Projective Memory and Resonant Bottlenecks in Passive Multiport Routing:
Exact Global Laws, Robust Error Floors, and High-Delay Border Phases

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Abstract

At distinct boundary frequencies ζ_i, let a square rational-inner multiport route one fixed input ray to prescribed output rays Y_i. Put $H = \mathrm{span}(Y_1, \dots, Y_L)$ and let δ_H be the least degree of a base-point-free projective map $R : \mathbb{P}^1 \to \mathbb{P}(H)$ satisfying $R(\zeta_i) = Y_i$. We prove the global identity

$$d_{\min} = \delta_H,$$

independently of the ambient port count. A scalar Fejér–Riesz factorization followed by a same-state-dimension rectangular lossless completion proves attainability. Conversely, a common realization denominator and the strict inequality $d_{\min} < L$ force every optimal routed column to lie identically in H, where it becomes a projective interpolant. This converts the full line- routing problem into a finite algebraic degree problem and contains the direct-sum, repeated-target, and coplanar laws as strata.

For orthogonal target bands we solve line-to-subspace incidence constraints. Full-support Lagrange kernels give every exact degree, while the generic law is a closed codimension formula. Exact memory takes the maximum shifted band cost, whereas border memory takes the minimum; this max–min split can be arbitrarily large.

For $r = \dim H$, orthonormal annihilators of the targets define an intrinsic incidence matrix A_d with $L(r-1)$ rows and $r(d+1)$ columns. Its minimum modulus gives the calibrated error floor

$$E_d \geq \frac{\sigma_{\min}(A_d)}{\sqrt{L(d+1) + \sigma_{\min}(A_d)^2}},$$

stable under operator-norm perturbations. More sharply, $E_d = 0$ if and only if A_d has a kernel. Hence border memory obeys both a linear rank law and the all-data base-point deletion formula

$$d_\partial = \min\{d : \ker A_d \neq 0\} = \min_{B \subseteq [L]} \big(|B| + \Delta(B^c)\big).$$

The universal dense-routing threshold is $\lfloor L(r-1)/r \rfloor$ and is generically sharp. In the planar stratum the zero-error closure is the determinantal locus cut out by the maximal minors of M_d, and all border degrees are found by one incremental exact elimination using $O(L^3)$ field operations.

The closure phase has an operational cost. If $d_\partial \leq d < d_{\min}$, then arbitrarily accurate degree-d routers exist but every such approximating sequence has divergent peak Wigner–Smith delay, $\|\operatorname{tr} Q\|_\infty \to \infty$. Under any finite peak-delay cap the best error is strictly positive and attained. Thus the exact trichotomy is physical as well as algebraic: robust positive-error phase, singular high-delay border phase, and exact finite-delay phase. In the planar stratum we additionally solve degree zero by the smallest Bloch-sphere cap, prove the exact binary occupancy law, classify all four-node tables by multiplicity and cross-ratio, and certify a strict 21% one-state error barrier. Exact artifacts and an independent verifier accompany the theorem fixtures. Classical rational interpolation and lossless completion are credited as ingredients; the result claimed here is their global state–error–delay synthesis.

Keywords: rational inner matrix; projective interpolation; McMillan degree; passive memory; border memory; resonant approximation; chordal error; lossless completion; Wigner–Smith delay.

Table 1: Classical interfaces and the paper-specific deductions. The first two columns are not claimed as new.

Classical interface	What it supplies	Deduction established here
Minimal rational interpolation [4, 5]	Rational/projective interpolants through finite data	Equality of projective degree and passive McMillan state count for every one-line routing table
Rectangular lossless completion [13]	Same-state square para-unitary completion	Exact lossless compiler attaining the projective lower bound in the ambient multiport
Base points and unattainable data [14, 15]	Common-factor and nonattainment mechanisms	Rank/error dichotomy, deletion law, determinantal closure, and the exact versus border max–min split
Potapov factors and group delay [16, 17, 18]	Factorization, winding and Poisson delay kernels	Divergent peak delay forced along every nonattained zero-error routing sequence

1 The global line-routing problem

Let S be an $N \times N$ rational matrix, analytic in $\mathbb{D}$, unitary almost everywhere on $\mathbb{T}$, and regular at distinct nodes $\zeta_1, \ldots, \zeta_L \in \mathbb{T}$. Its McMillan degree is the state dimension of a minimal conservative realization. Fix an input line $X \subset \mathbb{C}^N$ and target lines $Y_i \subset \mathbb{C}^N$. The routing task is

$$S(\zeta_i)X = Y_i, \qquad 1 \le i \le L. \tag{1}$$

Endpoint phases and bases are free. Write $d_{\min}$ for the least McMillan degree among all such S.

Put $H = \operatorname{span}(Y_1, \ldots, Y_L)$ and $r = \dim H$. Earlier exact occupancy laws cover direct-sum target families and give detector lower bounds on singular arrangements [1]. They do not give an equality theory for an arbitrary line table. The point of the present paper is that for one input line the missing invariant is the degree of a projective curve through the ordered data. Taking H to be the actual target span makes the resulting theorem global, not a fixed-rank special case.

Classical scalar, matrix, and tangential rational interpolation and their state-variable realizations describe minimal rational functions through finite data [2, 3, 4, 5, 6]. Minimal boundary interpolation by scalar unimodular functions and finite Blaschke products is also well developed [7, 8, 9]. Subspace Nevanlinna interpolation and boundary matrix all-pass construction provide the matrix context [10, 11, 12]. Rectangular lossless realization and same-state square completion are likewise classical [13]. Our contribution is not a new rational interpolation algorithm or a new completion theorem. It is the global identification of projective interpolation degree with passive state count, together with an intrinsic quantitative error operator, the exact border-memory law, and the divergent peak-delay theorem for every nonattained border phase. The complete planar and four-line classifications then become explicit singular strata of the global result.

The rank formula is the computational hinge. Exact memory asks whether a kernel vector polynomial is primitive and base-point-free, a nonlinear common- zero condition; border memory asks only whether a nonzero kernel vector exists. This separates exact realization from closure by one algebraic operation and turns the sign question for E_d into linear algebra.

2 Projective degree equals passive memory in every dimension

A vector polynomial $p = (p_1, \ldots, p_r)^T$ is *primitive* if its coordinates have no common projective zero, including at infinity. It then defines a morphism $[p] : \mathbb{P}^1 \to \mathbb{P}(H)$ whose degree is the maximum coordinate degree after removal of the common homogeneous factor.

Definition 2.1 (Global projective interpolation degree)**.** Let

$$\delta_H = \min\{\deg R : R : \mathbb{P}^1 \to \mathbb{P}(H), \quad R(\zeta_i) = Y_i \ (1 \le i \le L)\}.$$

For a subtable $A \subseteq [L]$ we write $\delta_H(A)$ for the same minimum on A, with $\delta_H(\varnothing) = 0$.

Lemma 2.2 (Exact-state lossless compiler)**.** *Let $p = (p_1, \dots, p_r)^T$ be a primitive polynomial vector of projective degree δ. Then there is an $r \times r$ rational-inner matrix S_p and a constant input vector u such that $S_p u$ represents $[p]$ and*

$$\deg p_a \le \delta, \qquad \deg h \le \delta, \qquad \deg_{\mathrm{McM}}(p/h) = \deg_{\mathrm{McM}} S_p = \delta,$$

where h is outer and $|h|^2 = \sum_a |p_a|^2$ on $\mathbb{T}$.

Proof. The positive Laurent polynomial $\sum_a |p_a|^2$ has bandwidth at most δ. Scalar Fejér–Riesz factorization therefore supplies an outer spectral factor h of degree at most δ, so $f = p/h$ is an inner $r \times 1$ column with at most δ states. If f admitted an n-state realization with $n < \delta$, clearing its common realization denominator would produce a polynomial projective representative of degree at most n, contrary to the minimal projective degree of $[p]$. Hence $\deg_{\mathrm{McM}} f = \delta$.

The completion interface is used with its convention made explicit. The function f is analytic in $\mathbb{D}$ and isometric on $\mathbb{T}$; reflect it to the exterior-disk coisometric row $f^\sharp(z) = f(1/\bar{z})^*$. The minimal realization completion [13, Theorem 2.1(ii)] extends $f^\sharp$ to a square para-unitary matrix with the same state dimension. Reflection back, followed by constant unitary changes of input and output bases, gives S_p without adding states. □

Theorem 2.3 (Global projective memory law)**.** *For every finite line-routing table at distinct boundary nodes,*

$$\boxed{d_{\min} = \delta_H.} \tag{2}$$

The equality is independent of the ambient port number. A minimum-degree projective interpolant compiles to a square lossless router of the same degree.

Proof. Vector Lagrange interpolation gives $\delta_H \le L - 1$. Choose a primitive polynomial representative $p = (p_1, \dots, p_r)^T$ of a minimum interpolant, all coordinates padded to degree δ_H. Lemma 2.2 compiles it to a square rational-inner router of exactly the same state dimension. An identity block embeds it in the ambient port space, proving $d_{\min} \le \delta_H$.

Conversely let S be a minimum-degree router, $d = \deg_{\mathrm{McM}} S$. The upper bound gives $d \le L - 1 < L$. For a unit vector x spanning the input line, a stable minimal realization writes the routed column as $F = Sx = n/\chi$, where every coordinate of the vector polynomial n and the common denominator χ has degree at most d, and $\chi(\zeta_i) \ne 0$. If ℓ annihilates H, then ℓn vanishes at all $L > d$ nodes, so $\ell n \equiv 0$. Thus F takes values identically in H. Cancel the common homogeneous factor from its r coordinate numerators. The resulting primitive projective map has degree at most d and interpolates every Y_i. Therefore $\delta_H \le d$. □

Corollary 2.4 (Span and direct-sum strata)**.** *Every table satisfies $\delta_H \ge r - 1$. If the L targets are linearly independent, then $r = L$ and $d_{\min} = L - 1$. For $r = 2$ the global invariant is the ordinary rational-map degree on $\mathbb{P}^1$ developed below.*

Proof. The image of a degree-d map $\mathbb{P}^1 \to \mathbb{P}(H)$ spans projective dimension at most d, because it is the linear projection of the degree-d Veronese curve. Hence $d \ge r - 1$. For independent targets, Lagrange interpolation gives the reverse bound $L - 1$. □

3 Block-subspace interpolation and full-support kernels

The line law admits a nontrivial incidence extension. Let

$$H = H_1 \oplus \cdots \oplus H_B, \qquad \dim H_b = r_b,$$

and partition the nodes into nonempty sets I_b, $|I_b| = L_b$. At a node $i \in I_b$, prescribe a nonzero subspace $Y_i \subseteq H_b$ of dimension s_i. The routed output is still one line, but it may be chosen freely in $\mathbb{P}(Y_i)$. Let $\Delta(J)$ be the least degree of a base-point-free vector polynomial satisfying the records in $J \subseteq [L]$, and set $\Delta(\varnothing) = 0$. Write $\Delta = \Delta([L])$; equivalently, Δ is the least degree of a base-point-free vector polynomial F satisfying

$$F(\zeta_i) \in Y_i \setminus \{0\}. \tag{3}$$

The proof of Theorem 2.3, with line equality replaced by incidence, gives $d_{\min} = \Delta$.

For each block, let $\ell_{b,i}$ be the scalar Lagrange polynomial on the nodes in I_b. If $0 \leq q \leq L_b - 1$, define

$$T_{b,q} : \bigoplus_{i \in I_b} Y_i \longrightarrow H_b \otimes \mathbb{C}^q$$

by taking the top q coefficient vectors of $\sum_{i \in I_b} \ell_{b,i}(z) u_i$. Put

$$q_b^* = \max\{q : \ker T_{b,q} \text{ contains } (u_i)_{i \in I_b} \text{ with every } u_i \neq 0\}. \tag{4}$$

The full-support clause in (4) is essential: an ordinary kernel may silently delete a record.

Lemma 3.1 (Hall–Vandermonde assignment). *Fix distinct nodes, r coordinate blocks, row demands $0 \leq c_i \leq r$, and $C = \sum_i c_i$. For polynomials of degree at most e, there is a choice of coordinate quotient spaces of dimensions c_i whose incidence matrix has rank*

$$\min\{C, r(e+1)\}.$$

Consequently maximal rank holds on a Zariski-open dense set of quotient spaces with the same dimensions.

Proof. View the c_i restrictions at node i as edges from that node to distinct coordinate blocks. If $C \leq r(e+1)$, seek a 0–1 assignment with row sums c_i and column sums at most $e+1$. Hall's capacitated condition holds: for a node subset J with $|J| \leq e+1$ its demand is at most $r|J|$, while for $|J| > e+1$ it is at most $C \leq r(e+1)$. The assigned evaluation rows split into coordinate Vandermonde blocks of at most $e+1$ rows, so they are linearly independent and the incidence matrix has full row rank.

If $C \geq r(e+1)$, first assign $e+1$ rows to every coordinate, using at most c_i rows at node i. For any set of k coordinates,

$$\sum_i \min\{c_i, k\} \geq \frac{k}{r} \sum_i c_i \geq k(e+1),$$

which is the dual Hall condition. Fill the remaining row demands arbitrarily. Each coordinate now contains an invertible $(e+1) \times (e+1)$ Vandermonde block, giving full column rank. In either case a maximal minor is nonzero for one coordinate configuration. Since that minor is polynomial in local Grassmannian coordinates, its nonvanishing locus is Zariski open and dense. $\square$

Theorem 3.2 (Exact full-support block law). *Every block-subspace table satisfies*

$$\boxed{d_{\min} = \Delta = \max_b\{L - L_b + L_b - 1 - q_b^*\} = L - 1 - \min_b q_b^*.} \tag{5}$$

For fixed dimensions (r_b, s_i), on a Zariski-open dense set of target subspaces,

$$d_{\min}^{\text{gen}} = \max_b \left\{ L - L_b + \left\lfloor \frac{\sum_{i \in I_b}(r_b - s_i)}{r_b} \right\rfloor \right\}. \tag{6}$$

For one r-dimensional block of line targets this becomes

$$d_{\min}^{\text{gen}} = \left\lfloor \frac{L(r-1)}{r} \right\rfloor. \tag{7}$$

Proof. Write $F = (F_b)_b$. At every node outside I_b, the component F_b must vanish. Hence

$$F_b(z) = Z_{\neg b}(z) G_b(z), \qquad Z_{\neg b}(z) = \prod_{i \notin I_b} (z - \zeta_i).$$

On I_b the scalar factor is nonzero. The unique polynomial G_b of degree at most $L_b - 1$ with chosen values $u_i \in Y_i$ is their Lagrange sum. It loses its top q coefficients precisely when $(u_i) \in \ker T_{b,q}$, and it serves every node precisely when all u_i are nonzero. At the maximal q it is primitive: division by a common scalar factor would preserve the nonzero node values and produce one additional vanished leading coefficient. The forced factor has degree $L - L_b$; simultaneous service of every block takes the maximum shifted degree. The global vector has no base point: its own block is nonzero at each record; away from all records every block is nonzero; and at infinity a maximum-degree block has nonzero leading vector. This proves (5). A full-support kernel exists exactly when no coordinate projection of $\ker T_{b,q}$ is zero, since finitely many proper linear subspaces cannot cover a complex vector space.

For genericity consider the incidence map at intrinsic degree e,

$$A_{b,e} : H_b \otimes \mathbb{C}[z]_{\leq e} \longrightarrow \bigoplus_{i \in I_b} H_b / Y_i.$$

Its generic rank is the smaller of its $r_b(e+1)$ columns and $C_b = \sum_i (r_b - s_i)$ rows by Lemma 3.1. The first generic kernel therefore occurs at $e = \lfloor C_b / r_b \rfloor$. There its dimension is $k = r_b(e + 1) - C_b \in \{1, \ldots, r_b\}$, whereas the kernel subspace vanishing at node i has generic dimension $\max\{0, k - s_i\} < k$. The finite union of these exceptional subspaces is not the whole kernel. The first full-support vector is primitive, or division would contradict minimality of e. Adding the forced shifts and taking the maximum proves (6); (7) is its one-block line specialization. □

For a fixed total line-record budget L, nonempty bands of dimensions r_b, and $R = \sum_b r_b$, optimizing the allocation gives the exact generic design law

$$d_{\text{opt}} = L - 1 - \left\lfloor \frac{L - B}{R} \right\rfloor. \tag{8}$$

It is attained whenever $L_b \geq r_b \lfloor (L - B)/R \rfloor + 1$ for every b; residual records may be assigned arbitrarily.

4 Incidence rank, robust error, and border memory

Choose for each Y_i an orthogonal quotient map $Q_i : H \to H \ominus Y_i$. On coefficient vectors of H-valued polynomials of degree at most d, define the intrinsic incidence matrix

$$A_d c = \big(Q_i p_c(\zeta_i) \big)_{i=1}^{L}. \tag{9}$$

Unitary basis changes do not change its singular values. Let E_d be the infimum worst-node chordal distance to the prescribed projective subspaces.

Theorem 4.1 (Rank–error dichotomy and base-point deletion)**.** *For every incidence table,*

$$E_d = 0 \quad \Longleftrightarrow \quad \ker A_d \neq \{0\}, \tag{10}$$

$$d_\partial := \min\{d : E_d = 0\} = \min_{B_0 \subseteq [L]} \big\{ |B_0| + \Delta(B_0^{\mathrm{c}}) \big\}. \tag{11}$$

If A_d has full column rank and $\sigma_d = \sigma_{\min}(A_d) > 0$, then

$$E_d \geq \boxed{\frac{\sigma_d}{\sqrt{L(d+1) + \sigma_d^2}}}. \tag{12}$$

If normalized data perturb A_d by operator norm at most $\eta < \sigma_d$, the same bound holds with σ_d replaced by $\sigma_d - \eta$.

In the block architecture of Theorem 3.2, write

$$\bar{e}_b = \min\{e : \ker A_{b,e} \neq 0\}.$$

Then

$$d_\partial = \min_b \{L - L_b + \bar{e}_b\}, \tag{13}$$

and generically

$$d_\partial^{\mathrm{gen}} = \min_b \left\{ L - L_b + \left\lfloor \frac{\sum_{i \in I_b} (r_b - s_i)}{r_b} \right\rfloor \right\}. \tag{14}$$

Thus exact and border memory use, respectively, the maximum and minimum of the shifted block costs.

Proof. If a router sequence has errors tending to zero, normalize the coefficients of the projected numerator and take a convergent subsequence. Quotient residuals converge to zero, producing a nonzero element of $\ker A_d$. Conversely take $0 \neq p \in \ker A_d$, let B_0 be the nodes where all coordinates vanish, and remove its scalar gcd. The reduced vector serves B_0^c and gives $|B_0| + \Delta(B_0^c) \leq d$.

For the reverse construction put $b = |B_0|$, $D = b + \Delta(B_0^c)$, choose a primitive minimum interpolant R of B_0^c, and set $P_0 = gR$ with $g(z) = \prod_{i \in B_0}(z - \zeta_i)$. The cases $B_0 = \varnothing$ and $\dim H = 1$ are immediate. Otherwise choose a functional λ nonzero on every Y_i, $i \in B_0$, representatives $a_i \in Y_i$ with $\lambda(a_i) = 1$, and their degree-at-most-$(b-1)$ Lagrange interpolant A_0. Then $\lambda(A_0) \equiv 1$. For $0 \neq v \in \ker \lambda$, the vector polynomial

$$A(z) = A_0(z) + g(z) z^{D-b} v$$

has exact degree D, serves B_0, has no finite base point because $\lambda(A) \equiv 1$, and has nonzero leading vector at infinity. Thus the projective pencil $P_0 + tA$ is not contained in the common-root locus: its point at $t = \infty$ is primitive. Properness of $\mathbb{P}^1$ makes the exceptional parameter set algebraic and finite. Choose nonzero $t_n \to 0$ outside it and also outside the single possible leading-cancellation value. The primitive degree-D maps $[P_0 + t_n A]$ serve B_0 exactly and converge to the required targets on B_0^c. Lossless completion proves (10)–(11).

For (12), write a routed unit column as n/χ and let c be the coefficients of its projection to H. If ϵ is the largest node error, orthogonal projection gives

$$\|Q_i p_c(\zeta_i)\| \leq |\chi(\zeta_i)|\epsilon, \qquad \|p_c(\zeta_i)\| \geq |\chi(\zeta_i)|\sqrt{1-\epsilon^2}.$$

Degree-d evaluation on $\mathbb{T}$ has norm $\sqrt{d+1}$. Summing and comparing $\|A_d c\| \geq \sigma_d \|c\|$ yields (12); Weyl's inequality gives the perturbation statement. Finally, a global block kernel may live in one component only, so the first rank loss takes the minimum shifted block cost, proving (13)–(14). $\square$

For two scalar bands with occupancies $(2, 4)$, the shifted costs are $(4, 2)$. Hence $d_{\min} = 4$ but $d_\partial = 2$. This exact fixture prevents the tempting but false assertion that generic exact and border memory always coincide.

5 The nonattained phase is necessarily high-delay

For a differentiable lossless boundary response set

$$Q_S(\theta) = -iS(e^{i\theta})^* \frac{d}{d\theta} S(e^{i\theta}), \qquad D(S) = \|\operatorname{tr} Q_S\|_\infty.$$

Let $E_{d,T}$ be the infimum node error among routers with $\deg_{\mathrm{McM}} S \leq d$ and $D(S) \leq T$.

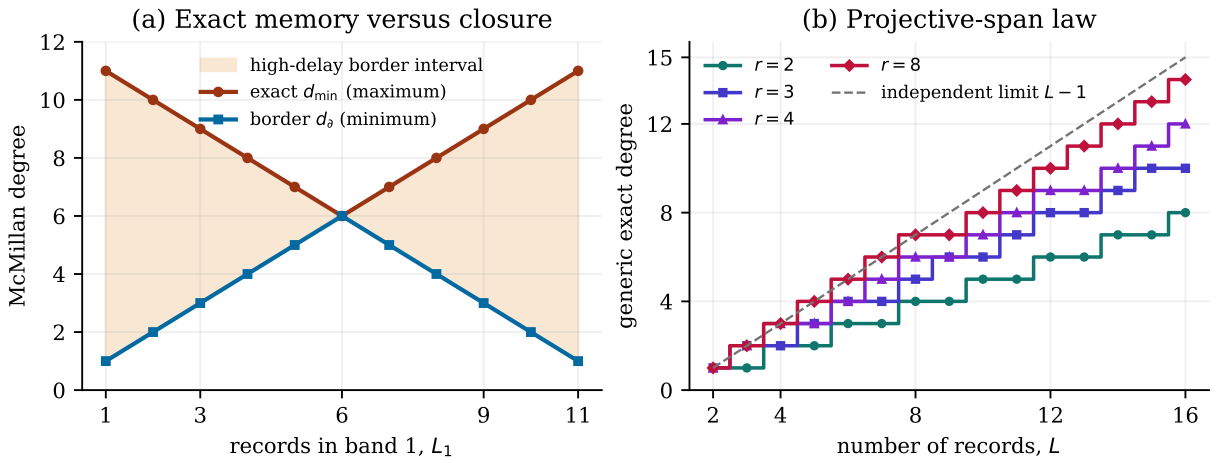

Figure 1: Two consequences of the global laws. (a) For two scalar bands with $L = 12$, exact memory is the maximum forced-zero count while border memory is the minimum; the shaded degrees are zero-error but nonattained and therefore high-delay. (b) For one r-dimensional target span, generic exact memory is $\lfloor L(r-1)/r \rfloor$ and approaches the independent-target limit as r grows. Both panels plot theorem values, not fitted numerics.

Table 2: Exact rational fixtures independently replayed with the manuscript.

Table	d_∂	$d_{\min}$	certificate
$\mathbb{C}^3$ subspaces of dimensions $(2,1,2)$	1	1	rank $A_0 = 3 = \#$col, rank $A_1 = 4 < 6$
Planar four-node $3+1$ collision	1	3	full-support loss
$\mathbb{C}^3$ targets $(2,1,2)\oplus$ two distinct $\mathbb{C}^2$ lines	3	4	shifted costs $(3,4)$
Two scalar bands, occupancies $(2,4)$	2	4	rank $A_1 = 4 = \#$col, rank $A_2 = 5 < 6$

Theorem 5.1 (Resonant border trichotomy). *For every finite incidence table:*

1. *if $d < d_\partial$, then $E_d > 0$, with the explicit floor* (12) *whenever A_d has full column rank;*
2. *if $d_\partial \le d < d_{\min}$, then $E_d = 0$ but $E_{d,T} > 0$ for every finite T, and every sequence with errors tending to zero satisfies $D(S_n) \to \infty$;*
3. *if $d \ge d_{\min}$, exact routing is attained by a finite-delay router.*

For fixed d, T, the constrained minimum defining $E_{d,T}$ is attained.

Proof. A degree-at-most-d rational inner matrix has a rank-one Blaschke–Potapov factorization [16, 13]

$$S(z) = \prod_{j=1}^{m} \left[I + (b_{\alpha_j}(z) - 1)v_j v_j^*\right] U, \qquad m \le d,$$

where U is unitary, $b_\alpha(z) = (z-\alpha)/(1-\bar{\alpha}z)$, and $\alpha_j \in \mathbb{D}$. Because the determinant is a constant phase times $\prod_j b_{\alpha_j}$, Jacobi's formula makes the trace delay the sum of nonnegative Poisson kernels,

$$\operatorname{tr} Q_S(\theta) = \sum_{j=1}^{m} \frac{1-|\alpha_j|^2}{|e^{i\theta}-\alpha_j|^2}.$$

The jth summand peaks at $(1+|\alpha_j|)/(1-|\alpha_j|)$. A finite delay cap therefore confines all zeros to a compact subdisk. The directions v_j and U also lie in compact spaces. For each $m = 0, \ldots, d$ the resulting parameter space is compact, and the capped degree-at-most-d class is the finite union of their continuous images. Thus the capped family is compact and evaluation at the fixed nodes is continuous. Its error minimum is attained.

If a capped sequence in the middle phase had errors tending to zero, a convergent subsequence would yield an exact router of degree at most d, contradicting $d < d_{\min}$. Hence every zero-error approximating sequence has unbounded $D(S_n)$; in fact the argument applies to every subsequence, so $D(S_n) \to \infty$. The remaining statements follow from Theorems 2.3 and 4.1. □

Theorem 5.1 turns algebraic nonattainment into an operational statement: a missing state can be approached only by a resonance whose peak dimensionless Wigner–Smith delay diverges. It does not assert an energy, resonator quality factor, linewidth, loss, or fabrication-cost law.

6 The planar stratum and its explicit lossless lift

Choose a basis of H and identify its lines with $\mathbb{P}(H) \simeq \mathbb{P}^1$. A rational map $R : \mathbb{P}^1 \to \mathbb{P}^1$ is represented by a coprime polynomial pair $[p : q]$, and $\deg R = \max(\deg p, \deg q)$. Poles of the affine ratio p/q are ordinary points of the projective map, not physical singularities.

Definition 6.1 (Projective interpolation degree)**.** For the ordered data (ζ_i, Y_i), let

$$\delta = \min\{\deg R : R : \mathbb{P}^1 \to \mathbb{P}(H),\ R(\zeta_i) = Y_i \text{ for every } i\}.$$

Common-denominator lemma. Let S be a rational-inner matrix of McMillan degree at most d, and let $F = Sx$ be any routed column. There are a polynomial χ and vector polynomial n such that

$$F(z) = \frac{n(z)}{\chi(z)}, \qquad \deg \chi \le d, \qquad \deg n_j \le d.$$

The denominator can be chosen nonzero at every regular interpolation node. Indeed, take a stable minimal realization $S(z) = D + zC(I - zA)^{-1}B$ of dimension at most d. With $\chi(z) = \det(I - zA)$, the adjugate formula gives the stated degree bounds for every entry of Sx. Stability makes χ zero-free on $\overline{\mathbb{D}}$; more generally, after cancelling any removable common factor, it is nonzero at each node where S is regular.

Theorem 6.2 (Exact planar reduction)**.** *If all target lines in* (1) *lie in one two-plane H, then*

$$\boxed{d_{\min} = \delta.} \tag{15}$$

The equality is independent of the ambient dimension and is constructive.

Proof. We first note that $\delta \le L - 1$. Apply a projective change of target chart whose point at infinity avoids the finite target set, interpolate the resulting affine values by a polynomial of degree at most $L - 1$, and undo the chart.

For the upper bound, let $[p : q]$ be a coprime representative of degree δ. For a polynomial r of degree at most δ, set

$$r^\sharp(z) = z^\delta \overline{r(1/\bar{z})}.$$

Because p and q are coprime, they have no common zero; in particular $|p|^2 + |q|^2 > 0$ on $\mathbb{T}$. Scalar Fejér–Riesz factorization [19] supplies an outer polynomial h, zero-free in $\mathbb{D}$, such that on $\mathbb{T}$

$$|h|^2 = |p|^2 + |q|^2.$$

Equivalently, $h^\sharp h = p^\sharp p + q^\sharp q$. Therefore

$$S_2(z) = \frac{1}{h(z)} \begin{pmatrix} p(z) & -q^\sharp(z) \\ q(z) & p^\sharp(z) \end{pmatrix} \tag{16}$$

is analytic in $\mathbb{D}$ and unitary on $\mathbb{T}$. Its first column represents $[p : q]$, and

$$\det S_2 = h^\sharp / h.$$

The strict positivity above also gives $h(\zeta_i) \neq 0$ at every routing node. With reversal taken at the displayed order δ, the scalar Blaschke product $h^\sharp/h$ has degree exactly δ; if $\deg h < \delta$, the missing reversed zeros occur at zero. For a rational-inner matrix the determinant winding equals its McMillan degree [16], so $\deg_{\mathrm{McM}} S_2 = \delta$. Constant unitaries align the input line and the basis of H, while a block identity embeds S_2 into dimension N without adding states. Hence $d_{\min} \leq \delta$.

Conversely, take a minimum-degree interpolant S with degree d. The upper construction already gives $d \leq L - 1 < L$. A stable conservative realization of S and the common-denominator lemma above give every entry of the routed column $F = Sx$ a common denominator of degree at most d, nonzero at the boundary nodes; the corresponding numerators have degree at most d. For every linear functional ℓ annihilating H, the numerator of ℓF vanishes at all $L > d$ distinct nodes. It is therefore zero identically. Thus $F(z) \in H$ as a rational identity. In a basis of H, cancel the common polynomial factor of the two coordinate numerators of F. The resulting coprime pair has degree at most d and interpolates every Y_i, so $\delta \leq d$. This proves (15). ☐

Remark 6.3 (Affine poles are harmless). The datum $R(z) = 1/z$ has an affine pole at zero, but the projective pair $[1 : z]$ is regular there. Formula (16) lifts this pair to an analytic inner column. Accordingly, pole tests on a chosen affine ratio cannot replace projective coprimality and Fejér–Riesz lifting.

7 Exact certificates and quantitative approximation barriers

Write $Y_i = [a_i : b_i]$ in the chosen basis of H. For $d \geq 0$, define the $L \times 2(d+1)$ matrix

$$M_d(i,:) = \big(b_i, b_i\zeta_i, \ldots, b_i\zeta_i^d, -a_i, -a_i\zeta_i, \ldots, -a_i\zeta_i^d\big). \tag{17}$$

A coefficient vector in $\ker M_d$ is exactly a polynomial pair (p, q) of degree at most d satisfying $[p(\zeta_i) : q(\zeta_i)] = [a_i : b_i]$, provided it has no base point at a node. Multiplying a homogeneous target representative by a nonzero scalar rescales one row, while changing the basis of H acts by an invertible block change on the coefficient columns. Hence every rank condition below is intrinsic to the ordered projective data.

Proposition 7.1 (The planar matrix is the intrinsic incidence operator)**.** *For normalized line representatives $y_i = (a_i, b_i)^T$, choose the unit annihilator $q_i = (-\overline{b}_i, \overline{a}_i)^T$ in* (9)*. In the monomial coefficient basis, A_d is M_d up to a diagonal unitary change of quotient coordinates and an inessential row sign. Consequently their ranks and singular values agree. Thus the planar results below are coordinate realizations of the global rank–error theory, rather than a second independent reduction.*

Proof. The quotient functional q_i^* sends $(p(\zeta_i), q(\zeta_i))^T$ to $-b_i p(\zeta_i) + a_i q(\zeta_i)$, the negative of row i of (17). Changing the phase of q_i only multiplies that row by a unit scalar. ☐

Proposition 7.2 (Fail-closed degree certificate)**.** *The number δ is the least d for which* $\ker M_d$ *contains a pair (p, q) generating the unit ideal, equivalently* $\gcd(p, q) = 1$*. For a nonconstant pair this can be certified by* $\mathrm{Res}(p, q) \neq 0$*; constant pairs are checked directly. Such a pair is an exact upper certificate. Full column rank of M_{d-1} is an exact lower certificate.*

Proof. The unit-ideal condition is precisely projective coprimality. When both polynomials form a nonconstant pair, it is equivalent to nonvanishing of the resultant; the constant pairs $[c : 0]$ and $[0 : c]$, $c \neq 0$, are checked by the unit-ideal condition itself. The claim follows from Definition 6.1 and (17). Full column rank excludes every nonzero pair of degree at most $d - 1$. ☐

The formulation is deliberately algebraic. A kernel vector alone is not enough: a common factor can inflate its displayed degree or introduce a base point. A Bézout identity $up + vq = 1$ is an equivalent independently checkable coprimality certificate.

We now normalize (a_i, b_i) to unit norm and choose an orthonormal basis of H. For a unit vector u and a target line $Y = \mathbb{C}y$, $\|y\| = 1$, write

$$\operatorname{dist}_{\mathrm{ch}}(\mathbb{C}u, Y) = \sqrt{1 - |\langle u, y\rangle|^2}$$

for complex-projective chordal distance.

Theorem 7.3 (Quantitative approximate-routing obstruction)**.** *Suppose M_d has full column rank and put $\sigma = \sigma_{\min}(M_d) > 0$. Every square rational-inner router S of McMillan degree at most d satisfies*

$$\max_{1\le i\le L} \operatorname{dist}_{\mathrm{ch}}\big(S(\zeta_i)X, Y_i\big) \;\ge\; \boxed{\frac{\sigma}{\sqrt{L(d+1)+\sigma^2}}}. \tag{18}$$

If a perturbed normalized data matrix $\widehat{M}_d$ obeys $\|\widehat{M}_d - M_d\|_2 \le \eta < \sigma$, the right-hand side remains valid with σ replaced by $\sigma - \eta$. Equivalently, the degree-constrained minimax error $E_d = \inf_{\deg_{\mathrm{McM}} S\le d} \max_i \operatorname{dist}_{\mathrm{ch}}(S(\zeta_i)X, Y_i)$ obeys the same lower bound.

Proof. Choose a unit input vector x and write $F = Sx$. On $\mathbb{T}$, $\|F\| = 1$. By the common-denominator lemma above, the two coordinates of $P_H F$ have the form $p/\chi, q/\chi$, with numerator degrees at most d. Let c collect their $2(d+1)$ coefficients. If $c = 0$, then F is orthogonal to every Y_i and the left-hand side of (18) is one, so assume $c \ne 0$.

Put $v_i = (p(\zeta_i), q(\zeta_i))$ and $e_i = \operatorname{dist}_{\mathrm{ch}}(\mathbb{C}F(\zeta_i), Y_i)$. Since $v_i = \chi(\zeta_i)P_H F(\zeta_i)$ and $y_i \in H$, the two-dimensional wedge identity gives

$$|(M_d c)_i| = |\det(y_i, v_i)| \le |\chi(\zeta_i)|e_i, \qquad \|v_i\| \ge |\chi(\zeta_i)|\sqrt{1-e_i^2}.$$

Evaluation of two degree-d polynomials at a unit-modulus node has operator norm $\sqrt{d+1}$, hence $\|v_i\| \le \sqrt{d+1}\|c\|$. If $\varepsilon = \max_i e_i < 1$, these inequalities imply

$$\|M_d c\|_2 \le \sqrt{L(d+1)}\,\|c\|\,\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}.$$

The reverse inequality $\|M_d c\|_2 \ge \sigma\|c\|$ and cancellation of $\|c\|$ give (18); the case $\varepsilon = 1$ is immediate. Finally, Weyl's inequality gives $\sigma_{\min}(\widehat{M}_d) \ge \sigma - \eta$, proving the perturbative claim. □

Corollary 7.4 (Exact-arithmetic coercivity certificate)**.** *Let $G_d = M_d^* M_d$, $m = 2(d+1)$, and*

$$\gamma_d = \det G_d \left(\frac{m-1}{\operatorname{tr} G_d}\right)^{m-1}.$$

If M_d has full column rank, then $\sigma^2 \ge \gamma_d > 0$, so the right-hand side of (18) *is at least*

$$\sqrt{\frac{\gamma_d}{L(d+1)+\gamma_d}}.$$

For Gaussian-rational nodes and targets, γ_d is an exact arithmetic certificate. Any stronger exact inequality $G_d \succeq \alpha I$ replaces γ_d by α.

Proof. If $0 < \lambda_1 \le \cdots \le \lambda_m$ are the eigenvalues of G_d, then $\det G_d = \lambda_1 \prod_{j>1}\lambda_j$. By arithmetic–geometric mean,

$$\prod_{j>1}\lambda_j \le \left(\frac{\sum_{j>1}\lambda_j}{m-1}\right)^{m-1} \le \left(\frac{\operatorname{tr} G_d}{m-1}\right)^{m-1}.$$

Thus $\sigma^2 = \lambda_1 \ge \gamma_d$. Apply Theorem 7.3. □

The normalization is essential: without unit homogeneous representatives and an orthonormal basis of H, σ changes under coordinate rescaling. The theorem now controls routing error itself, rather than only persistence of an exact rank obstruction.

8 The exact zero-memory frontier

The preceding obstruction is a lower bound for general d. At $d = 0$, however, the entire minimax problem has a closed geometric solution. For a unit target representative $y_i \in H$, write

$$\rho_i = y_i y_i^* = \frac{1}{2}\big(I + \mathbf{r}_i \cdot \boldsymbol{\sigma}\big), \qquad \mathbf{r}_i \in \mathbb{S}^2,$$

where $\boldsymbol{\sigma}$ is the Pauli triple. Define the smallest-cap radius

$$\phi_* = \min_{\mathbf{r}\in\mathbb{S}^2} \max_i \arccos(\mathbf{r} \cdot \mathbf{r}_i). \tag{19}$$

The spherical one-center problem and its support algorithms are classical [20]; the point here is their exact passive-routing consequence.

Theorem 8.1 (Exact zero-memory minimax law)**.** *For every finite planar line-routing table,*

$$\boxed{E_0 = \sin(\phi_*/2)}. \tag{20}$$

Moreover, an optimizing cap center has a first-order support certificate involving at most three active target lines: the origin lies in the convex hull of their tangent projections at the center. Consequently E_0 is computable by candidate centers generated by one-, two-, and three-target supports, together with a global containment check and the usual antipodal degeneracies.

Proof. Every constant unitary can send X to an arbitrary output line, and every output line extends to a constant unitary. It is enough to optimize over output lines in H. Indeed, if a unit output vector $u \notin H$ has $P_H u \neq 0$, then $v = P_H u / \|P_H u\|$ satisfies, for every $y_i \in H$,

$$|\langle v, y_i\rangle| = \frac{|\langle u, y_i\rangle|}{\|P_H u\|} \geq |\langle u, y_i\rangle|.$$

If $P_H u = 0$, every chordal error is one and any target line is no worse. Thus projection and normalization never increase the objective. If the resulting line has Bloch vector $\mathbf{r}$, then

$$|\langle y, y_i\rangle|^2 = \mathrm{tr}(\rho\rho_i) = \frac{1 + \mathbf{r}\cdot\mathbf{r}_i}{2}, \qquad \mathrm{dist}_{\mathrm{ch}}(\mathbb{C}y, Y_i) = \sin\left(\frac{1}{2}\arccos(\mathbf{r}\cdot\mathbf{r}_i)\right).$$

Since the last expression is increasing in the spherical angle, (20) follows from (19).

For the support statement, maximize $\tau(\mathbf{r}) = \min_i \mathbf{r}\cdot\mathbf{r}_i$ on the sphere and let A be the active set at a maximizer $\mathbf{r}_*$. If zero were outside the convex hull of the tangent projections $P_{\mathbf{r}_*^\perp}\mathbf{r}_i$, $i \in A$, a strictly separating tangent direction would increase every active scalar product to first order. The inactive constraints have positive slack, contradicting maximality. Planar Carathéodory therefore supplies a balancing subset of at most three active projections. Enumerating those support equations and retaining only centers whose caps contain every target gives the stated finite computation. □

Corollary 8.2 (Two-target frontier and sharp collision limit)**.** *For two distinct target lines with Fubini–Study angle $\alpha = \arccos|\langle y_1, y_2\rangle| \in (0, \pi/2]$,*

$$E_0 = \sin(\alpha/2).$$

If $t = \tan(\alpha/2)$, then

$$E_0^2 = \frac{t^2}{1+t^2}, \qquad \left(\frac{\sigma_{\min}(M_0)}{\sqrt{2 + \sigma_{\min}(M_0)^2}}\right)^2 = \frac{t^2}{1+2t^2}.$$

Hence the ratio between the singular-value lower bound of Theorem 7.3 and the exact optimum tends to one as $t \downarrow 0$.

Proof. The two Bloch vectors are separated by angle 2α, so their smallest cap is centered at the geodesic midpoint and has radius α. This proves the first formula. The two normalized rows of M_0 have Gram eigenvalues $1 \pm |\langle y_1, y_2\rangle|$, whence $\sigma_{\min}(M_0)^2 = 1 - \cos\alpha$. Substitution and the half-angle identity give the displayed rational formulas and their limit. □

9 Fiber multiplicity and binary routing

Projective interpolation makes collisions transparent. If a nonconstant rational map has degree d, no fiber contains more than d distinct points. This observation becomes a state lower bound after Theorem 6.2.

Proposition 9.1 (Fiber occupancy bound)**.** *Suppose at least two distinct target lines are used, and let* $n_{\max} = \max_Y \#\{i : Y_i = Y\}$*. Then*

$$d_{\min} \geq n_{\max}.$$

Proof. A constant projective map cannot hit two targets. For a nonconstant map $R = [p : q]$ of degree d, the equation $R(z) = Y$ is a nonzero polynomial equation of degree at most d. It has at most d distinct solutions. Apply this to a most occupied target and use Theorem 6.2. □

For exactly two targets, the fiber obstruction is the whole answer.

Theorem 9.2 (Exact binary collision law)**.** *If the table uses exactly two distinct target lines with occupancies* n_0 *and* n_∞*, then*

$$d_{\min} = \max(n_0, n_\infty). \tag{21}$$

Proof. Normalize the two targets to 0 and ∞. Let I_0, I_∞ be their node sets. The coprime pair

$$p(z) = \prod_{i \in I_0} (z - \zeta_i), \qquad q(z) = \prod_{i \in I_\infty} (z - \zeta_i)$$

has degree $\max(n_0, n_\infty)$ and realizes the table. The reverse inequality is Proposition 9.1. □

The binary law is the collision-dominated extreme: the constant-detector bound is sharp. The opposite extreme has all targets distinct. Then fiber multiplicity contributes only one, but global projective compatibility produces a growing cost.

Table 3: Exact planar mechanisms. The projective reduction converts every entry in the last column directly into passive state count.

Target pattern	Exact or generic degree	Governing obstruction
One target	0	constant projective map
Two targets, occupancies n_0, n_∞	exact max; border min	fiber/base-point duality
L distinct, generic	$\lceil (L-1)/2 \rceil$	projective parameter count
Exceptional table at $d_0 = \lceil (L-1)/2 \rceil$	$E_{d_0} = 0$, possibly unattained	dense exact-routing locus
Four distinct, exceptional	1 or 2	ordered cross-ratio
Four targets, pattern $3+1$	3	triple fiber

10 The generic planar law and an unbounded gap

Lemma 10.1 (Nonempty coprime threshold locus)**.** *Fix distinct nodes and put* $d_0 = \lceil (L-1)/2 \rceil$*. There is a nonempty Zariski-open set of target data for which* M_{d_0} *contains a coprime kernel pair. This open set meets the locus of pairwise distinct targets.*

Proof. Work first in the affine target chart $a_i/b_i = y_i$ and use the witness $y_i = \zeta_i^{d_0}$. At this point the columns of M_{d_0} are evaluations of monomials with exponents $0, \ldots, 2d_0$, with exponent d_0 duplicated. Because $L \leq 2d_0 + 1$, a Vandermonde $L \times L$ minor D is nonzero. On the principal open set $U = \{D \neq 0\}$, this same pivot solves L coefficients by Gaussian elimination and expresses the remaining coefficients as free parameters with entries in the localization $\mathbb{C}[y_1, \dots, y_L]_D$. Hence $\ker M_{d_0}$ has an algebraic frame on U.

At the witness, $(z^{d_0}, 1)$ lies in the kernel. Choose the constant linear combination of the local frame that equals this vector there. Multiplying its two polynomial coordinates by a common power of D clears every parameter denominator. Their resultant is then a regular function of the target data; at the witness it is a nonzero scalar multiple of $\operatorname{Res}(z^{d_0}, 1) = 1$. Its nonvanishing and $D \neq 0$ therefore define a nonempty Zariski-open coprime locus. The pairwise target inequalities also define a nonempty Zariski-open subset of $(\mathbb{P}^1)^L$. Their intersection is nonempty because $(\mathbb{P}^1)^L$ is irreducible. □

Theorem 10.2 (Generic planar memory)**.** *For fixed distinct nodes and L target lines in a two-plane, there is a Zariski-open dense set of ordered target data on which*

$$d_{\min} = \left\lceil \frac{L-1}{2} \right\rceil. \tag{22}$$

The open set can be intersected with the locus of pairwise distinct targets.

Proof. Put $d_0 = \lceil (L-1)/2 \rceil = \lfloor L/2 \rfloor$. For $d < d_0$, the matrix M_d has at most L columns and is generically full-column-rank. A concrete rank witness is obtained from affine data $a_i/b_i = \zeta_i^{d_0}$: an equation $p(\zeta_i) = \zeta_i^{d_0} q(\zeta_i)$ at all nodes would give L roots to a polynomial of degree at most $L-1$ and then force $p = q = 0$ at degree $d_0 - 1$. Thus the nonvanishing of a maximal minor supplies the required nonempty open set.

At $d = d_0$, M_{d_0} has more columns than rows, so it always has nonzero kernel. Lemma 10.1 supplies a nonempty open locus on which that kernel contains a coprime pair, and the lower-rank open condition above can be intersected with it. Apply Theorem 6.2 and Proposition 7.2. The scalar threshold itself is classical; the new conclusion here is its exact conversion to passive state count. □

Definition 10.3 (Border memory)**.** For $A \subseteq [L] = \{1, \dots, L\}$, let $\delta(A)$ denote the exact projective interpolation degree of the subtable indexed by A, with $\delta(\varnothing) = 0$. Define

$$d_\partial := \min\{d \geq 0 : E_d = 0\}.$$

Thus d_∂ is the first state count at which exact target tables lie in the closure of the degree-bounded passive class; it need not be attained.

Theorem 10.4 (Linear rank and base-point deletion law)**.** *Every finite planar routing table satisfies*

$$\boxed{\begin{aligned} d_\partial &= \min\{d \geq 0 : \ker M_d \neq \{0\}\} = \min\{d \geq 0 : \operatorname{rank} M_d < 2(d+1)\} \\ &= \min_{B \subseteq [L]} \big\{ |B| + \delta(B^c) \big\}. \end{aligned}} \tag{23}$$

Equivalently, for every $d \geq 0$,

$$\boxed{E_d = 0 \quad \Longleftrightarrow \quad \mu_d := \min_{\|c\|=1} \|M_d c\| = 0.} \tag{24}$$

Consequently the complete sign-and-attainment diagram is

$$\begin{array}{c|c} d < d_\partial & E_d > 0, \\ d_\partial \leq d < \delta & E_d = 0 \textit{ and the infimum is not attained,} \\ d \geq \delta & E_d = 0 \textit{ and is attained.} \end{array}$$

In particular, the distinction between exact and approximate memory is itself an exactly computable projective invariant.

Proof. First suppose $E_d = 0$. Choose degree-at-most-d routers S_n whose worst-node errors tend to zero, and fix a unit vector spanning X. Project $S_n x$ orthogonally to H and use the common-denominator lemma to write its two coordinates as a polynomial pair (p_n, q_n) of degree at most d, up to a scalar denominator. For all large n this pair is not identically zero, because its values at the routing nodes approach nonzero lines in H. Normalize its coefficient vector and pass to a convergent subsequence. The limit is a nonzero pair (p, q) of degree at most d. At each node, chordal convergence bounds the determinant of this pair with the normalized target vector by the routing error times its evaluation norm. The latter is uniformly bounded after coefficient normalization, so the limiting determinant vanishes. Hence its coefficient vector is a nonzero element of $\ker M_d$.

Let B be the set of nodes at which p and q both vanish. Write $(p, q) = g(\widetilde{p}, \widetilde{q})$ with the reduced pair coprime, and put $b = \deg g$ and $r = \max(\deg \widetilde{p}, \deg \widetilde{q})$. Distinctness of the nodes gives $|B| \le b$, while coefficient convergence and projective continuity show that $[\widetilde{p} : \widetilde{q}]$ interpolates every target outside B. Therefore

$$|B| + \delta(B^c) \le b + r = \max(\deg p, \deg q) \le d.$$

This proves both that rank deficiency is necessary and that the subset minimum in (23) is no larger than d_∂.

Conversely fix B, put $b = |B|$, and choose a coprime pair (p_0, q_0) of degree $r = \delta(B^c)$ interpolating the complementary subtable. If $b = 0$, the exact lift already gives $E_r = 0$. If $b > 0$, let $g(z) = \prod_{i\in B}(z - \zeta_i)$ and first choose a coprime pair (a_0, c_0) of degree at most $b - 1$ interpolating the data restricted to B; ordinary polynomial interpolation supplies such a pair. Put $D = b + r$. Without changing its values on B, extend it to a coprime pair (a, c) of exact degree D. If $c_0 \ne 0$, take $c = c_0$ and $a = a_0 + \lambda g z^r$: only finitely many nonzero values of λ can make a vanish at a root of c_0. If $c_0 = 0$, coprimality forces a_0 to be a nonzero constant, and $(a, c) = (a_0, g z^r)$ works. Consider

$$P_t = (1 - t) g p_0 + t a, \qquad Q_t = (1 - t) g q_0 + t c.$$

At every node in B, $[P_t : Q_t] = [a : c]$ for $t \ne 0$; outside B the values converge to $[p_0 : q_0]$ as $t \to 0$. Coprimality with no common projective zero is detected by the homogeneous degree-D resultant: it vanishes for a common finite root and also for a common root at infinity $[1 : 0]$. Since $(P_1, Q_1) = (a, c)$ is coprime and has exact degree D, this resultant is a nonzero polynomial in t. Hence a sequence $t_n \to 0$, $t_n \ne 0$, can be chosen away from its finitely many zeros, with every (P_{t_n}, Q_{t_n}) coprime. Its degree is at most $b + r$. Applying the exact same-degree lossless lift proves $E_{b+r} = 0$, and hence equality with the subset minimum.

It remains only to connect an arbitrary deficient M_d back to the subset formula. Take a nonzero kernel pair (p, q), remove its gcd g, and let B be the interpolation nodes at which both coordinates vanish. With $b = \deg g$ and reduced degree r, the same argument as above gives $|B| + \delta(B^c) \le b + r \le d$. Conversely, the smoothing construction for any B produces coprime approximating pairs of degree $|B| + \delta(B^c)$. Thus the first deficient matrix, the subset minimum and d_∂ coincide. Since the minimum modulus μ_d vanishes exactly when M_d has a nontrivial kernel, (24) follows.

Finally, $E_d = 0$ is monotone in d. Zero is attained exactly when an exact router of degree at most d exists, which by Theorem 6.2 is equivalent to $d \ge \delta$. The three regimes follow. □

Corollary 10.5 (Incremental exact border certificate)**.** *The integer d_∂ is obtained by scanning the nested sequence*

$$M_0, M_1, \ldots, M_{\lfloor L/2 \rfloor}$$

and stopping at the first loss of full column rank. A single incremental column elimination computes all these ranks using $O(L^3)$ field operations. A nonzero kernel vector at the stopping degree is an upper certificate for zero infimum, even when it has base points; full column rank at every preceding degree is the lower certificate. Polynomial gcd and subresultant tests are needed for exact memory δ, but not for border memory.

Proof. For $d_0 = \lfloor L/2 \rfloor$, the matrix M_{d_0} has $2(d_0+1) > L$ columns and is automatically deficient, so the scan terminates. Rank elimination need not restart at each degree: M_{d+1} appends exactly two columns to M_d. Maintain a column-echelon basis and reduce each appended length-L column against at most L pivots. Each update costs $O(L^2)$ field operations and at most $L+1$ columns are processed before termination, for $O(L^3)$ operations in total. Once dependence appears it persists under zero-padding of a kernel vector. The certificate statements are exactly Theorem 10.4. All tests are exact over the coefficient field of the data. □

Corollary 10.6 (Exact rank–error dichotomy). *With normalized homogeneous target representatives, let* $\mu_d = \min_{\|c\|=1} \|M_d c\|$. *Then precisely one of the following holds:*

$$\begin{aligned} \mu_d = 0: &\quad E_d = 0, \\ \mu_d > 0: &\quad E_d \geq \frac{\mu_d}{\sqrt{L(d+1) + \mu_d^2}} > 0. \end{aligned}$$

Moreover, the positive-error conclusion survives every calibrated perturbation $\widehat{M}_d$ *with* $\|\widehat{M}_d - M_d\|_2 < \mu_d$*, replacing* μ_d *by the remaining margin. Thus one matrix supplies both the exact qualitative phase and a quantitative, stable error certificate.*

Proof. The zero case is (24). In the positive case M_d has full column rank, so μ_d is its usual smallest singular value and Theorem 7.3 applies, including its perturbation statement. □

Corollary 10.7 (Exact determinantal closure). *Fix the distinct nodes and let* $\mathcal{R}_d \subset (\mathbb{P}^1)^L$ *be the planar target tables routed exactly by a degree-at-most-d passive network. Then its Euclidean closure is*

$$\overline{\mathcal{R}_d} = \{(Y_i) : \operatorname{rank} M_d < 2(d+1)\}.$$

When $2(d+1) \leq L$*, this is the projective determinantal set cut out in local homogeneous charts by the maximal minors of* M_d*; when* $2(d+1) > L$*, it is the whole target space. The exact-routable locus is dense in this set, but may omit its base-point strata.*

Proof. By definition, membership in the metric closure of $\mathcal{R}_d$ is equivalent to $E_d = 0$. Theorem 10.4 identifies this with a nonzero kernel of M_d. Maximal-minor vanishing is exactly that rank condition. The smoothing construction in the proof of Theorem 10.4 produces coprime degree-at-most-d tables converging to every deficient table, proving density rather than only containment. □

Corollary 10.8 (Complete binary approximation phase diagram). *For two target lines with positive occupancies* n_0, n_∞,

$$d_\partial = \min(n_0, n_\infty), \qquad d_{\min} = \max(n_0, n_\infty). \tag{25}$$

Thus $E_d > 0$ *below the smaller occupancy,* $E_d = 0$ *but is unattained between the two occupancies, and exact routing begins at the larger occupancy. For a* $1 + (L-1)$ *split the exact-to-border memory ratio is* $L-1$ *and is unbounded.*

Proof. Deleting all nodes carrying the less frequent target leaves a constant subtable, so Theorem 10.4 gives the upper bound $d_\partial \leq \min(n_0, n_\infty)$. For the reverse bound, fix B in (23). If the reduced interpolant on B^c is constant, B must contain every occurrence of the other target and therefore $|B|$ is at least the smaller occupancy. If it is nonconstant of degree r, each of its two fibers has at most r remaining nodes. Adding the deleted nodes shows $|B| + r \geq \max(n_0, n_\infty)$, hence certainly at least the smaller occupancy. Minimizing over B proves the first identity; the second is Theorem 9.2, and the phase statement follows from Theorem 10.4. □

Theorem 10.9 (Sharp universal closure threshold). *Fix distinct nodes and put* $d_0 = \lceil (L-1)/2 \rceil$. *For every ordered planar target table,*

$$\boxed{E_{d_0} = 0}. \tag{26}$$

More generally $E_d = 0$ for every $d \geq d_0$. The infimum at d_0 is attained if and only if $\delta \leq d_0$. For every $d < d_0$, by contrast, there is a Euclidean-open dense set of target tables for which $E_d > 0$. Consequently d_0 is the sharp state threshold at which degree-bounded passive routers become dense in all planar routing data.

Proof. The matrix M_{d_0} has $2(d_0 + 1) > L$ columns, so it has a nontrivial kernel for every target table. The rank law (23) therefore gives $E_{d_0} = 0$ directly. Allowing more states cannot increase the infimum. If the infimum at d_0 is attained with value zero, the attaining router interpolates exactly, so Theorem 6.2 gives $\delta \leq d_0$; the converse is immediate. Finally, for $d < d_0$ the full-column-rank condition on M_d is a nonempty Zariski-open condition by the proof of Theorem 10.2. It is Euclidean open dense, and Theorem 7.3 supplies a strictly positive error there. □

Corollary 10.10 (Exact one-state infimum without a minimizer)**.** *For three distinct nodes and any planar target table, $E_1 = 0$. If the target multiplicity pattern is $2 + 1$, then $d_{\min} = 2$ and this zero infimum is not attained by a one-state router.*

Proof. Here $d_0 = 1$, so Theorem 10.9 gives $E_1 = 0$. The binary collision law gives $d_{\min} = 2$ for the $2 + 1$ pattern, and the attainment statement follows. For an explicit normalized example, choose a projective coordinate on the domain taking the nodes to $(0, 1, \infty)$ and choose limiting targets $(0, 0, 1)$ in an orthonormal affine chart of H. For $0 < \varepsilon < 1$,

$$R_\varepsilon(z) = \frac{\varepsilon z}{\varepsilon z + 1 - \varepsilon}$$

is a degree-one Möbius map with values $(0, \varepsilon, 1)$ and worst-node chordal error $\varepsilon/\sqrt{1 + \varepsilon^2}$. Its determinant $\varepsilon(1 - \varepsilon)$ is nonzero, so the same-degree lossless lift is valid. The errors tend to zero, whereas a nonconstant degree-one map is injective and cannot attain the repeated limiting target. □

Remark 10.11 (Why rank deficiency is not exact feasibility). At d_0, the matrix M_{d_0} always has a nonzero kernel, but exceptional tables may have only kernel pairs with a common factor or a base point. Theorem 10.4 shows that such defective pairs are limits of coprime pairs even when no coprime pair exists at the limiting data. Thus full column rank in Theorem 7.3 is a structural separation condition, not a technical artifact: once it fails at the universal threshold, a positive error floor cannot hold for the full degree-bounded class.

For distinct lines in a two-plane, the span bound is $\dim H - 1 = 1$. Likewise, the optimized constant-detector hierarchy from [1] equals one: the kernel of a nonzero functional on H contains at most one of the distinct target lines.

Corollary 10.12 (Unbounded detector gap)**.** *On the generic pairwise-distinct planar locus,*

$$\frac{d_{\min}}{\max\{\textit{span bound, constant-detector bound}\}} = \left\lceil \frac{L-1}{2} \right\rceil \longrightarrow \infty.$$

Thus the optimized constant-detector hierarchy of [1] cannot approximate the exact passive memory within a universal multiplicative factor on this single fixed-rank stratum.

11 The complete four-line phase diagram

For four ordered distinct nodes, write

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

with the usual projective conventions at infinity. The order matters because each frequency is paired with a specified target. The projective classification below is classical/elementary; our claim is its exact consequence for square passive rational-inner routers.

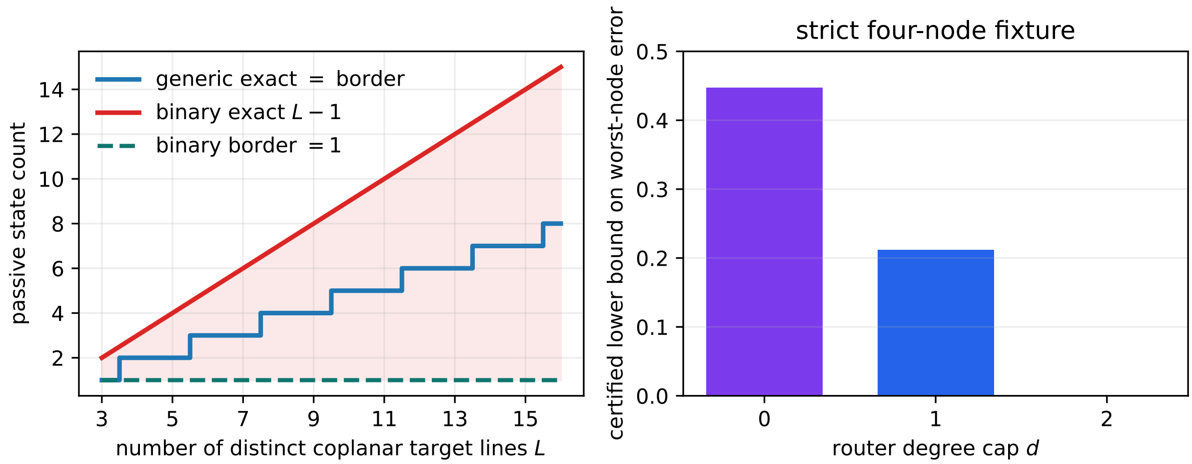

Figure 2: Left: generic exact/border memory and the binary $1+(L-1)$ split, whose exact memory is $L-1$ while its border memory remains one; the shaded exact-to-border gap is unbounded. Right: exact certified worst-node chordal-error barriers for the strict four-node fixture. Degree two attains zero error, whereas degrees zero and one remain quantitatively separated.

Theorem 11.1 (Four-line classification)**.** *For four target lines in one two-plane, the exact passive memory is*

$$d_{\min} = \begin{cases} 0, & \textit{all four targets coincide,} \\ 1, & \textit{all four are distinct and their ordered cross-ratio equals that of the nodes,} \\ 3, & \textit{the target multiplicities are } 3+1, \\ 2, & \textit{in every remaining case.} \end{cases} \tag{27}$$

Proof. A nonconstant degree-one map is a projective bijection, so it sends the four nodes to four distinct targets and preserves their ordered cross-ratio. These conditions are also sufficient by the uniqueness of a projective map through three ordered points.

If three nodes share a target and the fourth does not, a nonconstant map of degree at most two cannot work because every fiber has at most two distinct points. Polynomial interpolation gives the universal upper bound three.

It remains to show degree at most two in all other nonconstant cases. For four distinct targets with mismatched cross-ratio, normalize domain and target triples to $(0, \infty, 1)$, leaving $s \mapsto t$ with $s \neq t$. Then

$$R(z) = z(az + b), \qquad a = \frac{t-s}{s(s-1)}, \quad b = 1 - a$$

has degree two and realizes the data. For multiplicities $2+2$, after choosing a finite chart use

$$R(z) = c\frac{(z-z_1)(z-z_2)}{(z-z_3)(z-z_4)}.$$

For $2+1+1$, normalize the repeated nodes to $0, \infty$, the other nodes to $1, s$, and the targets to $\infty, \infty, 0, 1$. Choose $a \notin \{0, 1, s\}$ and set

$$q(z) = z, \qquad p(z) = A(z-1)(z-a), \qquad A = \frac{s}{(s-1)(s-a)}.$$

This is a coprime degree-two witness. None of these cases is constant or degree one, completing the lower bounds and the classification via Theorem 6.2. $\square$

Example 11.2 (Strict exact and approximate detector failure)**.** Take

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = (1, i, -1, -i), \qquad (Y_1, Y_2, Y_3, Y_4) = (0, 1, \infty, 2).$$

The ordered cross-ratios are 2 and 1/2, respectively. Hence $d_{\min} = 2$, while both constant lower bounds equal one. An exact witness is

$$p(z) = -\frac{(z-1)(3z-i)}{2}, \qquad q(z) = z+1, \qquad \operatorname{Res}(p,q) = -3 - i \neq 0.$$

It takes the four requested projective values exactly.

There is also a finite approximation gap below degree two. Normalize the four target representatives to unit norm. For $d = 1$, exact arithmetic gives

$$G_1 = M_1^* M_1 = \begin{pmatrix} 17/10 & 1+3i/10 & -9/10 & -i/10 \\ 1-3i/10 & 17/10 & i/10 & -9/10 \\ -9/10 & -i/10 & 23/10 & -1-3i/10 \\ i/10 & -9/10 & -1+3i/10 & 23/10 \end{pmatrix}.$$

The leading principal minors of $G_1 - \frac{3}{8}I$ are

$$\frac{53}{40}, \quad \frac{213}{320}, \quad \frac{637}{2560}, \quad \frac{213}{20480},$$

all positive. Sylvester's criterion yields $G_1 \succ \frac{3}{8}I$, and Theorem 7.3 therefore forces

$$\max_i \operatorname{dist}_{\mathrm{ch}}(S(\zeta_i)X, Y_i) > \sqrt{\frac{3}{67}} \approx 0.2116$$

for every router with at most one state. Likewise, $G_0 - I \succ 0$ gives the degree-zero barrier $1/\sqrt{5} \approx 0.4472$. The degree-two witness above attains zero, producing a certified finite-error staircase rather than only an exact degree jump.

12 Delay price and reproducibility

For a differentiable lossless boundary response, define the Wigner–Smith matrix

$$Q(\theta) = -iS(e^{i\theta})^* \frac{\mathrm{d}}{\mathrm{d}\theta} S(e^{i\theta}).$$

Jacobi's formula gives $\operatorname{tr}(S^{-1}\partial_\theta S) = \partial_\theta \log\det S$. Together with Potapov's factorization and winding count, this gives

$$\int_0^{2\pi} \operatorname{tr} Q(\theta)\, \mathrm{d}\theta = 2\pi \deg_{\mathrm{McM}} S$$

for finite rational-inner S [17, 18]. Combining it with Theorems 2.3 and 3.2 yields an exact action law.

Corollary 12.1 (Optimal integrated delay)**.** *For every line-routing table, and every block-subspace incidence table,*

$$\inf_S \int_0^{2\pi} \operatorname{tr} Q(\theta)\, \mathrm{d}\theta = 2\pi d_{\min}.$$

The projective lossless completion attains the infimum.

The computational artifact uses exact arithmetic over $\mathbb{Q}(i)$. It checks the strict witness in Example 11.2; exact representatives of the constant, cross-ratio-compatible, $2+2$, $2+1+1$, and $3+1$ strata; the normalized Gram matrices and Sylvester coercivity certificates behind the degree-zero and degree-one approximation barriers; the exact two-target zero-memory formulas and their sharp-collision ratios at four rational parameters; the explicit nonattained $E_1 = 0$ degeneration at four rational values of ε; full-column-rank exclusions below the generic threshold; coprime threshold witnesses for every $3 \leq L \leq 16$; and every nontrivial binary occupancy split through $L = 16$ (119 exact cases). For every $3 \leq L \leq 16$ it additionally checks the complete binary positive-error/nonattainment/exact phase partition and an explicit degree-one degeneration of the $1 + (L-1)$ family, whose exact-to-border ratio is $L - 1$. For all 119 binary tables it independently recomputes full column rank just below d_∂ and a nontrivial kernel at d_∂, directly auditing the polynomial-time rank formula rather than inferring it from occupancies. The frozen validator recomputes the content hash and checks declared coverage, phase degrees, ranks, and bounds. In CI the producer is replayed from scratch and its byte-identical JSON is compared with the frozen record; this replay is the algebraic check of cross-ratios, witness values, gcds, and resultants. These calculations audit examples and certificates; they are not substitutes for the proofs.

Limitation 12.2 (Exact scope). The global theorem assumes distinct frequency nodes, one input line, free endpoint phase, and square finite rational-inner competitors. Repeated nodes require confluent jets. The block theorem assumes a literal orthogonal direct sum of bands; overlapping target subspaces require a different arrangement law. Fixed endpoint frames, prescribed remaining columns, lossy or active architectures, and numerical values of the positive minimax errors beyond degree zero are not claimed. The perturbation theorem is only as meaningful as its calibrated operator-norm model. The high-delay theorem concerns differentiation with respect to the dimensionless phase θ; it is not a theorem about resonator quality factor, stored energy, linewidth, or fabrication cost.

Limitation 12.3 (Priority scope). Minimal scalar, vector, matrix, and tangential rational interpolation; unattainable points and common factors; generic rational-curve dimension counts; cross-ratios and fiber multiplicity; Fejér–Riesz factorization; rectangular lossless realization, all-pass completion, and Potapov factorization all predate this work [4, 5, 21, 14, 15, 22, 13]. The Poisson-kernel group-delay formula and the divergent peak produced by a Blaschke zero approaching $\mathbb{T}$ are classical as well. What is claimed here is that the projective border law forces this degeneration along every nonattained routing sequence. We do not claim a new interpolation algorithm or completion theorem. The claimed contribution is their synthesis for this passive architecture: the global conversion into state count (2), the full-support block specialization (5), the rank/error and deletion identities (10)–(11), the exact maximum/minimum split between multiband exact and border memory, and the resonant trichotomy of Theorem 5.1. The planar cap, binary, and cross-ratio results are explicit strata and audit fixtures, not claims of priority for their classical projective ingredients.

AI assistance disclosure. AI tools assisted with literature discovery, symbolic fixture generation, code review, and language editing. The author selected the claims, checked the proofs and exact certificates, and takes responsibility for the content.

Reproducibility record. The accompanying archive contains the manuscript source, the planar exact producer and independent validator, a separate rational-arithmetic verifier for the block-subspace laws, the frozen JSON objects, and a hash manifest. The archive is tested after extraction; the validators do not import each other's implementation. The source base is commit `b2a7f3268de19683573325c5a63d4ce0030ed955`. The frozen global and planar certificate hashes are, respectively, `ecfa47a4585bb646716888b93cc0ca4fca6fc2c33c5373786790c6d010af210a` and `09e3aa7dc2b2a80ce2222bae0a4a85357d0d36d09e257595b992ceb900db5c47`. From the extracted archive the fail-closed replay is

```
python verification/verify_global_projective_memory.py
  --check results/global_projective_memory/certificate.json
python verification/verify_planar_projective_memory.py
```

The archive is distributed beside the PDF as `Projective_Memory_Reproducibility.zip`; its final SHA-256 is recorded in the companion submission sheet and detached manifest, avoiding a self-referential archive hash inside the archive itself.

13 Conclusion

The exact state count of every finite line-routing table is the degree of one projective curve in the actual target span. This global identity replaces the former collection of direct-sum and coplanar strata by a single invariant. For block-subspace architectures, full-support Lagrange kernels give every nongeneric exact degree, while generic codimensions yield a closed design law. The closure problem is different: exact memory takes the most expensive band, whereas border memory can survive in the cheapest, producing an arbitrarily large separation already for scalar occupancies.

The incidence matrix makes this geometry falsifiable. Full column rank gives a calibrated positive chordal-error floor; first rank loss gives the exact zero-infimum boundary; base-point deletion explains every missing record. Most importantly, zero but unattained error is not free approximation. A bounded peak Wigner–Smith delay makes the fixed-degree lossless family compact, so the nonattained phase can approach zero only by driving a Potapov zero to the unit circle and creating a divergent delay peak. Passive line routing therefore has three operationally distinct phases: robust error, singular high-delay approximation, and exact finite-delay realization. The explicit planar cap, binary occupancy, and four-node cross-ratio theorems show how the global laws resolve the first collision strata without concealing their classical origins.

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