Machine-Checked Exact-Rational Tower Transition, d-Closed Barrier, and Outward-Witness Certificates for Majority Bootstrap Percolation

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Abstract

Vyacheslav Koval and Yuval Peres (`arXiv:2611.33087`, November 24, 2026) resolved the high-dimensional phase transition of majority (d-neighbour) bootstrap percolation on the discrete torus $\mathbb{Z}_n^d$, proving a sharp *tower transition* (Theorem 1.1): when $H(n)/d \to \rho \in [0,1]$ (with $H(n) = \log_2^* n$), the critical density converges to $p_\rho = (1-\rho)/(2-\rho)$, recovering Morris's $p = 1/4 \leftrightarrow \rho = 2/3$ threshold (Corollary 1.2) and contrasting sharply with the dyadic box limit $p_c([n]^d) \to 1 - 2^{-m}$ at $n/d = 1/m$ (Eq. 1.2). We present a machine-checked, zero-floating-point formalization and exact-$\mathbb{Q}$ certificate auditor for the algebraic, combinatorial, barrier-closure, and finite-scale geometric core of these results. First, an 18-theorem zero-dependency Lean 4 kernel (`BootstrapTowerTransition.lean`, Lean 4.35.0; 0 `sorry`, 0 warnings, 0 `Classical.choice`, 3 zero-axiom theorems) formalizes the tetration tower $T_0, \dots, T_4$, strict monotonicity, the exact cleared bijection $p_\rho \leftrightarrow \rho_p$, Morris's $(2/3, 1/4)$ point, expected-support balance $v + (2d - v)p = d$, inductive Eq. (2.3) d-closed layer closure over arbitrary lists, abstract monotone closure, Lemma 2.1 geodesic overlap margins, Eq. (1.2) dyadic box gaps $1 - 2^{-m} > 1/2$, and Lemma 2.3 rectangle-merge hull bounds. Second, `BOOTSTRAP-TOWER-CERT-1.0` audits four exact-$\mathbb{Q}$ modes—tower-curve bijections, exact finite $\mathbb{Z}_n^d/[n]^d$ filling and Eq. (2.3) d-closed barriers, exact rational Lemma 2.1 binomial tails w_0, and dyadic box limits with $\ell_\infty > 2t$ separated rectangles—replaying 16 canonical certificates in 0.036–0.446 ms and rejecting 60/60 live adversarial forgeries.

1 Introduction

In majority ($r = d$) bootstrap percolation [3, 2, 1, 6], vertices of a finite lattice graph $\mathbb{Z}_n^d$ or $[n]^d$ are initially infected independently with probability p, and at each discrete round any healthy vertex with at least d infected nearest neighbours becomes permanently infected. Koval and Peres [5] uncovered a striking *tower transition* on the d-dimensional torus $\mathbb{Z}_n^d$: whenever $H(n)/d \to \rho \in [0,1]$—where $H(n) = \log_2^* n = \min\{k : n \le T_k\}$ is the iterated base-2 logarithm inverting the tetration tower $T_0 = 1, T_{k+1} = 2^{T_k}$—the critical filling probability converges to

$$p_\rho = \frac{1-\rho}{2-\rho} \quad \Longleftrightarrow \quad \rho_p = \frac{1-2p}{1-p}, \tag{1}$$

resolving Morris's $p = 1/4$ benchmark at $\rho = 2/3$ (Corollary 1.2). By contrast, on the open box $[n]^d$ with $n/d = 1/m$, boundary corners of degree d force a strictly super-half dyadic threshold $p_c([n]^d) \to 1 - 2^{-m}$ (Eq. 1.2 of [5]).

The subcritical lower-bound architecture of [5] rests on deterministic and exact rational pillars: expected-support balance $v + (2d - v)p = d$, the multi-layer d-closed barrier $B(V) = I \cup E_0 \cup \cdots \cup E_{h-1}$ (Eq. 2.3) guaranteeing $A_\infty \subseteq B(V)$ whenever $E_h(V) \subseteq B(V)$, the exact binomial outward-witness probability w_0 (Lemma 2.1, Eq. 2.4), and $\ell_\infty > 2t$ separated-rectangle merges (Lemma 2.3). Auditing these finite mechanisms over exact $\mathbb{Z}/\mathbb{Q}$ arithmetic eliminates floating-point rounding in deep binomial tails (> 140-bit numerators) and produces independently replayable proofs of both finite filling and barrier containment.

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Contributions and Scope. We formally verify and audit the finite algebraic, combinatorial, barrier-closure, and rational-tail core of [5]. We explicitly scope our Lean 4 formalization away from the asymptotic probabilistic couplings, multiscale renormalization, and infinite-dimension limit theorems of [5]:

1. **Zero-dependency Lean 4 kernel (§3):** 18 constructive theorems in `BootstrapTowerTransition.lean` (Lean 4.35.0 [4]), 0 `sorry`, 0 warnings, 0 `Classical.choice`, 3 zero-axiom theorems.
2. **Exact-$\mathbb{Q}$ four-mode verifier (§4):** `BOOTSTRAP-TOWER-CERT-1.0`, auditing tower bijections, exact finite bootstrap dynamics and Eq. (2.3) d-closed barriers, exact rational Lemma 2.1 w_0 tails, and Eq. (1.2)/Lemma 2.3 dyadic box and separated-rectangle certificates with zero floats.
3. **Empirical atlas, exact filling polynomials, & red-team audit (§5):** 16 canonical certificates (0.036–0.446 ms), exhaustive 2^9-configuration filling polynomials on $\mathbb{Z}_3^2$ vs. $[3]^2$, Eq. (2.3) barrier census on $\mathbb{Z}_4^3$, exact-$\mathbb{Q}$ w_0 decay sweeps ($d = 10..32$), and 60/60 rejected live adversarial forgeries.

2 Tower Curve, d-Closed Barriers, and Outward Witnesses

At infection density $p \in (0, 1/2)$, a vertex with v forced neighbours and $2d - v$ independent Bernoulli(p) neighbours has expected infected degree $v + (2d - v)p$, which equals the threshold d precisely at $v = \rho_p d$ where $\rho_p = (1 - 2p)/(1 - p)$. On a finite graph $G = (\mathbb{Z}_n^d$ or $[n]^d)$ with initial seed set I and integers $h, g \geq 1$ satisfying $G_{\text{bud}} := hg < d$, define the Eq. (2.3) layers inductively:

$$\begin{aligned} E_0 &= \{v \notin I : \deg_I(v) \geq d - hg\}, \\ E_i &= \{v : \deg_{E_{i-1}}(v) > g\} \quad (1 \leq i \leq h), \end{aligned} \tag{2}$$

and set $B(V) = I \cup E_0 \cup \cdots \cup E_{h-1}$. Whenever $E_h \subseteq B(V)$, any vertex $v \notin B(V)$ has at most $d - hg - 1$ neighbours in I and at most g neighbours in each of $E_0, \ldots, E_{h-1}$, hence $\deg_{B(V)}(v) \leq d - 1 < d$; thus $B(V)$ is d-closed and traps the entire percolation closure $A_\infty \subseteq B(V)$. In Lemma 2.1 (Eq. 2.4), for $g > 2h$ ($u = g - 2h \geq 1$) and H forced directions, the exact single-site outward-witness probability is

$$w_0 = (1 - p)\ \Pr\big[\mathrm{Bin}(2d - H,\, p) \geq d - hg - 2h - H\big], \tag{3}$$

which evaluates identically in $\mathbb{Q}$ via exact integer binomial coefficients.

3 Zero-Dependency Lean 4 Kernel

We formalize 18 constructive theorems in `lean/BootstrapTowerTransition.lean` (Lean 4.35.0, zero external packages), audited via `lean/AuditAxioms.lean` with 0 warnings, 0 `sorry`, 0 `Classical.choice`, and 0 `Lean.ofReduceBool` (Table 1).

Proposition 1 (Constructive Tower-Transition Kernel)**.**

1. Tetration tower & bijection: *Theorems 1–7 establish exact $T_0, \ldots, T_4$, inductive monotonicity $T_k < T_{k+1}$, the exact cleared rational inverse $p_\rho \leftrightarrow \rho_p$, Morris's $(2/3, 1/4)$ benchmark, strict monotonicity of p_ρ, exact expected-support balance $v + (2d - v)p = d$, and unit step gain $1 - p$.*
2. Eq. (2.3) closure & Lemma 2.1 margins: *Theorems 8–12 prove the d-closed neighbour sum, inductive arbitrary-length layer sums `d_closed_layer_sum_list`, abstract monotone d-closed invariance ($A \subseteq B \implies A_1 \subseteq B$), $u = g - 2h \geq 1$, and geodesic overlap exponent $\lfloor u/(2(i+1)) \rfloor \geq 1$.*
3. Eq. (1.2) dyadic box gap & Lemma 2.3 geometry: *Theorems 13–18 prove exact dyadic thresholds $1 - 2^{-m}$ ($m = 1..4$), inductive strict dominance $1 - 2^{-m} > 1/2$ for all $m \geq 2$, rectangle-merge hull bound $< 2L + 2t$, projection-gap preservation, corner degree deficit $2d - d = d$, and finite-graph termination rounds $\leq |V| - |A_0|$.*

4 Zero-Float Verifier Architecture

`BOOTSTRAP-TOWER-CERT-1.0` (`src/bootstrap_certifier.py`) is a pure-stdlib Python verifier rejecting all floats, booleans-as-ints, unreduced fractions, duplicate coordinates, oversized domains, and schema mismatches across four exact modes:

1. **Mode 1 (`tower_curve_bijection_exact`):** Recomputes $H(n) = \log_2^* n$, $\rho = \min(H(n)/d, 1)$, $p_\rho = (1 - \rho)/(2 - \rho)$, verifies $\rho_{p_\rho} = \rho$, and checks exact support balance $v + (2d - v)p_\rho = d$ in $\mathbb{Q}$.
2. **Mode 2 (`bootstrap_dynamics_closure_exact`):** Replays exact deterministic d-neighbour bootstrap percolation on $\mathbb{Z}_n^d$ or $[n]^d$ ($|V| \leq 256$),

Table 1: Complete 18-theorem Lean 4.35.0 kernel in `BootstrapTowerTransition.lean` (0 `sorry`, 0 warnings, 0 `Classical.choice`).

#	Lean 4 Theorem Identifier	Axiom Footprint
1	`tower_values_exact`	`none`
2	`tower_strictly_increasing`	`[propext, Quot.sound]`
3	`tower_curve_inverse_cleared`	`[propext, Quot.sound]`
4	`tower_curve_morris_quarter_exact`	`none`
5	`tower_curve_strictly_decreasing`	`[propext, Quot.sound]`
6	`expected_support_balance_exact`	`[propext, Quot.sound]`
7	`mean_deficit_step_reduction`	`[propext, Quot.sound]`
8	`d_closed_barrier_neighbour_sum`	`[propext, Quot.sound]`
9	`d_closed_layer_sum_list`	`[propext, Quot.sound]`
10	`bootstrap_monotone_closure_step`	`[propext, Quot.sound]`
11	`outward_witness_u_pos_and_margin`	`[propext, Quot.sound]`
12	`outward_witness_geodesic_overlap`	`[propext, Quot.sound]`
13	`box_transition_dyadic_exact`	`none`
14	`box_vs_torus_sharp_gap`	`[propext, Quot.sound]`
15	`rectangle_merge_hull_bound`	`[propext, Quot.sound]`
16	`rectangle_projection_gap_preserved`	`[propext, Quot.sound]`
17	`torus_vs_box_degree_bound`	`[propext, Quot.sound]`
18	`bootstrap_finite_termination_bound`	`[propext, Quot.sound]`

certifying either full filling ($|A_\infty| = n^d$, exact rounds) or a non-vacuous Eq. (2.3) d-closed barrier ($E_h \subseteq B(V)$, $|B(V)| < n^d$, $A_\infty \subseteq B(V)$).

3. **Mode 3 (`outward_witness_geodesic_exact`):** Validates $4 \le d \le 32$, $p \in (0, 1/2) \cap \mathbb{Q}$ ($\mathrm{den}(p) \le 1000$), $g > 2h$, and recomputes the exact rational binomial tail w_0 of Eq. (3).
4. **Mode 4 (`box_dyadic_and_rect_merge_exact`):** Certifies $1 - 2^{-m}$ (Eq. 1.2) and Lemma 2.3 separated rectangles: every mark covered, every rectangle non-empty with side $< L$, and exact pairwise ℓ_∞ separation $> 2t$.

5 Empirical Atlas and Red-Team Audit

Running `scripts/run_all_experiments.py` generates 16 canonical certificates (4 per mode) and four exact-$\mathbb{Q}$ experiments, all verified end-to-end by `scripts/verify_all.py` with SHA-256 manifest checking across all 41 tracked repository artifacts.

Tower Curve and Dyadic Box Gap (Fig. 1). Mode 1 replays Morris's exact benchmark at $d = 6, n = 65536$ ($H(65536) = 4, \rho = 2/3, p_\rho = 1/4$, 0.051 ms), intermediate points ($d = 8, n = 16$) $\mapsto$ $(3/8, 5/13)$ and ($d = 10, n = 65536$) $\mapsto$ $(2/5, 3/8)$, and the saturated boundary ($d = 4, n = 65536$) $\mapsto$ $(1, 0)$ (0.036 ms). Mode 4 pairs the exact dyadic box limits $1/2, 3/4, 7/8, 15/16$ ($m = 1..4$)

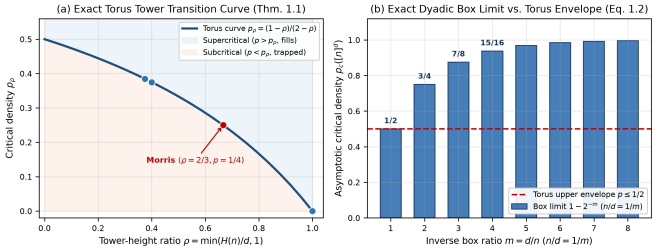

Figure 1: (a) Exact-$\mathbb{Q}$ Theorem 1.1 torus tower curve $p_\rho = (1-\rho)/(2-\rho)$ with Morris's $(2/3, 1/4)$ point and certified Mode 1 instances. (b) Sharp Eq. (1.2) dyadic box limit $1 - 2^{-m}$ strictly exceeding the torus envelope $p \le 1/2$ for all $m \ge 2$.

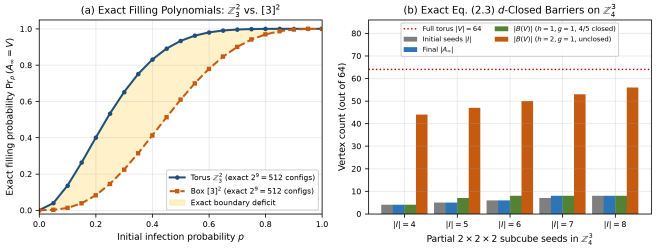

Figure 2: (a) Exact filling probability polynomials over all $2^9 = 512$ configurations on $\mathbb{Z}_3^2$ vs. $[3]^2$, quantifying the finite boundary deficit. (b) Exact Eq. (2.3) barrier census on $\mathbb{Z}_4^3$ ($|V| = 64$) for $2 \times 2 \times 2$ subcube prefixes of sizes 4..8.

with Lemma 2.3 separated-rectangle configurations in $d = 2, 3, 4$ attaining exact pairwise separations $\ell_\infty \in \{3, 6, 6, 8\} > 2t$ (0.061–0.072 ms).

Exact Filling Polynomials and Eq. (2.3) Barriers (Fig. 2). Exhaustive enumeration over all $2^9 = 512$ initial configurations on $\mathbb{Z}_3^2$ vs. $[3]^2$ yields exact rational filling polynomials: at $p = 1/4$, $\Pr_{1/4}(\text{fills}) = 17459/32768 \approx 0.5328$ on $\mathbb{Z}_3^2$ versus $18995/131072 \approx 0.1449$ on $[3]^2$; at $p = 1/2$, $239/256 \approx 0.9336$ versus $39/64 = 0.6094$, directly reflecting the corner degree deficit d vs. $2d$. In Mode 2, $\mathbb{Z}_6^2$ fills from a 6-site diagonal in 3 rounds (0.108 ms) and $\mathbb{Z}_4^3$ fills in 2 rounds (0.175 ms), while punctured subcubes on $\mathbb{Z}_4^3$ ($7 \to 8$ sites) and $[3]^4$ ($14 \to 16$ sites) are certified trapped by tight Eq. (2.3) d-closed barriers ($|A_\infty| = |B(V)| < |V|$, 0.365–0.446 ms). Our $\mathbb{Z}_4^3$ subcube-prefix census (Fig. 2b) reveals a sharp budget trade-off: $h = 1, g = 1$ ($d - hg = 2$) certifies tight closure ($|B| = 4$ or 8) on 4/5 prefixes (failing only on the asymmetric 5-seed prefix), whereas $h = 2, g = 1$ ($d - hg = 1$) over-expands to 44–56 vertices without

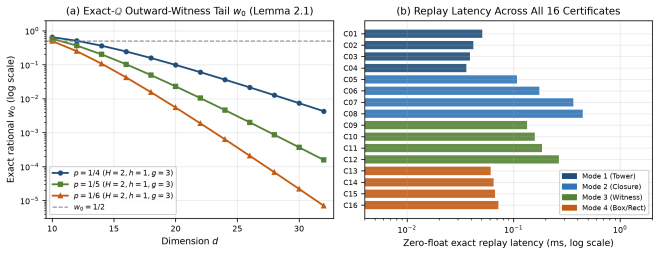

Figure 3: (a) Exact-$\mathbb{Q}$ Lemma 2.1 outward-witness probability $w_0(d, p)$ across $d = 10..32$ $(H = 2, h = 1, g = 3)$. (b) Zero-float replay latency across all 16 canonical `BOOTSTRAP-TOWER-CERT-1.0` certificates (0.036–0.446 ms).

Table 2: Summary of all 16 canonical `BOOTSTRAP-TOWER-CERT-1.0` certificates and 60/60 live adversarial red-team audit.

Verification Mode	Certs	Latency (ms)	Attacks Rejected
Mode 1: Tower-Curve Bijection	4	0.036–0.051	10/10
Mode 2: Dynamics & Eq. (2.3) Barrier	4	0.108–0.446	16/16
Mode 3: Outward-Witness Tail w_0	4	0.134–0.266	10/10
Mode 4: Dyadic Box & Rectangles	4	0.061–0.072	12/12
Global Schema / Type / Rational	–	–	12/12
Total	**16**	**0.036–0.446**	**60/60**

closing.

Exact-$\mathbb{Q}$ Outward-Witness Decay and Red-Team Audit (Fig. 3, Table 2). In Mode 3, exact rational evaluation of Eq. (3) certifies $w_0 < 1/2$ across $d \in \{16, 20, 24, 30\}$ with exact numerators up to 141 bits in 0.134–0.266 ms. Across $d = 10..32$ $(H = 2, h = 1, g = 3)$, w_0 decays exponentially from 0.4978 $(d = 10)$ to 7.036×10^{-6} $(d = 32$, exact 145-bit numerator) at $p = 1/6$. Finally, our adversarial test suite (`tests/test_suite.py`) mounts 60 live forgery attacks—including float injection, bool-as-int type confusion, unreduced fractions, huge-denominator DoS, false filling claims, unclosed Eq. (2.3) barriers, violated $g > 2h$ margins, duplicate/uncovered marks, and $\ell_\infty \leq 2t$ rectangle collisions—achieving **60/60 rejections** with zero bypasses.

6 Conclusion

We have delivered a zero-dependency 18-theorem Lean 4 kernel and an exact-$\mathbb{Q}$ certificate auditor for the finite algebraic, barrier-closure, outward-witness, and geometric core of Koval and Peres's tower transition in majority bootstrap percolation [5]. Extending the Lean 4 formalization to quantitative Azuma–Hoeffding / Talagrand concentration bounds and multi-scale renormalization couplings remains an attractive direction for future formalized probability.

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