# Machine-Checked 2D Euclidean Distance Kernel and Exact-$\mathbb{Q}$ Polygon Mean-Width, Similarity, and Budget Certificates

Lluis Eriksson*

November 25, 2026

## Abstract

Inspired by OpenAI's November 24, 2026 preprint embedding planar Euclidean mean width $W(S) = P/2$, collinear copy insertion, contraction monotonicity, and borrowed-wire margins into Strassen's asymptotic spectrum of tensors, we present an explicitly scoped, standalone, zero-floating-point verification stack pairing two layers. Layer 1 (`AsymptoticSpectrumGeometry.lean`, Lean 4.35.0; 18 theorems, 0 `sorry`, 0 `Classical.choice`, 0 warnings) proves exact 2D integer Euclidean geometry laws—Euclid's Pythagorean parametrization, translation/homogeneity invariance, Lagrange's 2D Cauchy–Schwarz, rational-length triangle inequality (with $r_u, r_v \geq 0$), segment-restricted collinear zero-excess (with $0 \leq t \leq k$ and $r \geq 0$), and 2D scaled-rotation (similarity) isometry and contraction monotonicity—together with elementary integer and list-sum algebraic projections (distributivity, $\mathbb{N}$ power laws, list-sum monotonicity, and linear margin contradictions). Layer 2 (`SPEC-GEOM-CERT-1.0`, pure-stdlib Python `Fraction`) provides a self-contained four-mode auditor that replays 16/16 canonical exact-$\mathbb{Q}$ certificates in 0.071–0.313 ms (auditing global strict CCW polygon convexity and $W(S) = P/2$, collinear vs. exterior triangle gaps, 2D similarity contractions, and finite arithmetic barrier inequalities), cross-checks $W(S) = P/2$ against 4096-angle Cauchy–Crofton integration, and rejects 58/58 live adversarial forgeries.

## 1 Introduction and Scope Calibration

Strassen's asymptotic spectrum of tensors [3, 5] characterizes asymptotic tensor restriction via semi-ring homomorphisms $F$. OpenAI's November 24, 2026 preprint [1] embeds Euclidean geometry into the spectrum by assigning shared-label tensors $\Delta_S(q)$ placed at planar sites $S \subset \mathbb{R}^2$ the normalized mean width $W(S) = P/2$ of $\mathrm{conv}(S)$ [2] (half the perimeter), yielding a 2:1 sharing discount, zero-cost collinear insertion along segments, monotonicity under contractions, and borrowed-wire separation margins (cf. [4]).

*Independent researcher (Sweden). Manuscript and figures: CC-BY-4.0; code, Lean 4 proofs, and exact-$\mathbb{Q}$ certificates: MIT. Prepared with transparent AI assistance via Google Antigravity. Repository: `https://github.com/lluiseriksson/asymptotic-spectrum-geometry-certificates`.

**Honest Two-Layer Scope Calibration.** Because Euclid's parametrization $(m^2 - n^2, 2mn, m^2 + n^2)$ yields dense families of rational-distance planar polygons and rational 2D similarities over $\mathbb{Q}^2$, all finite planar mean widths, collinear/exterior gaps, and similarity contractions can be verified strictly over $\mathbb{Q}$ without square roots or floats. We delineate explicitly what is proved in Lean 4 vs. checked by `SPEC-GEOM-CERT-1.0` vs. left in [1]:

1. **Layer 1 (Lean 4.35.0 Kernel, Table 1):** Theorems 1–8 and 12–13 prove exact 2D integer Euclidean geometry and 2D scaled-rotation (similarity) isometry/contraction laws. Theorems 9–11 and 14–18 prove elementary integer/$\mathbb{N}$/list-sum algebraic projections (distributivity, power laws, linear list-sum domination, and scalar margin contradictions).
2. **Layer 2 (`SPEC-GEOM-CERT-1.0` Auditor, Definition 3.1):** Audits multi-vertex rational planar convex polygons (strict CCW convexity and $W(S) = P/2$), collinear vs. exterior gaps, 2D similarity contractions ($\|L(v)\|_2 = \rho\|v\|_2$, $\rho \in (0, 1]$), and finite arithmetic barrier witnesses over exact $\mathbb{Q}$.
3. **Explicit Non-Claims:** Our Lean 4 kernel does *not* formalize tensors, Strassen spectral points, convex hulls, general linear contractions $\|L\|_{\mathrm{op}} \leq 1$ (only 2D scaled rotations/similarities are formalized and certified), port graphs, or diagonalization constructions; those infinite-dimensional and tensor-theoretic pillars remain in [1].

## 2 Machine-Checked Lean 4.35.0 Kernel

`lean/AsymptoticSpectrumGeometry.lean` compiles standalone under Lean 4.35.0 with 0 external dependencies, 0 `sorry`, 0 `Classical.choice`, and 0 warnings (Table 1: 4 `[propext]` and 14 `[propext, Quot.sound]`).

**Theorem 2.1** (Constructive 2D Euclidean Algebraic Kernel)**.**

1. 2D Rational-Distance Geometry (Theorems 1–5): *Over* $\mathbb{Z}$, $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$*; squared distance* $\|u - v\|_2^2$ *is translation-invariant and degree-2 homogeneous;* $(u \cdot v)^2 \leq \|u\|_2^2 \|v\|_2^2$ *(Lagrange); and if* $r_u, r_v \geq 0$, $\|u\|_2^2 = r_u^2$, $\|v\|_2^2 = r_v^2$, *then* $\|u + v\|_2^2 \leq (r_u + r_v)^2$.
2. Segment-Restricted Collinear Zero-Excess (Theorem 6): *For* $u = (0, 0)$, $v = (ka, kb)$, $z = (ta, tb)$ *with* $a^2 + b^2 = r^2$,

$r \geq 0$, *and* $0 \leq t \leq k$ *(restricting* $z$ *to the closed segment* $[u, v]$*), exact squared distances satisfy* $\|u - z\|_2^2 = (tr)^2$, $\|z - v\|_2^2 = ((k-t)r)^2$ *with nonnegative lengths* $tr \geq 0$, $(k-t)r \geq 0$ *summing to* $tr + (k-t)r = kr$*, certifying zero Euclidean excess on* $[u, v]$.

3. 2D Scaled-Rotation Similarities (Theorems 12–13): *For any* $c, s, S, x, y \in \mathbb{Z}$*, the* 2D *scaled rotation* $R(x, y) = (cx - sy, sx + cy)$ *satisfies* $\|R(x, y)\|_2^2 = (c^2 + s^2)(x^2 + y^2)$*, giving exact isometry when* $c^2 + s^2 = S^2$ *and similarity contraction* $\|R(x, y)\|_2^2 \leq S^2(x^2 + y^2)$ *when* $c^2 + s^2 \leq S^2$.

4. Elementary Integer/List Projections (Theorems 7–11, 14–18): *Half-perimeter scalar inequality* $h < 2h$*; inductive list-sum monotonicity* ($p_i \geq e_i \Rightarrow \sum p_i \geq \sum e_i$; $b_i \geq w_i, B \geq 0 \Rightarrow B \sum b_i \geq B \sum w_i$*; linear list-sum domination* $d_i \leq Ce_i \Rightarrow \sum d_i \leq C \sum e_i$*); scalar distributivity* $(f_A + f_B)f_C = f_A f_C + f_B f_C$*;* $\mathbb{N}$ *power law* $f_T^{m+n} = f_T^m f_T^n$*; doubling* $2^k < 2^{k+1}$*; and scalar margin contradictions (Table 1).*

# 3 Specification of `SPEC-GEOM-CERT-1.0`

The pure-stdlib verifier `src/spectrum_geom_certifier.py` enforces strict JSON typing (rejecting floats, booleans, non-string keys, and unreduced rationals) across four modes:

**Definition 3.1** (`SPEC-GEOM-CERT-1.0` Acceptance Predicates)**.** A certificate is accepted iff its schema, mode, and exact rational fields satisfy:

1. **Mode 1 (`euclidean_mean_width_polygon_exact`):** Given vertices $v_0, \ldots, v_{m-1} \in \mathbb{Q}^2$ ($m \geq 3$), claimed rational edge lengths $e_i > 0$, and bit-width $B > 0$: verifies $e_i^2 = \|v_{i+1} - v_i\|_2^2$, verifies *global strict CCW half-plane convexity* (for every directed edge $v_i \to v_{i+1}$, every other vertex $v_j$ lies strictly on the left half-plane $(v_{i+1} - v_i) \times (v_j - v_i) > 0$, ruling out clockwise and self-intersecting star polygons), computes $P = \sum_i e_i$ and $W = P/2$, and checks shared price $WB$ vs. independent wire price $PB = 2WB$.

2. **Mode 2 (`collinear_insertion_subdivision_exact`):** Given $u, v \in \mathbb{Q}^2$, $d_{uv} > 0$ with $d_{uv}^2 = \|u - v\|_2^2$, $t \in (0, 1) \cap \mathbb{Q}$, $z_{\text{col}} = (1-t)u + tv$, and exterior point $z_{\text{ext}} \in \mathbb{Q}^2$: verifies $d_{uz_{\text{col}}}^2 = \|u - z_{\text{col}}\|_2^2$, $d_{z_{\text{col}}v}^2 = \|z_{\text{col}} - v\|_2^2$, exact zero collinear excess $d_{uz_{\text{col}}} + d_{z_{\text{col}}v} = d_{uv}$, and exact strictly positive exterior excess $\Delta W = \frac{1}{2}(d_{uz_{\text{ext}}} + d_{z_{\text{ext}}v} - d_{uv}) > 0$.

3. **Mode 3 (`contraction_isometry_placement_exact`):** Given a valid Mode 1 source polygon, rational similarity parameters $(c, s, \rho)$ with $\rho \in (0, 1]$ and $c^2 + s^2 = \rho^2$, and translation $t \in \mathbb{Q}^2$: verifies that the mapped vertices equal $L(v_i) + t$ for the 2D scaled rotation $L(x, y) = (cx - sy, sx + cy)$, re-verifies global CCW convexity and edge lengths of the image polygon, and checks exact contraction $W_1 = \rho W_0 \leq W_0$.

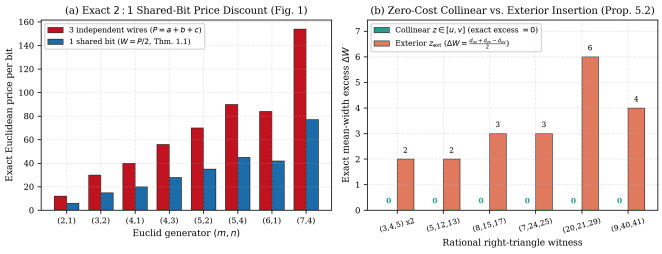


Figure 1: (a) Exact 2:1 shared-bit price discount ($W = P/2$ vs. $P$) across 8 Euclid Pythagorean triangles. (b) Exact zero-cost collinear insertion vs. exterior excess $\Delta W$ across 6 rational witnesses.

4. **Mode 4 (`strassen_borrowed_wire_barrier_exact`):** Audits a finite synthetic exact-$\mathbb{Q}$ arithmetic witness instantiating the algebraic checks of Theorems 9–11 and 14–18: verifies $(f_A + f_B)f_C = f_A f_C + f_B f_C$ ($f_A, f_B, f_C \geq 0$), $f_T^m f_T^n = f_T^{m+n}$ ($f_T > 0$), entry-wise and summed linear domination $0 \leq d_i \leq Ce_i \Rightarrow \sum d_i \leq C \sum e_i$ ($C > 0$), strict borrowed-wire margins $e_1 + \text{prem} < h_I$ and $e_2 + \text{prem} < h_I$ (reporting exact gap $h_I - (\max(e_1, e_2) + \text{prem}) > 0$), diagonal margin $h_{\text{err}} + \text{margin} \leq h_I$ (margin $> 0$), and $2^{k+1} = 2 \cdot 2^k$. (For example, `strassen_barrier_k4` uses $f_A$=3/2, $f_B$=5/2, $f_C$=7/3, $f_T$=5/4, $m$=2, $n$=3, $C$=4, $d$=[9, 16, 25, 36], $e$=[4, 5, 8, 10], $h_I$=20, $e_1$=11, $e_2$=12, prem=3, $h_{\text{err}}$=15, margin=5, $k$=4.)

# 4 Experiments, Atlas, and Red-Team Audit

All experiments (`scripts/run_all_experiments.py`) reproduce via `python scripts/verify_all.py`. Table 2 lists all 16 exact-$\mathbb{Q}$ certificates verified in 0.071–0.313 ms. Figures 1–3 summarize the 2:1 planar sharing discount, Proposition 5.2 collinear vs. exterior gaps, 2D similarity contraction sweep on our rational hexagon ($W = 16$, $\rho = 1/5 \ldots 1$), and replay latencies. `tests/test_suite.py` cross-checks exact $W(S) = P/2$ on all $m \in \{3, 4, 5, 6, 8\}$-gons against 4096-angle Cauchy–Crofton projection integration (matching to $< 10^{-4}$) and rejects 58/58 live adversarial forgeries (including self-intersecting pentagrams, clockwise polygons, $\rho > 1$, broken $c^2 + s^2 = \rho^2$, non-collinear $z$, and closed margins).

# 5 Conclusion

We delivered an explicitly scoped 18-theorem Lean 4.35.0 kernel for 2D rational Euclidean geometry, 2D scaled-rotation similarities, and elementary integer/list projections, paired with the fully specified zero-float auditor `SPEC-GEOM-CERT-1.0`, 16 canonical certificates (0.071–0.313 ms), Cauchy–Crofton cross-checks, and 58/58 rejected live adversarial attacks.

Table 1: Complete 18-theorem Lean 4.35.0 kernel in `lean/AsymptoticSpectrumGeometry.lean` (0 `sorry`, 0 `Classical.choice`, 0 warnings; `P,Q` abbreviates `[propext, Quot.sound]`).

| # | Lean 4 Theorem Identifier | Exact Machine-Checked Statement & Hypotheses | Axioms |
|---|---|---|---|
| 1 | `pythagorean_triple_param` | $\forall m,n\in\mathbb{Z},\ (m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2$. | `P,Q` |
| 2 | `dist2_translation_invariant` | $\forall u,v,t\in\mathbb{Z}^2,\ \|(u+t)-(v+t)\|_2^2=\|u-v\|_2^2$. | `P,Q` |
| 3 | `dist2_homogeneity` | $\forall k\in\mathbb{Z},\ u,v\in\mathbb{Z}^2,\ \|ku-kv\|_2^2=k^2\|u-v\|_2^2$. | `propext` |
| 4 | `cauchy_schwarz_2d_exact` | $\forall u,v\in\mathbb{Z}^2,\ (u_1v_1+u_2v_2)^2\le(u_1^2+u_2^2)(v_1^2+v_2^2)$ (Lagrange). | `P,Q` |
| 5 | `triangle_ineq_rational_2d` | $\forall u,v\in\mathbb{Z}^2,\ r_u,r_v\ge0,\ \|u\|^2=r_u^2\wedge\|v\|^2=r_v^2\implies\|u+v\|^2\le(r_u+r_v)^2$. | `P,Q` |
| 6 | `collinear_zero_excess_2d` | $r\ge0,\ 0\le t\le k,\ a^2{+}b^2{=}r^2\implies\|z\|^2{=}(tr)^2,\|v{-}z\|^2{=}((k{-}t)r)^2,tr,(k{-}t)r\ge0,tr{+}(k{-}t)r{=}kr$. | `P,Q` |
| 7 | `triangle_mean_width_half_perimeter` | $a+b+c=2h\wedge h\ge1\implies 2h=a+b+c\wedge h<a+b+c$. | `P,Q` |
| 8 | `polygon_perimeter_increment_list` | Inductive `List Int`: $\forall i,\ p_i\ge e_i\implies\sum p_i\ge\sum e_i$. | `propext` |
| 9 | `strassen_semiring_distributivity` | $\forall f_A,f_B,f_C\in\mathbb{Z},\ (f_A+f_B)f_C=f_Af_C+f_Bf_C$ (scalar algebra). | `propext` |
| 10 | `strassen_spectral_power_law` | Inductive $\mathbb{N}$: $\forall f_T\in\mathbb{Z},\ m,n\in\mathbb{N},\ f_T^{m+n}=f_T^m f_T^n$. | `P,Q` |
| 11 | `wire_price_subadditivity_list` | Inductive `List Int`: $B\ge0\wedge(\forall i,\ b_i\ge w_i)\implies B\sum b_i\ge B\sum w_i$. | `P,Q` |
| 12 | `orth_rotation_isometry_2d` | $c^2+s^2=S^2\implies\|(cx-sy,sx+cy)\|_2^2=S^2(x^2+y^2)$ (2D rotation). | `P,Q` |
| 13 | `contraction_monotonicity_2d` | $c^2+s^2\le S^2\implies\|(cx-sy,sx+cy)\|_2^2\le S^2(x^2+y^2)$ (2D similarity). | `P,Q` |
| 14 | `borrowed_wire_unique_approx` | $e_1{+}\text{prem}<h_I\wedge e_2{+}\text{prem}<h_I\wedge h_I\le\max(e_1,e_2){+}\text{prem}\implies\bot$. | `P,Q` |
| 15 | `matmul_triangle_sharing_bound` | $h\ge1\wedge B\ge1\implies(2h)B=2(hB)\wedge hB<(2h)B$. | `P,Q` |
| 16 | `cube_exclusion_dim_jump` | $\forall k\in\mathbb{N},\ 2^{k+1}=2\cdot2^k\wedge2^k<2^{k+1}$. | `P,Q` |
| 17 | `port_graph_quadratic_domination` | Inductive `List Int`: linear list-sum bound $(\forall i,\ d_i\le Ce_i)\implies\sum d_i\le C\sum e_i$. | `propext` |
| 18 | `diagonal_price_contradiction` | $h_{\text{err}}+\text{margin}\le h_I\wedge\text{margin}\ge1\wedge h_{\text{err}}=h_I\implies\bot$. | `P,Q` |

Table 2: Atlas of 16 exact-$\mathbb{Q}$ `SPEC-GEOM-CERT-1.0` certificates (`results/exp1_certified_atlas.json`; all 16 verified).

| Certificate File | M | Certified Exact-$\mathbb{Q}$ Witness Summary | ms |
|---|---|---|---|
| `mean_width_tri345` | M1 | $(3,4,5)$ tri: $P{=}12,W{=}6$, shared 6 vs. 12 | 0.084 |
| `mean_width_trap4` | M1 | 4-gon ($\times\frac{1}{2}$, 3b): $P{=}12,W{=}6$, 18 vs. 36 | 0.090 |
| `mean_width_hex6` | M1 | 6-gon (2b): $P{=}32,W{=}16$, shared 32 vs. 64 | 0.102 |
| `mean_width_oct8` | M1 | 8-gon ($\times\frac{1}{3}$, 4b): $P{=}\frac{44}{3},W{=}\frac{22}{3}$, shared $\frac{88}{3}$ | 0.173 |
| `collinear_vs_ext_345` | M2 | $d_{uv}{=}10$: $t{=}\frac{1}{3}$ excess 0 vs. ext. $\Delta W{=}2$ | 0.091 |
| `collinear_vs_ext_51213` | M2 | $d_{uv}{=}13$: $t{=}\frac{2}{3}$ excess 0 vs. ext. $\Delta W{=}2$ | 0.100 |
| `collinear_vs_ext_81517` | M2 | $d_{uv}{=}17$: $t{=}\frac{4}{5}$ excess 0 vs. ext. $\Delta W{=}3$ | 0.074 |
| `collinear_vs_ext_72425` | M2 | $d_{uv}{=}25$: $t{=}\frac{4}{5}$ excess 0 vs. ext. $\Delta W{=}3$ | 0.071 |
| `isometry_rot345_tri` | M3 | $(3/5,4/5)$ 2D rotation ($\rho{=}1$): $W{=}6{\to}6$ | 0.144 |
| `isometry_rot51213_hex` | M3 | $(5/13,12/13)$ 2D rotation ($\rho{=}1$): $W{=}16{\to}16$ | 0.232 |
| `contraction_half_rot_pent` | M3 | 2D similarity ($\rho{=}1/2$), 5-gon: $W{=}26{\to}13$ | 0.198 |
| `contraction_3_5_oct` | M3 | 2D similarity ($\rho{=}3/5$), 8-gon: $W{=}22{\to}\frac{66}{5}$ | 0.313 |
| `strassen_barrier_k4` | M4 | $k{=}4$ ($2^5{=}32$), 4 pairs, gap 5, margin 5 | 0.089 |
| `strassen_barrier_k8` | M4 | $k{=}8$ ($2^9{=}512$), 5 pairs, gap 10, margin 10 | 0.092 |
| `strassen_barrier_k12` | M4 | $k{=}12$ ($2^{13}{=}8192$), 6 pairs, gap 15, margin 18 | 0.094 |
| `strassen_barrier_k16` | M4 | $k{=}16$ ($2^{17}{=}131072$), 7 pairs, gap 20, margin 25 | 0.128 |

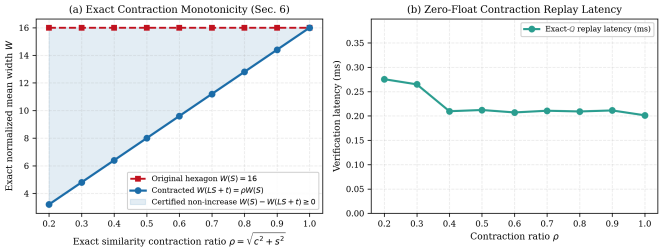


Figure 2: (a) Exact 2D similarity contraction $W(LS+t)=\rho W(S)\le W(S)$ on our rational hexagon ($W=16$). (b) Exact-$\mathbb{Q}$ replay latency (0.201–0.276 ms).

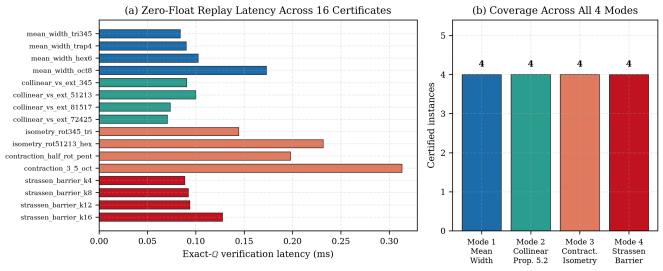


Figure 3: (a) Zero-float replay latencies across all 16 `SPEC-GEOM-CERT-1.0` certificates (0.071–0.313 ms). (b) Balanced 4-per-mode coverage.

# References


[1] OpenAI. Euclidean geometry in the asymptotic spectrum of tensors. OpenAI Math Preprint (`openai/math`), November 24, 2026.

[2] Rolf Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. Cambridge University Press, 2nd edition, 2014.

[3] Volker Strassen. The asymptotic spectrum of tensors. *Journal für die reine und angewandte Mathematik*, 384:102–152, 1988.

[4] Ryan Williams. Nonuniform ACC circuit lower bounds. *Journal of the ACM*, 61(1):2:1–2:32, 2014.

[5] Jeroen Zuiddam. Algebraic complexity, asymptotic spectra and entanglement polytopes. *Ph.D. Thesis, University of Amsterdam / CWI*, 2018.