $m = q\,2^i + r$ ($0 \le r < 2^i$); then $q < 2^{k-i-2}$ since $2^{k-i-2} \cdot 2^i = 2^{k-2} > m$. If $r < 2^{i-1}$, $(p, j) = (q, r)$ gives $p\,2^{i+1} + 2j + 2m = 4m = n \equiv 0$ in $A^{(i)}_{k-1}$; if $r \ge 2^{i-1}$, $(s, j) = (q+1, 2^i - r)$ gives $-s\,2^{i+1} + 2j + 2m = 0$ in $B'^{(i)}_{k-1}$. For $D^{(i)}_{\text{rev}}$ ($1 \le i \le k-2$), dividing $m = q'\,2^{k-i-1} + r'$ places 0 in $A^{(i)}_{\text{rev},k-1}$ when $r' < 2^{k-i-2}$ and in $B'^{(i)}_{\text{rev},k-1}$ when $r' \ge 2^{k-i-2}$. □

A third obstruction occurs for neighbors of vertex 0 across the entire first half of $(2^{k-1}, 2^k)$:

Theorem C (First-Half Neighbor Shortcut Collision)**.** *For every $k \ge 4$ and every even n satisfying $2^{k-1} + 2 \le n \le 3 \cdot 2^{k-2} - 2$, from originator $u = 2^{k-1} - 1$ under Theorem 4.1's reverse-shifted scheme $D^{(1)}_{\text{rev}} = (k-2, \ldots, 1, k-1, k-2)$, the shortcut target $w = 2^{k-2} - 1$ already belongs to $B'^{(1)}_{\text{rev},k-1}$ via $s = (3 \cdot 2^{k-2} - n)/2 \in [1, 2^{k-3}]$. Hence redirecting $(u \to 0)$ to $(u \to w)$ at round $k-1$ fails to inform a new vertex, yielding $|I_{k-1}| < 2^{k-1}$.*

Proof. Since n is even and $2^{k-1} + 2 \le n \le 3 \cdot 2^{k-2} - 2$, $s = (3 \cdot 2^{k-2} - n)/2 \in [1, 2^{k-3}]$. In (5) shifted by u at $i = 1, j = 1$, the element $2^{k-1} - 2s + u \in B'^{(1)}_{\text{rev},k-1}$ equals $2^{k-1} - (3 \cdot 2^{k-2} - n) + (2^{k-1} - 1) = n + (2^{k-2} - 1) \equiv w \pmod n$. □

3 Constructive Repairs and Gap-Free Upper-Band MATBGs

Despite Theorem B, a single non-dimensional call across the existing shortcut edge $(2^{k-1} - 1, 2^{k-2} - 1) \in E(\text{KG}'(n-1))$ repairs Sub-case 5 universally:

Theorem D (Universal Sub-case 5 Shortcut Rerouting)**.** *Let $k \ge 4$, $2^{k-1} < n < 2^k$ be even, and u be even with $\max(2^{k-2}+2, n-2^{k-1}+2) \le u \le \min(2^{k-1}-2, n-2^{k-2})$ (which holds automatically for every Sub-case 5 originator whenever $n \ge 3 \cdot 2^{k-2}$). Under the collision-free standard scheme $D^{(0)}_{\text{std}} = (1, \ldots, k-1, 1)$ on* KG(n)*:*

1. $0 \notin I_{k-2}$;
2. $v_{\text{top}} = 2^{k-1} - 1 \in B_{k-2}$ *via* $s_1 = (u + 2^{k-1} - n)/2 \in [1, 2^{k-3}]$;
3. *the shortcut partner* $v_{\text{pair}} = 2^{k-2} - 1 \notin I_{k-1}$;
4. $1 = \phi_1(0) \in B_{k-1}$ *via* $s_2 = (u+2)/2 \in [1, 2^{k-2}]$.

Therefore, rerouting the single round-$(k-1)$ call $(v_{\text{top}} \to 0)$ across the shortcut edge $(v_{\text{top}}, v_{\text{pair}}) \in E(\text{KG}'(n-1))$ produces a valid telephone broadcast tree T_u in KG$'(n-1)$ *with exact doubling $|I_t| = 2^t$ ($0 \le t \le k-1$), full coverage $|I_k| = n - 1$, and exact minimum average time $b_a(T_u) = \beta_a(n-1)$.*

Proof. (1) Elements of B_{k-2} are odd mod n while $0 < u \le 2p + u \le 2^{k-2} - 2 + u < n$ for $p < 2^{k-3}$, so $0 \notin A_{k-2} \cup B_{k-2}$. (2) For $s_1 = (u + 2^{k-1} - n)/2 \in [1, 2^{k-3}]$, $2s_1 - 1 + n - u = 2^{k-1} - 1 < n$, so $v_{\text{top}} \in B_{k-2}$. (3) A_{k-1} is even mod n

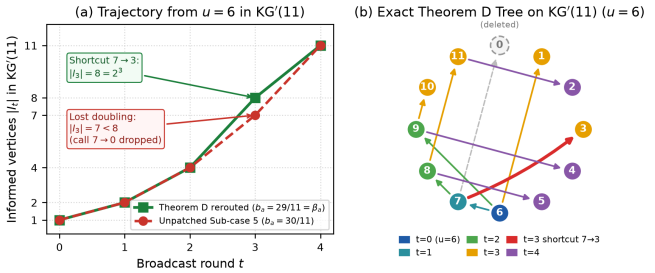

Figure 3: Exact comparison on KG′(11) from antipodal originator $u = 6$ (certificate `C09`). (a) Unpatched Sub-case 5 drops call $7 \to 0$ at round 3 ($|I_3| = 7 < 8$, $b_a = 30/11$), whereas Theorem D rerouting $7 \to 3$ restores $|I_3| = 8$ and $\beta_a(11) = 29/11$. (b) Complete valid telephone tree in KG′(11) with the red round-3 shortcut $7 \to 3$.

while $v_{\text{pair}} = 2^{k-2} - 1$ is odd. If $2s - 1 + n - u \equiv 2^{k-2} - 1 \pmod{n}$ with $1 \le s \le 2^{k-2}$, then either $2s - 1 + n - u < n$ giving $u = 2s + n - 2^{k-2} > n - 2^{k-2}$ (contradiction), or $2s - 1 - u = 2^{k-2} - 1$ giving $2s = u + 2^{k-2} > 2^{k-1}$ (contradiction). Thus $v_{\text{pair}} \notin I_{k-1}$. (4) For $s_2 = (u + 2)/2 \in [1, 2^{k-2}]$, $2s_2 - 1 + n - u = 1 + n \equiv 1 \pmod{n}$, so $1 \in B_{k-1}$. By Theorem A(1), $D^{(0)}_{\text{std}}$ doubles without collision through round $k - 1$. Replacing $(v_{\text{top}} \to 0)$ by $(v_{\text{top}} \to v_{\text{pair}})$ at round $k - 1$ yields $I'_{k-1} = (I_{k-1} \setminus \{0\}) \cup \{v_{\text{pair}}\}$ of exact cardinality 2^{k-1}, and preserves round-k full coverage because 0's unique round-k target $1 = \phi_1(0)$ already lies in I'_{k-1}. Finally, $(v_{\text{top}}, v_{\text{pair}}) \in E(\text{KG}'(n-1))$ for every $k \ge 4$ regardless of the parity of $k - 1$. □

Figure 3 displays the exact Theorem D repair on KG′(11) from $u = 6$, restoring $b_a = 29/11 = \beta_a(11)$. On the sole remaining first-half Sub-case 5 boundary $u = 2^{k-2}$, a 2-adic shift avoids vertex 0 entirely (for worst-case k-round broadcasting; note that shifted schemes in the first half may still collide in KG(n) per Theorem A, e.g. $|I_6| = 58 < 64$ in `C12`):

Theorem E (2-Adic Boundary Shift for $u = 2^{k-2}$)**.** *For every $k \ge 4$ and $n = 2^{k-1} + 2m$ with $1 \le m < 2^{k-3}$, write $m = 2^{s-1}(2q + 1)$ where $s = 1 + v_2(m) \in [1, k - 3]$. Then from originator $u = 2^{k-2}$ under $D^{(s)}_{\text{std}}$, vertex $0 \notin I_{k-1} \pmod{n}$.*

Proof. Since u is even, $0 \notin B^{(s)}_{k-1} \cup A'^{(s)}_{k-1}$ by parity. For any $0 \le p < 2^{k-s-2}$ and $0 \le j < 2^{s-1}$, we have $0 < p\,2^{s+1} + 2j + u \le 2^{k-1} - 2^{s+1} + 2^s - 2 + 2^{k-2} < 3 \cdot 2^{k-2} < 2n$, so congruence to 0 $\pmod{n}$ forces exact equality $p\,2^{s+1} + 2j + 2^{k-2} = n = 2^{k-1} + 2m$. Dividing by 2 gives $p\,2^s + j = 2^{k-3} + 2^{s-1}(2q + 1)$; reducing mod 2^s (since $s \le k - 3 \implies 2^s \mid 2^{k-3}$) yields $j \equiv 2^{s-1} \pmod{2^s}$, contradicting $j < 2^{s-1}$. Similarly, for $1 \le t \le 2^{k-s-2}$ and $1 \le j \le 2^{s-1}$, $0 < n + 2j + u - t\,2^{s+1} < 2n$, so congruence to 0 $\pmod{n}$ forces $t\,2^{s+1} = 2j + 2^{k-2}$, which reduces mod 2^s to $0 \equiv j \pmod{2^s}$, contradicting $1 \le j \le 2^{s-1}$. □

Combining Theorem A(3), Theorem D, and universal neighbor-partner separation yields a complete, gap-free MATBG theorem on the upper band of [7]:

Theorem F (Gap-Free MATBG Proof for $2^k - 5$ and $2^k - 3$)**.** *For every $k \ge 4$, both $\text{KG}'(2^k - 3)$ and $\text{KG}'(2^k - 5)$ are minimum average time broadcast graphs with exact edge counts $B_a(n_{\text{odd}}) = \lceil n_{\text{odd}}(k-1)/2 \rceil$. Conversely, for $n_{\text{odd}} = 2^k - 1$ ($k \ge 3$), Labahn's lower bound [6] satisfies $2(k+1)\lceil n_{\text{odd}}(k-1)/2 \rceil < (k+1)n_{\text{odd}}(k-1) + n_{\text{odd}}$, strictly excluding $2^k - 1$.*

Proof. Let $n \in \{2^k - 4, 2^k - 2\}$ ($k \ge 4$, so $n \ge 3 \cdot 2^{k-2}$). By Theorem A(3), all $2(k-1)$ standard and reverse shifts double without collision through round $k - 1$ in KG(n), and by [2] cover KG(n) in k rounds. (i) *Non-neighbors of 0 outside Sub-case 5:* By Proposition 3.7 of [1], the assigned shift avoids 0 through round $k - 1$, so deleting 0 leaves $|I_t| = 2^t$ ($t \le k - 1$) and $|I_k| = n - 1$ intact. (ii) *Sub-case 5:* For $n = 2^k - 2$, Sub-case 5 is empty ($n - 2^{k-1} + 2 = 2^{k-1} > 2^{k-1} - 2$). For $n = 2^k - 4$, Sub-case 5 consists solely of $u = 2^{k-1} - 2 = n/2 > 2^{k-2}$, which achieves exact doubling and $\beta_a(n-1)$ by Theorem D. (iii) *Neighbors of 0 ($u = 2^i - 1$, $1 \le i \le k - 1$):* By the automorphism f_u for odd u, an odd vertex w lies in $I_{k-1}(u)$ iff $w - u \in A_{k-1}(0) \cup B'_{k-1}(0) \pmod{n}$. If $i \le k - 2$ is odd, Theorem 4.1 uses $D^{(i)}_{\text{std}}$ and shortcut partner $w = 2^{i+1} - 1$, so $w - u = 2^i \in (0, n)$. Here $p\,2^{i+1} + 2j \equiv 2j \not\equiv 2^i \pmod{2^{i+1}}$ ($j < 2^{i-1}$) excludes $A^{(i)}_{k-1}(0)$, while $n - s\,2^{i+1} + 2j' \ge n - 2^{k-1} + 2 \ge 2^{k-2} + 2 > 2^i$ excludes $B'^{(i)}_{k-1}(0) + n$. If $i \ge 2$ is even or $i = k - 1$ is odd, Theorem 4.1 uses $D^{(r)}_{\text{rev}}$ ($r = k - i \in [1, k-2]$) and $w = 2^{i-1} - 1$, so $w - u \equiv n - 2^{i-1} \pmod{n}$. In (5), $j\,2^i + 2p \equiv 2p \not\equiv 2^{i-1} \pmod{2^i}$ ($p < 2^{i-2}$) excludes $A^{(r)}_{\text{rev},k-1}(0) + n$, while $j'\,2^i - 2s \le 2^{k-1} - 2 < 2^{k-1} \le n - 2^{i-1}$ excludes $B'^{(r)}_{\text{rev},k-1}(0)$. Because D is collision-free and $\phi_{\alpha_{k-1}}(u) = 0$, necessarily $0 \notin I_{k-2}$; redirecting $(u \to 0)$ to $(u \to w)$ at round $k - 1$ preserves $|I_{k-1}| = 2^{k-1}$ and places 0's unique round-k target $\phi_{\alpha_k}(0) = w$ already into I'_{k-1}. Finally, $(nd)/2 - d + \lceil d/2 \rceil = \lceil (n-1)d/2 \rceil$ matches the MATBG degree lower bound. For $n_{\text{odd}} = 2^k - 1$ ($k \ge 3$), $2\lceil n_{\text{odd}}(k-1)/2 \rceil \le n_{\text{odd}}(k-1) + 1$; multiplying by $k + 1$ and applying $k + 1 < 2^k - 1 = n_{\text{odd}}$ gives $2(k+1)\lceil n_{\text{odd}}(k-1)/2 \rceil < (k+1)n_{\text{odd}}(k-1) + n_{\text{odd}}$, so $B_a(2^k - 1) \ge B(2^k - 1) > \lceil (2^k - 1)(k-1)/2 \rceil$ by [6]. □

4 Machine-Checked Lean 4 Kernel and Zero-Float Audit

We separate universal symbolic verification from finite explicit schedule validation:

Lean 4.35.0 Structural Kernel. The file `lean/KnodelMatbgCert.lean` defines the modular Knödel neighbor map `kgNeighbor` directly and proves all 24 universal theorems in Table 1 without Mathlib or external dependencies.

Table 1: All 24 universal modular-arithmetic theorems formally verified in `lean/KnodelMatbgCert.lean` (Lean 4.35.0, zero external dependencies, 0 `sorry`, 0 custom axioms, 0 `Classical.choice`, 0 warnings; axioms strictly `[propext, Quot.sound]`).

#	Lean 4 Theorem Identifier	Universally Quantified Formal Statement (all variables in $\mathbb{N}$)
1	`kg_neighbor_involution`	$0 < n \wedge v < n \implies \phi_d(\phi_d(v)) = v$
2	`lemma31_step_A_to_B`	$t \geq 1, p < 2^{t-1} \implies \phi_{t+1}((2p+u) \bmod n) = (2(2^t - p) - 1 + n - u) \bmod n$
3	`lemma31_step_B_to_A`	$t \geq 1, 1 \leq s \leq 2^{t-1} \implies \phi_{t+1}((2s-1+n-u) \bmod n) = (2(2^t - s) + u) \bmod n$
4	`lemma31_collision_free`	$2 \mid n, t \leq k-1, 2^{k-1} < n \implies A_t, B_t$ injective and disjoint mod n
5	`theoremA_sharp_collision_witness`	$k \geq 4, 1 \leq i \leq k-2, 2^i + 4 < 2^{k-1} \implies$ exact $A_{k-1} \cap (B'_{k-1} + n)$ witness at $n = 2^k - 2^i - 4$
6	`theoremA_collision_free_above_threshold`	$n > 2^k - 2^i - 4 \implies p\,2^{i+1} + 2j + s\,2^{i+1} < n + 2j'$ (std shift zero collisions)
7	`theoremA_extreme_half_loss`	$k \geq 4, 1 \leq p \leq 2^{k-3} - 1, n = 2^{k-1} + 2 \implies 4p = n + 2 - 4(2^{k-3} + 1 - p)$
8	`theoremB_antipodal_std_rev_hits_zero`	$k \geq 4, 2^{k-1} < 4m < 2^k \implies m < 2^{k-2} \wedge (2m + 2m) \bmod 4m = 0 \wedge (2m - 2m) \bmod 4m = 0$
9	`theoremB_antipodal_all_std_shifts_hit_zero`	$k \geq 4, 4m < 2^k, 1 \leq i \leq k-2 \implies \exists$ witness in $A^{(i)}_{k-1}$ or $B'^{(i)}_{k-1}$ hitting 0 (mod $4m$)
10	`theoremB_antipodal_all_rev_shifts_hit_zero`	$k \geq 4, 4m < 2^k, 1 \leq i \leq k-2 \implies \exists$ witness in Lemma 3.4 A_{k-1} or B'_{k-1} hitting 0 (mod $4m$)
11	`theoremC_first_half_shortcut_already_informed`	$k \geq 4, 2 \mid n, 2^{k-1} + 2 \leq n \leq 3 \cdot 2^{k-2} - 2 \implies s = (3 \cdot 2^{k-2} - n)/2 \in [1, 2^{k-3}]$ hits $w = 2^{k-2} - 1$
12	`theoremD_subcase5_zero_not_in_I_km2`	Sub-case 5 hypotheses $\implies 0 \notin A_{k-2} \cup B_{k-2}$ (mod n)
13	`theoremD_subcase5_sender_in_I_km2`	Sub-case 5 hypotheses $\implies s_1 = (u + 2^{k-1} - n)/2 \in [1, 2^{k-3}]$ places $2^{k-1} - 1 \in B_{k-2}$
14	`theoremD_subcase5_shortcut_target_uninformed`	$2^{k-2} < u \leq n - 2^{k-2}, 2 \mid n, 2 \mid u \implies 2^{k-2} - 1 \notin A_{k-1} \cup B_{k-1}$ (mod n)
15	`theoremD_subcase5_one_in_I_km1`	$2 \mid u, u \leq 2^{k-1} - 2, 2^{k-1} < n \implies s_2 = (u+2)/2 \in [1, 2^{k-2}]$ places $1 \in B_{k-1}$
16	`theoremE_2adic_shift_avoids_zero`	$k \geq 4, 1 \leq s \leq k-3, m = 2^{s-1}(2q+1) < 2^{k-3} \implies 0 \notin A^{(s)}_{k-1} \cup B'^{(s)}_{k-1}$ (mod n)
17	`theoremF_kg_prime_edge_count`	$2 \mid n, n \geq 2, d \geq 1 \implies (nd)/2 - d + (d+1)/2 = ((n-1)d + 1)/2$
18	`theoremF_labahn_lower_bound_exceeds_ceiling`	$k \geq 3, n = 2^k - 1 \implies 2(k+1)\lceil n(k-1)/2 \rceil < (k+1)n(k-1) + n$
19	`theoremF_upper_band_subcase5_characterization`	$k \geq 4 \implies$ Sub-case 5 empty at $2^k - 2$; unique $u = 2^{k-1} - 2 > 2^{k-2}$ at $2^k - 4$
20	`theoremD_second_half_eliminates_boundary`	$k \geq 4, 3 \cdot 2^{k-2} \leq n, n - 2^{k-1} + 2 \leq u \implies 2^{k-2} < u$
21	`theoremA_rev_collision_free_above_threshold`	$n > 2^k - 2^{k-i-1} - 4 \implies j'\,2^{k-i} + j\,2^{k-i} + 2p < n + 2s$ (rev shift zero collisions)
22	`lemma33_rev_unshifted_injective`	$1 \leq t \leq k-1, 2^{k-1} < n \implies A^{\mathrm{rev}}_t = \{(n - p\,2^{k-t}) \bmod n\}$ injective mod n
23	`theoremF_neighbor_partner_uninformed_std`	$n \geq 3 \cdot 2^{k-2}, 1 \leq i \leq k-2 \implies 2^i \notin A^{(i)}_{k-1}(0)$ and $2^i < n + 2j' - s\,2^{i+1}$
24	`theoremF_neighbor_partner_uninformed_rev`	$n \geq 3 \cdot 2^{k-2}, 2 \leq i \leq k-1 \implies 2^{i-1} \notin -A^{(k-i)}_{\mathrm{rev}}(0)$ and $j'\,2^i - 2s < n - 2^{i-1}$

Table 2: Canonical certificate suite (`C01`–`C16`) audited by the AST-verified zero-float Python engine `KNODEL-MATBG-CERT-1.0`.

ID	Mode / Target Theorem	Instance Parameters	Exact Audited Metric	Outcome
`C01`	Mode 1 (Thm. A sharp)	$k = 5, n = 26, i = 1$ (std)	$\lvert I_4 \rvert = 14 < 16$, witness $x = 12$ ($p = 3, j = 0, s = 4, j' = 1$)	PASS
`C02`	Mode 1 (Thm. A sharp)	$k = 6, n = 52, i = 3$ (std)	$\lvert I_5 \rvert = 30 < 32$, witness $x = 22$ ($p = 1, j = 3, s = 2, j' = 1$)	PASS
`C03`	Mode 1 (Thm. A half-loss)	$k = 7, n = 66, i = 1$ (std)	$\lvert I_6 \rvert = 34$, exact loss $30 = 2^5 - 2$ (15 even + 15 odd)	PASS
`C04`	Mode 1 (Thm. A clean band)	$k = 7, n = 124$ (all 12 shifts)	$\lvert I_t \rvert = 2^t$ ($t \leq 6$), 0 collisions across all 12 shifts	PASS
`C05`	Mode 2 (Thm. B antipodal)	$k = 4, n = 12, u = 6$	6/6 dimensional schemes hit 0 by round 3	PASS
`C06`	Mode 2 (Thm. B antipodal)	$k = 6, n = 52, u = 26$	10/10 dimensional schemes hit 0 by round 5	PASS
`C07`	Mode 2 (Thm. B antipodal)	$k = 7, n = 124, u = 62$	12/12 dimensional schemes hit 0 by round 6	PASS
`C08`	Mode 2 (Thm. C first-half)	$k = 6, n = 46, u = 31, w = 15$	$w = 15 \in I_5$ ($s = 1$); 0 uninformed neighbors of u	PASS
`C09`	Mode 3 (Thm. D reroute)	$k = 4, n = 12, u = 6$ ($\mathrm{KG}'(11)$)	Shortcut $7 \to 3$: $b_a = 29/11 = \beta_a(11)$ vs unpatched 30/11	PASS
`C10`	Mode 3 (Thm. D reroute)	$k = 6, n = 52, u = 26$ ($\mathrm{KG}'(51)$)	Shortcut $31 \to 15$: $b_a = 81/17 = \beta_a(51)$ vs 244/51	PASS
`C11`	Mode 3 (Thm. D reroute)	$k = 7, n = 124, u = 62$ ($\mathrm{KG}'(123)$)	Shortcut $63 \to 31$: $b_a = 734/123 = \beta_a(123)$ vs 245/41	PASS
`C12`	Mode 3 (Thm. E 2-adic)	$k = 7, n = 72, u = 32$ ($m = 4$)	Shift $s = 1 + v_2(4) = 3$: $0 \notin I_6$, $\lvert I_7 \rvert = 72$	PASS
`C13`	Mode 4 (Thm. F MATBG)	$\mathrm{KG}'(11)$ ($2^4 - 5$, 17 edges)	All 11/11 originators exact $\beta_a(11) = 29/11$ (1 reroute)	PASS
`C14`	Mode 4 (Thm. F MATBG)	$\mathrm{KG}'(27)$ ($2^5 - 5$, 54 edges)	All 27/27 originators exact $\beta_a(27) = 104/27$ (1 reroute)	PASS
`C15`	Mode 4 (Thm. F MATBG)	$\mathrm{KG}'(59)$ ($2^6 - 5$, 148 edges)	All 59/59 originators exact $\beta_a(59) = 291/59$ (1 reroute)	PASS
`C16`	Mode 4 (Thm. F MATBG)	$\mathrm{KG}'(61)$ ($2^6 - 3$, 153 edges)	All 61/61 originators exact $\beta_a(61) = 303/61$ (0 reroutes)	PASS

Table 1 (#5) enforces $2^i + 4 < 2^{k-1}$ (excluding the degenerate point $(k, i) = (4, 2)$ where $2^4 - 2^2 - 4 = 8 = 2^3$ lies outside $(2^3, 2^4)$); rows #21–#24 formally verify reverse-shift separation, $D^{(0)}_{\mathrm{rev}}$ injectivity, and neighbor-partner separation across all $n \geq 3 \cdot 2^{k-2}$. Every theorem passes `#print axioms` using only `[propext, Quot.sound]`—zero `sorry`, zero custom axioms, zero `Classical.choice` (using a constructive Euclidean division lemma in place of Lean's classical `Nat.div_lt_of_lt_mul`), and zero compiler warnings.

Zero-Float Python Verifier and Adversarial Suite. The engine `src/knodel_matbg_verifier.py` enforces strict integer and lowest-terms `fractions.Fraction` arithmetic via an AST audit banning float/complex literals, `float()`, `complex()`, true-division `/`, exponentiation `**`, and floating-point libraries. Across Modes 1–4 (Table 2), it builds explicit graphs $\mathrm{KG}(n)$ and $\mathrm{KG}'(n-1)$, validates telephone-model matching constraints end-to-end (including both unpatched and rerouted schedules in Mode 3), and checks all 16 certificates (`C01`–`C16`). The test suite `tests/test_verifier.py` executes 48 checks in under 1 s, including 23 adversarial forgery-rejection tests (rejecting `bool`-as-`int`, unreduced fractions, double calls per round, non-edge calls, excluded $n_{\mathrm{odd}} = 15$, non-bool optional flags, `**` operators, and falsely claiming Theorem 4.1's dimensional schedule is a MATBG on the colliding order $\mathrm{KG}'(25)$), plus a first-half regression check on $\mathrm{KG}'(17)$.

5 Scope, Open Direction, and Reproducibility

Scope and Open Problem Status. Our results confirm that Altay, Fakharan, and Harutyunyan's broadcast-function bound $B(n-1) \leq \lceil (n-1)(k-1)/2 \rceil$ (even $n \neq 2^k$) holds, while demonstrating that Section 5's MATBG claim is unproved for intermediate orders $n \in [2^{k-1}+2, 2^k-6]$ where shifted dimensional schemes collide in $\mathrm{KG}(n)$. Theorem F rigorously closes the MATBG problem on the upper band $n_{\mathrm{odd}} \in \{2^k-5, 2^k-3\}$ for all $k \geq 4$ (relying on Bermond et al. [2] for k-round coverage of dimensional schemes on $\mathrm{KG}(n)$ and Proposition 3.7 of [1] for $0 \notin I_{k-1}$ on non-neighbor originators outside Subcase 5, with all collision-freedom, antipodal rerouting, and neighbor-partner separation theorems machine-checked in Lean 4). Constructing a universal collision-free MATBG family across the remaining interior band $[3 \cdot 2^{k-2}, 2^k-6]$ remains an open problem.

Reproducibility. All Lean 4 proofs, Python verification code, 16 JSON certificates with `MANIFEST.sha256`, 48 unit and forgery tests, and figure scripts are archived under MIT license at `github.com/lluiseriksson/knodel-matbg-obstruction-repair` (`v1.0.0`) and reproduce via `lake build`, `python src/knodel_matbg_verifier.py certificates`, and `pytest -q` under Lean 4.35.0 and Python 3.12+.

References

[1] Sirma Cagil Altay, Mohammadhossein Fakharan, and Hovhannes A. Harutyunyan. A general upper bound on broadcast function $B(n)$ using Knödel graph, November 2026. arXiv:2611.34167v1.

[2] Jean-Claude Bermond, Hovhannes A. Harutyunyan, Arthur L. Liestman, and Stéphane Perennes. A note on the dimensionality of modified Knödel graphs. *International Journal of Foundations of Computer Science*, 8(2):109–116, 1997.

[3] Arthur M. Farley. Minimal broadcast networks. *Networks*, 9(4):313–332, 1979.

[4] Hovhannes A. Harutyunyan. An efficient vertex addition method for broadcast networks. *Internet Mathematics*, 5(3):211–225, 2008.

[5] Walter Knödel. New gossips and telephones. *Discrete Mathematics*, 13(1):95, 1975.

[6] Roger Labahn. A minimum broadcast graph on 63 vertices. *Discrete Applied Mathematics*, 53(1–3):247–250, 1994.

[7] Arthur L. Liestman and Nataša Pržulj. Minimum average time broadcast graphs. *Parallel Processing Letters*, 8(2):139–147, 1998.