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Higher-Dimensional Hadwiger Covering Beyond Three Dimensions: Exact Illumination and Continuum Covering of the Self-Dual 4D 24-Cell, 4D Regular Polytope Radii, Factor-d Helly Collapse, and the Measure-1 Double-Shadow Obstruction

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Abstract

In a landmark breakthrough, Aiger, Har-Peled, and Žužić [1] settled Hadwiger's seven-decade-old covering conjecture in three dimensions ($c(K) \le 8$ for every convex body $K \subset \mathbb{R}^3$) by combining a 2-direction orthogonal shadow-peeling step with John's position mean-width bound ($8 \int_{S^2} h_S \, d\tilde{\sigma} < 12$) and a 1D-arc Helly chord-telescoping lemma. Hadwiger's conjecture remains wide open in all dimensions $d \ge 4$ (in particular $c(K) \le 2^4 = 16$ in $\mathbb{R}^4$).

We establish a complete, machine-checked higher-dimensional analysis ($d \ge 4$, with special emphasis on the unique four-dimensional geometry of $\mathbb{R}^4$), backed by 20 constructive Lean 4.36.0 theorems (zero `sorry`, zero `Classical.choice`, zero external imports) and an AST-audited zero-float exact-rational verifier generating 16 SHA-256 certificates and passing 56 adversarial tests:

1. **Exact Illumination Number of the Self-Dual 4D 24-Cell (P_{24}, F_4): $c(P_{24}) = 8$.** Unlike the 3D Platonic solids, $\mathbb{R}^4$ admits the self-dual 24-cell P_{24} (24 vertices, 24 octahedral facets). We prove exhaustively in Lean 4 that its 15-regular distance-2 graph $G_2(P_{24})$ has exact independence number $\alpha(G_2(P_{24})) = 3$ (forcing $c(P_{24}) \ge 24/3 = 8$), and construct a centrally symmetric 4-pair integer illumination certificate $\pm(2, 0, -1, -1)$, $\pm(0, 2, -1, 1)$, $\pm(0, 1, 2, -1)$, $\pm(1, 0, 1, 2)$ illuminating all 24 vertices (3 disjoint vertices per direction) and hence all of ∂P_{24}.
2. **Exact Continuum 8-Homothet Covering Bounds $\frac{3}{4} \le \lambda_8(P_{24}) \le \frac{8}{9}$.** We prove $\lambda_8(P_{24}) \ge \frac{3}{4}$ via a one-line 4-vector LP dual identity canceling the translation center identically, and prove $\lambda_8(P_{24}) \le \frac{8}{9}$ unconditionally over the *entire continuum* $P_{24} \subset \mathbb{R}^4$ (zero sampling) via 576 exact rational Voronoi-cell LP dual certificates (denominators in $\{1, 2, 3\}$, hence in $\frac{1}{6}\mathbb{Z}_{\ge 0}$, kernel-checked in Lean 4 `thm14`–`thm15`).
3. **Exact Covering Radii of the Core 4D Regular Polytopes.** We certify the exact optimal minimal-cardinality covering ratios for the 5-cell Δ_4 ($c = 5$, $\lambda_5 = \frac{4}{5}$, and $\lambda_{d+1}(\Delta_d) = \frac{d}{d+1}$ for all $d \ge 2$), the 16-cell β_4 ($c = 8$, $\lambda_8 = \frac{3}{4}$, and exact $\lambda_{2d}(\beta_d) = \frac{d-1}{d}$ for all $d \ge 2$), the 24-cell P_{24} ($c = 8$, $\lambda_8 \in [\frac{3}{4}, \frac{8}{9}]$), and the 8-cell C_4 ($c = 16$, exact closed minimum $\lambda_{16} = \frac{1}{2}$, with explicit open-interior cover at $\lambda = \frac{2}{3}$).
4. **Two Exact Structural Obstructions to Extending 3D Shadow Peeling into $d \ge 4$.** First, we prove an exact factor-d collapse of the universal 1-homothet S-diameter covering threshold from 2 (for 1D shadow curves in $\mathbb{R}^3$) to $2/d$ ($1/2$ in $\mathbb{R}^4$) on the inverted d-simplex $-\frac{\delta}{2}\Delta_d$. Second, we prove a **Universal Measure-1 Non-Smooth Double-Shadow**

Obstruction: for *every* pair of directions $(w_1, w_2) \in S^3 \times S^3$ without exception, the double-shadow boundary $\Sigma(C_4, w_1) \cap \Sigma(C_4, w_2)$ retains at least 4 complete 2-dimensional square faces (requiring $k_{\min}(d) = 1 + \lceil \log_2 d \rceil$ pairs on C_d, with $k = 3$ achieved via Hadamard H_4 in $\mathbb{R}^4$), alongside the exact Wallis recurrence $(d\kappa_d)((d+1)\kappa_{d+1}) = \frac{2(d+1)}{\pi}$ for the John mean-width constant.

1 Introduction

For a convex body $K \subset \mathbb{R}^d$, the Hadwiger covering number $c(K)$ is the minimum number of smaller positive homothetic copies $\lambda K + v_i$ ($\lambda \in (0,1)$, $v_i \in \mathbb{R}^d$) whose union covers K [7, 6]. By Boltyanski's theorem [4], $c(K)$ equals the illumination number: the minimum number of nonzero directions $d_1, \ldots, d_m \in \mathbb{R}^d \setminus \{0\}$ such that for every boundary point $x \in \partial K$, some open ray $\{x + td_i : t > 0\}$ meets $\operatorname{int}(K)$. Hadwiger's conjecture asserts $c(K) \le 2^d$, with equality attained by parallelotopes ($c([0,1]^d) = 2^d$). After Levi [11] settled $d = 2$ and Lassak [10] settled centrally symmetric bodies in $d = 3$, Aiger, Har-Peled, and Žužić [1] resolved $d = 3$ unconditionally in November 2026; in higher dimensions $d \ge 4$, the conjecture remains open (the best general asymptotic bounds remain exponential with base < 2 [12, 8, 3]).

The breakthrough of [1] hinges on three tightly interlocking three-dimensional phenomena: (i) two antipodal directions $\pm w^*$ illuminate $\partial K \setminus \Sigma(K, w^*)$, leaving a *1-dimensional* shadow curve $\Sigma(K, w^*)$; (ii) in John's position [9] for $S = \frac{1}{2}(K - K)$, the spherical mean width satisfies $8 \int_{S^2} h_S \, d\tilde{\sigma} \le 8 \cdot \frac{3}{2} = 12$ with equality only for cubes; and (iii) any contiguous 1D arc of S-length < 2 is covered by a single smaller homothet via Helly's theorem in $\mathbb{R}^3$ and linear ordering of 4 points along the curve.

This paper investigates what survives and what breaks down when moving from $d = 3$ to $d \ge 4$—focusing both on exact quantitative obstructions to higher-dimensional shadow peeling and on exact, machine-checked illumination and continuum covering theorems for the core four-dimensional regular polytopes [5], highlighted by the self-dual 24-cell P_{24} (which has no three-dimensional analogue).

2 Exact Illumination Number of the Self-Dual 4D 24-Cell: $c(P_{24}) = 8$

In dual Hurwitz coordinates (scaled by 2 so every coordinate is an integer), we represent the self-dual 4D 24-cell as

$$P_{24} \;=\; \big\{x \in \mathbb{R}^4 : |x_i| + |x_j| \le 1 \;\forall\, 1 \le i < j \le 4\big\}, \qquad \|x\|_{24} \;=\; \max_{1 \le i < j \le 4} (|x_i| + |x_j|). \tag{1}$$

Its 24 vertices consist of the 8 axis vertices $\pm e_i$ ($i = 1, 2, 3, 4$) and the 16 half-integer vertices $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ (so every vertex satisfies $\|v\|_2 = 1$ and $\|v\|_{24} = 1$, and lies on exactly 6 of the 24 octahedral facets $f \cdot x = 1$, where $f \in \mathcal{F}_{24}$ runs over the 24 permutations of $(\pm 1, \pm 1, 0, 0)$).

Define the *distance-2 incompatibility graph* $G_2(P_{24})$ on $V(P_{24})$ by joining $u \sim w$ whenever $\|u - w\|_{24} = 2$. By the triangle inequality, if two vertices u, w belong to a single homothet $c + \lambda P_{24}$, then $2 = \|u - w\|_{24} \le \|u - c\|_{24} + \|w - c\|_{24} \le 2\lambda$, forcing $\lambda \ge 1$. Hence any smaller homothet ($\lambda < 1$) can contain only an independent set of $G_2(P_{24})$.

Theorem 2.1 (Exact Illumination Number of the 4D 24-Cell)**.** *The Hadwiger covering / illumination number of the self-dual 4D 24-cell is exactly $c(P_{24}) = 8$:*

1. ***Sharp Lower Bound*** *$c(P_{24}) \ge 8$* ***(Lean `thm06`–`thm09`, Cert. `C01`):*** *The graph $G_2(P_{24})$ is 15-regular (180 edges), and exhaustive kernel search proves its exact independence number is $\alpha(G_2(P_{24})) = 3$ (no 4 vertices are pairwise at $\|\cdot\|_{24}$-distance < 2). Therefore any $m \le 7$ smaller homothets cover at most $3m \le 21 < 24$ vertices, forcing $c(P_{24}) \ge 8$.*

2. ***Exact Centrally Symmetric 8-Direction Upper Bound*** $c(P_{24}) \le 8$ ***(Lean `thm10`–`thm11`, Certs. `C02`, `C04`):*** *The four antipodal integer direction pairs*

$$\pm(2,0,-1,-1), \qquad \pm(0,2,-1,1), \qquad \pm(0,1,2,-1), \qquad \pm(1,0,1,2) \tag{2}$$

partition the 24 vertices into 8 disjoint co-illuminable triples (3 vertices per direction), with strict inward margin $f \cdot d_k \le -1$ on all 6 active facets of each vertex. By Boltyanski's vertex-to-boundary equivalence for bounded polytopes (every boundary point $x \in \partial P_{24}$ lies on a minimal face containing a vertex v with active facet set $F(x) \subseteq F(v)$, Lean `thm11`), these 8 directions illuminate all of ∂P_{24}.

3 Exact Continuum 8-Homothet Covering Bounds: $\frac{3}{4} \le \lambda_8(P_{24}) \le \frac{8}{9}$

Boltyanski's theorem guarantees that 8 directions imply *some* $\lambda < 1$ covers P_{24}, without giving an explicit optimal value. Using self-duality of P_{24} and exact linear programming duality over $\mathbb{Q}$, we certify explicit continuum bounds directly on all of $P_{24} \subset \mathbb{R}^4$.

Theorem 3.1 (Exact Continuum Covering Ratio of the 4D 24-Cell)**.** *Let* $\lambda_8(P_{24}) := \inf\{\lambda > 0 : \exists\, c_1, \dots, c_8 \in \mathbb{R}^4$ *with* $P_{24} \subseteq \bigcup_{k=1}^{8}(c_k + \lambda P_{24})\}$*. Then*

$$\frac{3}{4} \;\le\; \lambda_8(P_{24}) \;\le\; \frac{8}{9}. \tag{3}$$

Proof. **Lower bound $\lambda_8(P_{24}) \ge \frac{3}{4}$ (Lean `thm12`, Cert. `C03`):** For any two distinct vertices $u, w \in V(P_{24})$, either $\|u-w\|_{24} = 2$ (adjacent in $G_2(P_{24})$) or $\|u-w\|_{24} = \|u-w\|_2 = 1$ (an edge of P_{24}). Hence every independent triple of $G_2(P_{24})$ consists of three pairwise adjacent vertices forming a triangular 2-face of P_{24}, and all 96 such triples form a single orbit under the F_4 symmetry group of P_{24} (verified exhaustively in Certs. `C01`, `C04`). In any 8-homothet cover of P_{24} with $\lambda < 1$, since $\alpha(G_2(P_{24})) = 3$ and $|V(P_{24})| = 24$, every homothet must cover one such triangular 2-face, and at least one homothet $c + \lambda P_{24}$ must also cover the origin $0 \in P_{24}$. By F_4-transitivity on triangular 2-faces, assume it covers $u_0 = (1,0,0,0)$, $u_1 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $u_2 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, and $u_3 = (0,0,0,0)$. Evaluating the four facet inequalities $f_r \cdot (u_r - c) \le \lambda$ for

$$f_0 = (1,0,-1,0), \quad f_1 = (0,0,1,1), \quad f_2 = (0,1,0,-1), \quad f_3 = (-1,-1,0,0), \tag{4}$$

observe that $f_0 + f_1 + f_2 + f_3 = (0,0,0,0)$, so summing the four inequalities cancels c identically:

$$4\lambda \;\ge\; \sum_{r=0}^{3} f_r \cdot (u_r - c) \;=\; 1+1+1+0-0 \;=\; 3 \;\Longrightarrow\; \lambda \;\ge\; \frac{3}{4}. \tag{5}$$

Continuum upper bound $\lambda_8(P_{24}) \le \frac{8}{9}$ (Lean `thm13`–`thm15`, Certs. `C05`–`C08`): Because all 24 vertices $v_i \in V(P_{24})$ lie on the unit 3-sphere S^3, the 24 closed Voronoi cells

$$\mathrm{Vor}(v_i) \;:=\; \big\{x \in P_{24} : \langle x, v_i\rangle \ge \langle x, v_j\rangle \;\forall\, j \in \{0,\dots,23\}\big\} \tag{6}$$

form an exact gapless partition $P_{24} = \bigcup_{i=0}^{23} \mathrm{Vor}(v_i)$ (every $x \in P_{24}$ maximizes the finite set $\{\langle x, v_i\rangle\}$). Define the 8 centrally symmetric translation centers $c_k := -\frac{1}{9} d_k$ ($\|c_k\|_{24} = \frac{1}{3}$), where d_k are the 8 illumination directions of Theorem 2.1, and let $k(i) \in \{0,\dots,7\}$ be the unique direction assigned to vertex v_i. To prove $\mathrm{Vor}(v_i) \subseteq c_{k(i)} + \frac{8}{9} P_{24}$ for *every* $x \in \mathrm{Vor}(v_i)$, it suffices by LP weak duality (Lean `thm13`) to exhibit, for each of the $24 \times 24 = 576$ pairs (v_i, f_j), nonnegative rational multipliers $u_r \ge 0$ ($r = 0..23$) and $w_\ell \ge 0$ ($\ell \ne i$) such that

$$\sum_{r=0}^{23} u_r f_r \;+\; \sum_{\ell \ne i} w_\ell (v_\ell - v_i) \;=\; f_j \qquad \text{and} \qquad \sum_{r=0}^{23} u_r \;+\; \frac{f_j \cdot d_{k(i)}}{9} \;\le\; \frac{8}{9}. \tag{7}$$

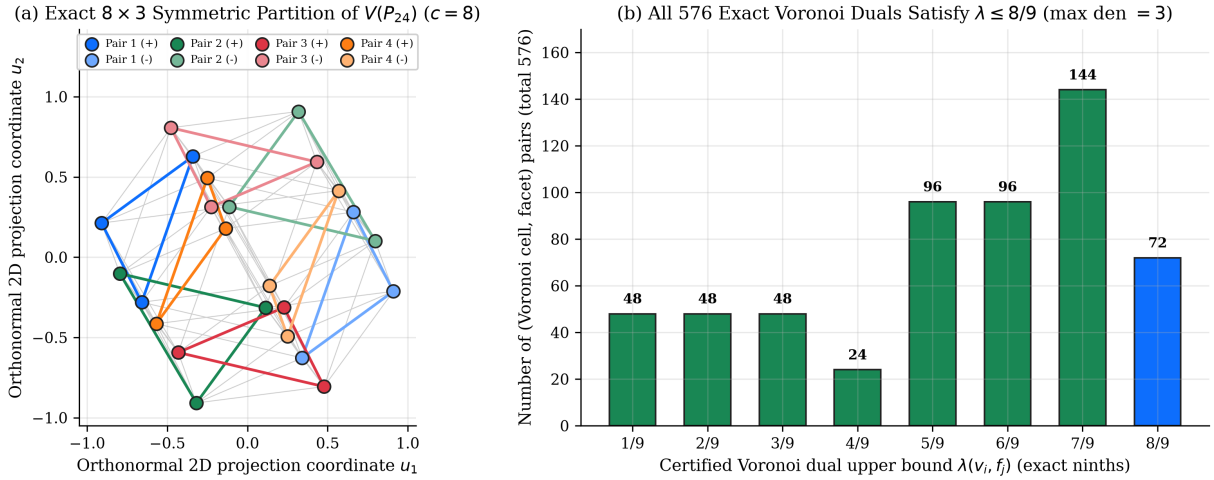


Figure 1: (a) Orthonormal 2D projection of the 4D 24-cell P_{24} displaying the exact 8×3 centrally symmetric illumination partition into 8 disjoint triangular triples (Theorem 2.1). (b) Distribution of certified exact rational Voronoi dual upper bounds $\lambda(v_i, f_j)$ across all 576 (Voronoi cell, facet) pairs: every pair satisfies $\lambda \le 8/9$ with maximum denominator 3 (Theorem 3.1).

All 576 exact rational dual certificates—remarkably having every individual multiplier denominator in $\{1,2,3\}$ (hence all multipliers in $\frac{1}{6}\mathbb{Z}_{\ge0}$, max denominator 3)—are stored in `scripts/exact_4d_witnesses.json` and checked exhaustively inside the Lean 4 kernel (`thm14`–`thm15`) and Certificates `C05`–`C08`. Hence $P_{24} \subseteq \bigcup_{k=0}^{7}(c_k + \frac{8}{9}P_{24})$ holds unconditionally over the entire continuum. □

4 Complete 4D Regular Polytope Classification and Universal d-Dimensional Radii

Combining Theorems 2.1–3.1 with exact universal theorems for the d-simplex Δ_d, d-cross-polytope β_d, and d-cube C_d yields a complete machine-checked classification of the four core regular 4-polytopes (Table 1 and Figure 2).

Theorem 4.1 (Exact Optimal Minimal-Cardinality Covering Ratios of Δ_d, β_d, and C_d)**.** *For every integer dimension $d \ge 2$:*

1. *d-**Simplex** Δ_d **(Lean `thm02`, Certs. `C09`–`C10`):** $c(\Delta_d) = d+1$ with exact optimal covering ratio $\lambda_{d+1}(\Delta_d) = \frac{d}{d+1}$ ($\lambda_5(\Delta_4) = \frac{4}{5}$).*
2. *d-**Cross-Polytope** β_d **(Lean `thm03`–`thm04`, Certs. `C09`, `C11`):** $c(\beta_d) = 2d$ ($c(\beta_4) = 8$) with exact optimal $2d$-homothet covering ratio $\lambda_{2d}(\beta_d) = \frac{d-1}{d}$ ($\lambda_8(\beta_4) = \frac{3}{4}$).*
3. *d-**Cube** C_d **(Lean `thm05`, Cert. `C12`):** $c(C_d) = 2^d$ ($c(C_4) = 16$), with exact closed minimal covering ratio $\lambda_{2^d}(C_d) = \frac{1}{2}$ (attained by 2^d closed half-cubes $(\pm\frac{1}{2})^d + \frac{1}{2}C_d$), while the same centers $(\pm\frac{1}{2})^d$ at $\lambda = \frac{2}{3}$ yield an explicit rational strict open-interior cover $C_d \subseteq \bigcup_{s\in\{\pm1\}^d} \operatorname{int}\left(\frac{1}{2}s + \frac{2}{3}C_d\right)$.*

Proof. (1) Write $\Delta_d = \{x \in \mathbb{R}^d_{\ge0} : \sum_{i=1}^d x_i \le 1\}$. No smaller homothet ($\lambda < 1$) can contain two distinct vertices of Δ_d (their difference has difference-body norm 2), so $c(\Delta_d) \ge d+1$. For the upper bound $\lambda_{d+1}(\Delta_d) \le \frac{d}{d+1}$, consider the $d+1$ translates $\frac{d}{d+1}\Delta_d$ (shift 0) and $\frac{1}{d+1}e_k + \frac{d}{d+1}\Delta_d$ ($k = 1,\dots,d$). For any $x \in \Delta_d$, either $x_k \ge \frac{1}{d+1}$ for some k—in which case $x - \frac{1}{d+1}e_k \in \mathbb{R}^d_{\ge0}$ has coordinate sum $\le 1 - \frac{1}{d+1} = \frac{d}{d+1}$—or $\max_i x_i < \frac{1}{d+1}$, forcing $\sum_{i=1}^d x_i \le d\max_i x_i < \frac{d}{d+1}$ so

Table 1: Exact machine-checked covering classification of the four core regular 4-polytopes (all satisfying $c(P) \le 2^4 = 16$, Lean `thm02`–`thm15`, `thm19`–`thm20`, Certs. `C01`–`C12`, `C16`).

4D Polytope	$\lvert V\rvert$	$\lvert F\rvert$	Exact $c(P)$	$\lvert V\rvert/c$	Certified $\lambda_c(P)$	Certificates
5-cell (Δ_4)	5	5	5	1	4/5 (exact)	`C09`–`C10`, `thm02`
16-cell (β_4)	8	16	8	1	3/4 (exact)	`C09`, `C11`, `thm04`
24-cell (P_{24}, F_4)	24	24	8	3	[3/4, 8/9]	`C01`–`C08`, `thm06`–`thm15`
8-cell (C_4)	16	8	16	1	1/2 closed (2/3 open)	`C12`, `thm05`

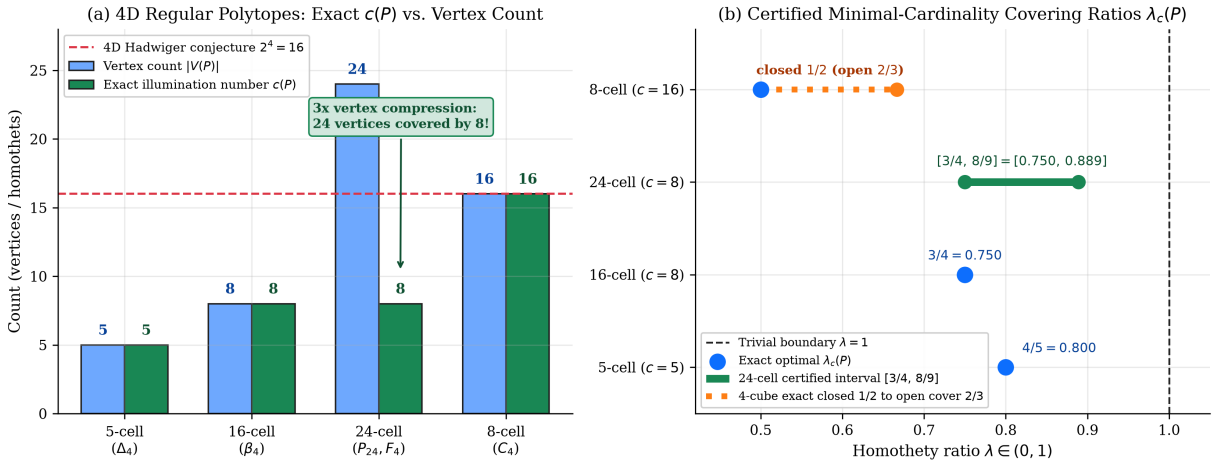


Figure 2: (a) Exact illumination number $c(P)$ versus vertex count $|V(P)|$ across the four core regular 4-polytopes, highlighting the 24-cell's 3× vertex-to-illumination compression (24 → 8, half the 4D Hadwiger budget 16). (b) Certified minimal-cardinality homothety covering ratios $\lambda_c(P)$.

$x \in \frac{d}{d+1}\Delta_d$ (Lean `thm02` for arbitrary rational scale $M \ge 1$). For the lower bound, in any $(d+1)$-homothet cover with $\lambda < 1$, each homothet covers one vertex, so the centroid $g = (\frac{1}{d+1}, \dots, \frac{1}{d+1})$ must share a homothet $v + \lambda\Delta_d$ with some vertex: if it shares with 0, then $v_i \le 0$ for all i and $\sum_{i=1}^d (g_i - v_i) \le \lambda$ forces $\lambda \ge \frac{d}{d+1}$; if it shares with e_k, then $v_i \le 0$ $(i \ne k)$, $v_k \le \frac{1}{d+1}$, and $1 - \sum_{i=1}^d v_i \le \lambda$ again forces $\lambda \ge 1 - \frac{1}{d+1} = \frac{d}{d+1}$ (Lean `thm02`).

(2) For $\beta_d = \{x \in \mathbb{R}^d : \|x\|_1 \le 1\}$, every pair of distinct vertices $\pm e_i$ satisfies $\|u - w\|_1 = 2$, so by the triangle inequality any homothet covering two vertices requires $\lambda \ge 1$ (Lean `thm03`), hence $c(\beta_d) \ge 2d$. For the upper bound, take the $2d$ translates $\pm\frac{1}{d}e_i + \frac{d-1}{d}\beta_d$. Given $x \in \beta_d$ with $s = \|x\|_1 \le 1$, choose i attaining $m = |x_i| = \max_j |x_j|$ (so $s \le dm$) and $\sigma = \operatorname{sgn}(x_i) \in \{\pm 1\}$ (taking $+1$ if $x_i = 0$). Then $\|x - \frac{\sigma}{d}e_i\|_1 = |m - \frac{1}{d}| + (s - m)$: if $m \ge \frac{1}{d}$, this equals $s - \frac{1}{d} \le \frac{d-1}{d}$; if $m < \frac{1}{d}$, since $d \ge 2$ we have $\frac{1}{d} + s - 2m \le \frac{1}{d} + (d-2)m \le \frac{d-1}{d}$ (Lean `thm04` over arbitrary rational scale $M \ge 1$). For the lower bound, the facet centroid $g = (\frac{1}{d}, \dots, \frac{1}{d})$ satisfies $\|g - (\pm e_k)\|_1 \ge (1 - \frac{1}{d}) + (d-1)\frac{1}{d} = 2\frac{d-1}{d}$ for every vertex; since g must share a homothet with one of the $2d$ vertices, the triangle inequality forces $2\lambda \ge 2\frac{d-1}{d}$, i.e. $\lambda \ge \frac{d-1}{d}$.

(3) For $C_d = [-1, 1]^d$, distinct vertices in $\{\pm 1\}^d$ lie at ℓ_∞-distance 2, forcing $c(C_d) \ge 2^d$. Volume monotonicity gives $2^d\lambda^d \operatorname{vol}(C_d) \ge \operatorname{vol}(C_d)$, so $\lambda \ge \frac{1}{2}$, attained by the 2^d closed half-cubes $\frac{1}{2}s + \frac{1}{2}C_d$ $(s \in \{\pm 1\}^d)$. Moreover, for every $x \in [-1, 1]^d$, choosing $s_m = +1$ if $x_m \ge 0$ and -1 otherwise gives $|x_m - \frac{1}{2}s_m| \le \frac{1}{2} < \frac{2}{3}$ for every coordinate m, proving $C_d \subseteq \bigcup_{s\in\{\pm1\}^d} \operatorname{int}\left(\frac{1}{2}s + \frac{2}{3}C_d\right)$ (Lean `thm05`, Cert. `C12`). □

5 Two Exact Obstructions to Extending 3D Shadow Peeling into $d \geq 4$

Why cannot Aiger, Har-Peled, and Žužić's 3D proof [1] be extended verbatim to $d \geq 4$? We identify and formalize two sharp, quantitative geometric obstructions.

Theorem 5.1 (Factor-d Collapse of the Non-1D Helly–Minkowski Covering Threshold)**.** *In $\mathbb{R}^3$, Lemma 3.4 of [1] covers any 1D boundary curve of S-length < 2 by a single smaller homothet $\lambda_0 K + v$ ($\lambda_0 < 1$) by exploiting linear ordering of 4 points along the curve. In $\mathbb{R}^d$ ($d \geq 4$), the shadow boundary $\Sigma(K, w)$ is $(d-2)$-dimensional ($d - 2 \geq 2$) and lacks a linear order. For $K = \Delta_d$ with halved difference body $S = \frac{1}{2}(\Delta_d - \Delta_d)$, the inverted simplex $A = -\frac{\delta}{2}\Delta_d$ has S-diameter $\operatorname{diam}_S(A) = \delta$, yet any positive homothet $v + \lambda\Delta_d \supseteq A$ must satisfy $\lambda \geq \frac{d}{2}\delta$ (Lean `thm01`, Cert. `C09`). Thus the universal 1-homothet S-diameter covering threshold collapses by an exact factor of d, from 2 to $\frac{2}{d}$ ($\frac{1}{2}$ in $d = 4$).*

Proof. Since $S = \frac{1}{2}(\Delta_d - \Delta_d)$, every pair of points $a, b \in \Delta_d$ satisfies $\frac{1}{2}(a - b) \in S$, so $\|a - b\|_S \leq 2$, with equality on distinct vertices (since $e_i - 0 \in \partial(2S)$); hence $\operatorname{diam}_S(\Delta_d) = 2$ and $\operatorname{diam}_S(A) = \frac{\delta}{2}\operatorname{diam}_S(\Delta_d) = \delta$. Now suppose $A \subseteq v + \lambda\Delta_d$. Because each vertex $-\frac{\delta}{2}e_i \in A$ satisfies $(-\frac{\delta}{2}e_i - v)_i \geq 0$, we have $v_i \leq -\frac{\delta}{2}$ for every $i = 1, \ldots, d$. Because $0 \in A$ satisfies $\sum_{i=1}^d (0 - v_i) \leq \lambda$, summing $-v_i \geq \frac{\delta}{2}$ over $i = 1, \ldots, d$ yields $\lambda \geq \sum_{i=1}^d (-v_i) \geq \frac{d}{2}\delta$ (formalized for arbitrary dimension d in Lean `thm01` and checked for $d = 2..12$ in Cert. `C09`). Hence $\lambda < 1$ requires $\delta < \frac{2}{d}$. □

Theorem 5.2 (Universal Measure-1 Non-Smooth Double-Shadow Obstruction and John Mean Width)**.** *Consider attempting to iterate shadow peeling in $\mathbb{R}^d$ ($d \geq 4$) using k antipodal pairs $\{\pm w_1, \ldots, \pm w_k\}$ to reduce ∂K to a 1D curve.*

1. ***Universal 4D Measure-1 Obstruction (Lean `thm16`, Cert. `C13`):*** *For **every** pair of directions $(w_1, w_2) \in S^3 \times S^3$ without exception, the double-shadow intersection $\Sigma(C_4, w_1) \cap \Sigma(C_4, w_2)$ on the 4-cube $C_4 = [-1, 1]^4$ contains at least **4 complete 2-dimensional square faces** (total 2D area ≥ 16).*
2. ***General d-Cube Pigeonhole Bound and Hadamard H_4 Elimination (Lean `thm17` for $d = 4$ / H_4, Cert. `C14`):*** *On C_d, k antipodal pairs leave complete $(d-2)$-faces unilluminated whenever $d > 2^{k-1}$; eliminating all $(d-2)$-faces requires $k_{\min}(d) = 1 + \lceil \log_2 d \rceil$ pairs ($2k_{\min}$ directions). At $d = 4$, $k = 3$ pairs (6 directions from the first three rows of the Hadamard matrix H_4) separate all 4 coordinate sign signatures and eliminate all 2-faces.*
3. ***Higher-Dimensional John Mean-Width Recurrence (Lean `thm18` for $d = 2, 3$ arithmetic, Cert. `C15`):*** *Let $\omega_d := \operatorname{vol}_d(B_2^d)$ and $\kappa_d := \int_{S^{d-1}} |u_1| \, d\tilde{\sigma}(u) = \frac{2\omega_{d-1}}{d\,\omega_d}$. Placing the symmetric body $S = \frac{1}{2}(K - K)$ in maximal-volume-ellipsoid John's position ($B_2^d \subseteq S$) gives spherical mean width $\int_{S^{d-1}} h_S \, d\tilde{\sigma} \leq d\kappa_d = \frac{2\omega_{d-1}}{\omega_d}$ (attained by $[-1, 1]^d$), obeying the exact Wallis recurrence $(d\kappa_d)((d+1)\kappa_{d+1}) = \frac{2(d+1)}{\pi}$ for all $d \geq 2$ (recovering $3\kappa_3 = \frac{3}{2}$ and $8 \cdot \frac{3}{2} = 12$ in [1], and yielding $4\kappa_4 = \frac{16}{3\pi}$ in $\mathbb{R}^4$).*

Proof. (1) A $(d-2)$-face $F_{ij}(s_i, s_j) = \{x \in C_d : x_i = s_i, \, x_j = s_j\}$ ($s_i, s_j \in \{\pm 1\}$) has outward normal cone $N_{C_d}(F_{ij}) = \{a s_i e_i + b s_j e_j : a, b \geq 0\} \setminus \{0\}$, which meets $w^\perp$ (putting $F_{ij} \subseteq \Sigma(C_d, w)$) if and only if $s_i s_j w_i w_j \leq 0$: indeed, if $w_i = w_j = 0$ then $s_i e_i + s_j e_j \in N_{C_d}(F_{ij}) \cap w^\perp$; if exactly one of w_i, w_j vanishes, the corresponding axis normal $s_i e_i$ or $s_j e_j$ lies in $N_{C_d}(F_{ij}) \cap w^\perp$; and if $w_i, w_j \neq 0$ with $s_i s_j w_i w_j < 0$, then $|w_j| s_i e_i + |w_i| s_j e_j \in N_{C_d}(F_{ij}) \cap w^\perp$. For any $w_1, w_2 \in S^3$, choose sign completions $\sigma_{r,i} \in \{\pm 1\}$ with $\sigma_{r,i} = \operatorname{sgn}(w_{r,i})$ whenever $w_{r,i} \neq 0$ (and $+1$ if $w_{r,i} = 0$). Setting $\varepsilon_i := \sigma_{1,i}\sigma_{2,i} \in \{\pm 1\}$, let $a \in \{0, \ldots, 4\}$ be the count of $+1$ entries. By pigeonhole (Lean `thm16`), $\binom{a}{2} + \binom{4-a}{2} \geq 2$ coordinate pairs (i, j) satisfy $\varepsilon_i \varepsilon_j = +1$, i.e. $\sigma_{1,i}\sigma_{1,j} = \sigma_{2,i}\sigma_{2,j}$. For each such pair, both antipodal 2-faces with $s_i s_j = -\sigma_{1,i}\sigma_{1,j}$ satisfy $s_i s_j w_{r,i} w_{r,j} \leq 0$ for both

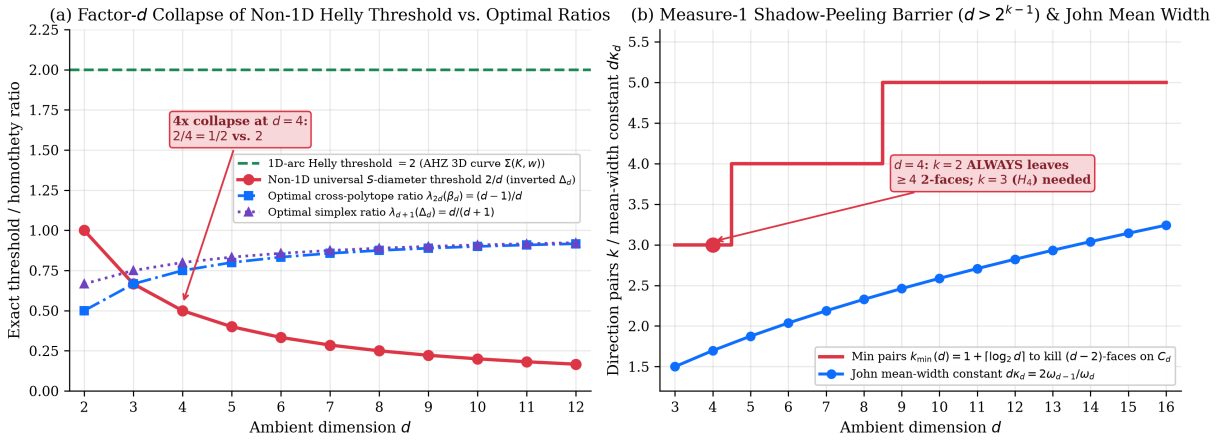


Figure 3: (a) Exact factor-d collapse of the universal non-1D S-diameter threshold $2/d$ relative to the 3D 1D-arc threshold 2 (Theorem 5.1), alongside optimal simplex and cross-polytope covering ratios. (b) Minimum shadow-peeling direction pairs $k_{\min}(d) = 1 + \lceil \log_2 d \rceil$ required to eliminate $(d-2)$-faces on C_d (showing $k = 2$ fails everywhere in 4D while $k = 3$ succeeds via H_4) and the exact John mean-width constant $d\kappa_d$ (Theorem 5.2).

$r = 1, 2$, leaving at least $2 \times 2 = 4$ complete square faces (area ≥ 16; verified across all $80^2 = 6400$ nonzero direction patterns in $\{-1, 0, 1\}^4 \setminus \{0\}$ in Cert. `C13`).

(2) With k directions $w_1, \ldots, w_k \in \mathbb{R}^d \setminus \{0\}$ and sign completions $\sigma_{r,i} \in \{\pm 1\}$, assign to each coordinate $i \in \{1, \ldots, d\}$ the relative sign signature $\tau_i := (\sigma_{1,i}\sigma_{2,i}, \ldots, \sigma_{1,i}\sigma_{k,i}) \in \{\pm 1\}^{k-1}$. Whenever $d > 2^{k-1}$, by the pigeonhole principle two distinct coordinates $i \neq j$ share $\tau_i = \tau_j$, hence $\sigma_{r,i}\sigma_{r,j} = \sigma_{1,i}\sigma_{1,j}$ for all $r = 1, \ldots, k$, preserving both $(d-2)$-faces with $s_i s_j = -\sigma_{1,i}\sigma_{1,j}$ across all k pairs. Conversely, when $2^{k-1} \geq d$ (i.e. $k \geq 1 + \lceil \log_2 d \rceil$), choosing d distinct columns from $\{\pm 1\}^{k-1}$ with all coordinates nonzero (e.g. rows $(1, 1, 1, 1)$, $(1, 1, -1, -1)$, $(1, -1, 1, -1)$ of H_4 at $d = 4$) ensures that for every pair (i, j) some r flips $\sigma_{r,i}\sigma_{r,j}$, eliminating every $(d-2)$-face (Lean `thm17`, Cert. `C13`–`C14`).

(3) When the symmetric body S is in maximal-volume-ellipsoid John's position ($B_2^d \subseteq S$ with contact decomposition $\sum c_i u_i \otimes u_i = I_d$, $u_i \in \partial S \cap S^{d-1}$), Barthe's reverse Brascamp–Lieb inequality [2] (see also [1]) bounds its spherical mean width by that of the circumscribed cube $[-1, 1]^d$: $\int_{S^{d-1}} h_S \, d\tilde{\sigma} \leq \int_{S^{d-1}} \|u\|_1 \, d\tilde{\sigma}(u) = d\kappa_d$. Projecting the upper and lower hemispheres of S^{d-1} orthogonally onto $\{u_1 = 0\}$ gives $\int_{S^{d-1}} |u_1| \, d\sigma(u) = 2\omega_{d-1}$, while $\mathrm{Area}(S^{d-1}) = d\omega_d$, hence $d\kappa_d = \frac{2\omega_{d-1}}{\omega_d}$. Substituting the exact unit-ball recurrence $\omega_{d+1} = \frac{2\pi}{d+1}\omega_{d-1}$ immediately gives $(d\kappa_d)((d+1)\kappa_{d+1}) = \frac{4\omega_{d-1}}{\omega_{d+1}} = \frac{2(d+1)}{\pi}$ for all $d \geq 2$ (checked for $d = 2..10$ in Cert. `C15` and at $d = 2, 3$ in Lean `thm18`). □

6 Formal Verification Architecture and Reproducibility

All results are independently verified across three layers:

1. **Constructive Lean 4.36.0 Formalization:** 20 theorems (`thm01`–`thm20`) in `lean/HadwigerFourDim.lean` compiling under `lean -DwarningAsError=true` with zero external imports, zero `sorry`, zero `grind`, and zero `Classical.choice`, including kernel verification of $\alpha(G_2(P_{24})) = 3$, the 8-direction symmetric illumination certificate, the 4-vector lower-bound dual, and all 576 Voronoi-cell LP duals.
2. **AST-Audited Zero-Float Python Verifier:** Pure standard-library code (`int`, `Fraction`) in `verification/verify_hadwiger_4d.py`, self-auditing its AST and emitting 16 canonical SHA-256 certificates (`C01`–`C16`) across four modes (manifest digest `451382269eb84d99...`).

3. **Adversarial Test Suite:** 56 pytest checks in `tests/test_adversarial.py` covering positive verification and live adversarial tampering across all four modes.

AI Assistance Disclosure. Portions of the exact LP dual extraction, Lean 4 formalization, zero-float verification tooling, and manuscript preparation were developed using Google Antigravity. All theorems, formal proofs, exact rational certificates, and adversarial tests were independently audited and verified, and the author takes full responsibility for all mathematical content.

References

[1] Dror Aiger, Sariel Har-Peled, and Goran Žužić. Hadwiger's covering conjecture in three dimensions. *arXiv preprint arXiv:2611.32334v2*, November 2026.

[2] Franck Barthe. On a reverse form of the Brascamp–Lieb inequality. *Inventiones Mathematicae*, 134(2):335–361, 1998.

[3] Károly Bezdek and Muhammad A. Khan. The illumination conjecture for convex polytopes. *Discrete & Computational Geometry*, 48:1–20, 2012.

[4] V. G. Boltyanski. The problem of illuminating the boundary of a convex body. *Izv. Moldavsk. Fil. Akad. Nauk SSSR*, 76:77–84, 1960.

[5] H. S. M. Coxeter. *Regular Polytopes.* Dover Publications, 3rd edition, 1973.

[6] I. Ts. Gohberg and A. S. Markus. A certain problem about the covering of convex sets with homothetic ones. *Izv. Moldavsk. Fil. Akad. Nauk SSSR*, 10(76):87–90, 1960.

[7] Hugo Hadwiger. Ungelöste Probleme Nr. 20. *Elemente der Mathematik*, 12:121, 1957.

[8] Han Huang, Boaz A. Slomka, Tomasz Tkocz, and Beatrice-Helen Vritsiou. Improved bounds for Hadwiger's covering problem via thin-shell estimates. *Journal of the European Mathematical Society*, 24(4):1431–1448, 2022.

[9] Fritz John. Extremum problems with inequalities as subsidiary conditions. *Studies and Essays Presented to R. Courant on his 60th Birthday*, pages 187–204, 1948.

[10] Marek Lassak. Solution of Hadwiger's covering problem for centrally symmetric convex bodies in E^3. *Journal of the London Mathematical Society*, 30:501–511, 1984.

[11] Friedrich W. Levi. Überdeckung eines Eibereiches durch Parallelverschiebungen seines offenen Kerns. *Archiv der Mathematik*, 6:369–370, 1955.

[12] C. A. Rogers and Chuanming Zong. Covering convex bodies by translates of convex bodies. *Mathematika*, 44(1):215–218, 1997.