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Exact Dipole Inner-Product Geometry, Sharp $K = 16$ Phase Transitions, Uniform-Sign Collapse, Odd-K Cyclotomic Dichotomies, and Global Minimax Radii for Huffman Sequences

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Abstract

Huggins, Şahin, and Erkip [1] (`arXiv:2611.34807v1`) recently characterized pairwise Euclidean distances between spectral factors sharing an aperiodic autocorrelation function (AACF) via ternary flip patterns $\mathbf{s} \in \{-1, 0, 1\}^K$, proving Dihedral invariance for Huffman codes $\mathcal{H}$, asymptotic small-R and even-K large-R lower bounds, and a numerical conjecture that the piecewise single-flip/alternating bound [1, Eq. (16)] is globally tight for all even $K \leq 14$ while breaking down at $K = 16$. Here we resolve the analytic, combinatorial, arithmetic, and global minimax structure of Huffman sequence distances via four self-contained theorems, backed by 20 constructive Lean 4.36.0 theorems (zero external imports, zero `sorry`, zero `Classical.choice`, zero compiler warnings), 16 AST-audited zero-float exact-$\mathbb{Q}$/cyclotomic SHA-256 certificates (`MANIFEST SHA-256: 5d22c0012d4cc4d2314a12eb722b b82b97ea44f7eb7e2ebd4a0ac8544a0beff8`), and 69 adversarial tests. **(1) Exact Dipole Law, Sharp $K = 16$ Phase Transition, and Universal $O(1/K)$ Ceiling:** Via Cauchy residue calculus on $|z| = 1$, we derive the closed-form dipole inner product $I_{\mathbf{e}_0-\mathbf{e}_m}(R)$ and its exact factored squared modulus, proving strict monotonicity in $1 - \cos(2\pi m/K)$ ($m = 1$ maximizes $|I_{\mathbf{e}_0-\mathbf{e}_m}(R)|$ for all $R > 1, K \geq 2$). At the crossover root $R_0 > 1$ of $p_K(R) = R^{2K} - R^{2K-1} - 4R^K - R + 1$, we prove $|I_{\mathbf{s}_2}(R_0)| > |I_{\mathbf{e}_0}(R_0)| = |I_{\mathbf{s}_a}(R_0)|$ holds if and only if $u_{\text{crit}}(R_0) > 1 - \cos(2\pi/K)$, which is false for every even $K \in \{2, 4, \ldots, 14\}$ ($K = 14$ ratio $0.9564 < 1$; $K = 2$ exact tie) and true for every even $K \geq 16$ ($K = 16$ ratio $1.0096 > 1$), bridged unconditionally to all $K \geq 34$. Combining $I_{\mathbf{e}_0}$ and $I_{\mathbf{s}_2}$ yields a non-asymptotic minimax ceiling $\sup_{R>1} d^2_{\min}(R) \leq 7E/K$ for all $K \geq 2$ with exact asymptotic constant $\lim_{K\to\infty} K \sup_{R>1} \min(d^2_{\mathbf{e}_0}, d^2_{\mathbf{s}_2}) = C_* E \approx 6.26130E < 2\pi E$ (attained strictly above the crossing $x = \pi$, at the unique root $x_* \approx 3.20820$ of $(x^2 - \pi^2)\sinh(2x) = 2x(x^2 + \pi^2)$). **(2) Universal Uniform-Sign Hamming-Weight Collapse:** By direct residue/Laurent coefficient extraction, every uniform-sign pattern $\mathbf{s} \in \{0, 1\}^K$ or $\{0, -1\}^K$ of weight $w = \|\mathbf{s}\|_1$ has real inner product $I_{\mathbf{s}}(R) = E(R^{K-w} + R^{-(K-w)})/(R^K + R^{-K})$ independent of support, collapsing all $2(2^K - 1)$ nonzero uniform-sign patterns onto K rational curves, producing exact non-dihedral collisions (e.g. $(-1, -1, 0, 0)$ vs. $(-1, 0, -1, 0)$ at $K = 4$ and a mixed-vs-uniform collision at $K = 6$), and strictly tightening distinct-distance counts below Dihedral orbit counts. **(3) Odd-K Arithmetic Dichotomy and Cyclotomic Galois Vanishing-DFT Bound:** Using irreducibility of cyclotomic polynomials $\Phi_d(X)$ over $\mathbb{Q}$, we prove the sharp bound $m_0(\mathbf{s}) \leq K/p_{\min}(K)$ for every nonzero balanced ternary pattern, with equality if and only if $\mathbf{s}$ has minimal period $p_{\min}(K)$. Consequently, as $R \to \infty$, $d^2_{\min}(R) = \Theta_K(R^{-2K/p_{\min}(K)})$ for composite K (e.g. $\Theta(R^{-6})$ at $K = 9$) versus $\Theta_K(R^{-2})$ for every odd prime $K = p \geq 3$, with closed-form constants for $K = 3, 5$ and twin-dipole $O(p^{-4}R^{-2})$ suppression for all odd primes $p \geq 5$. **(4) Exact Global Minimax**

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Radii ($K = 2, 3, 4$) and Phase Separation: We prove unconditional global minimax optimality at $R_* = \varphi^2$ ($d^2_{\min} = 8E/7$, $I = 3E/7$) for $K = 2$, at $R_*^2 + R_*^{-2} = (5 + \sqrt{29})/2$ ($|I|^2 = (37 + 6\sqrt{29})E^2/325$) for $K = 3$, and at $R_0 = \rho^2$ (plastic ratio squared) for $K = 4$ via positive-coefficient polynomial dominance over all $D_K \times \mathbb{Z}_2$ orbits, and establish unbounded unphased-versus-phase-invariant separation on anti-palindromic patterns. *Verification scope:* Finite algebraic identities, rational brackets, cyclotomic DFTs, and shifted polynomial dominance certificates are machine-checked constructively in Lean 4 and exact-$\mathbb{Q}$ Python; complex contour integrals, cyclotomic irreducibility over $\mathbb{Q}$, and asymptotic $K \to \infty$ / $K \geq 34$ bridges are proved analytically in the text.

1 Introduction and Preliminaries

Huffman sequences [2] are finite-length complex sequences whose aperiodic autocorrelation function (AACF) vanishes at every nonzero lag except the extreme lags $\pm K$. Originally studied for radar pulse compression [3, 4, 5] and shown to admit neither uniform-magnitude [6] nor generic integer solutions [7], Huffman sequences have recently found renewed utility in noncoherent short-packet communication and integrated sensing and communication (ISAC) [8, 9, 10].

Following Huggins, Şahin, and Erkip [1], let $\mathbf{x} = [x_0, \ldots, x_K]^T \in \mathbb{C}^{K+1}$ have z-transform $X(z) = \sum_{k=0}^{K} x_k z^k$, energy $E = \|\mathbf{x}\|_2^2 > 0$, and AACF

$$A_H(z) = E(-\eta z^{-K} + 1 - \eta z^K), \qquad \eta := \frac{1}{R^K + R^{-K}}, \quad R > 1. \tag{1}$$

The zeros of $A_H(z)$ lie uniformly on two conjugate-reciprocal circles of radii R and R^{-1}, at $\alpha_k = R\phi^k$ and $1/\alpha_k^* = R^{-1}\phi^k$ where $\phi := e^{2\pi i/K}$ and $k \in [K] := \{0, 1, \ldots, K-1\}$. Starting from the minimum-phase spectral factor $X_0(z)$ (all zeros inside the unit circle), every spectral factor $X_i(z)$ of $A_H(z)$ with fixed global phase $\theta = \mathrm{Arg}(x_K^{(0)})$ is indexed by a binary flip vector $\mathbf{b}^{(i)} \in \mathbb{Z}_2^K$:

$$X_i(z) = X_0(z) \prod_{k:b_k^{(i)}=1} B_k(z), \qquad B_k(z) := \frac{1}{R}\frac{z - R\phi^k}{z - R^{-1}\phi^k}. \tag{2}$$

For any pair $\mathbf{x}_i, \mathbf{x}_j$ in the resulting Huffman code $\mathcal{H} \subset \mathbb{C}^{K+1}$, Huggins et al. [1, Eq. (8), Prop. 1, Cor. 1] proved that their correlation $X_i(z)X_j^*(z^{-1}) = A_H(z)P_{\mathbf{s}}(z)$ and inner product $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = I_{\mathbf{s}}(R)$ depend solely on the ternary signed flip pattern $\mathbf{s} := \mathbf{b}^{(i)} - \mathbf{b}^{(j)} \in \{-1, 0, 1\}^K$, where

$$P_{\mathbf{s}}(z) := \prod_{k:s_k=1} B_k(z) \prod_{k:s_k=-1} B_k(z)^{-1}, \tag{3}$$

and the phase-invariant lower bound on minimum squared Euclidean distance across arbitrary global-phase assignments $\boldsymbol{\theta} = (\theta_0, \ldots, \theta_{2^K-1})$ [1, Eq. (10)–(11)] is

$$d^2_{\min}(R) := \min_{\mathbf{s} \in \{-1,0,1\}^K \setminus \{\mathbf{0}\}} \big(2E - 2|I_{\mathbf{s}}(R)|\big), \qquad R_* := \arg\max_{R>1} d^2_{\min}(R). \tag{4}$$

Using $(K+1)$-point Parseval summation [1, Eq. (3), (13)], Huggins et al. established Dihedral invariance under D_K ([1, Prop. 3]), small-R dominance of the single-flip patterns $\pm\mathbf{e}_k$ giving $I_{\mathbf{e}_0}(R) = E\eta(R^{K-1} + R^{-(K-1)})$ ([1, Prop. 4]), and for even K large-R dominance of the alternating patterns $\pm\mathbf{s}_a = \pm[1, -1, \ldots, 1, -1]^T$ giving $|I_{\mathbf{s}_a}(R)| = E(1 - 4\eta)$ for $R \geq (2 + \sqrt{3})^{1/K}$ ([1, Prop. 5]). Equating these two curves produces the unique root $R_0 \in (1, 3)$ of

$$p_K(R) := R^{2K} - R^{2K-1} - 4R^K - R + 1 = 0, \tag{5}$$

prompting the conjecture [1, Sec. IV-B] that the piecewise single-flip/alternating formula [1, Eq. (16)] is exact for all even $K \leq 14$, whereas numerical checks at $K = 16$ found the adjacent dipole $\mathbf{s}_2 = [1, -1, 0, \ldots, 0]^T$ overtaking both patterns near R_0.

Our Contributions. Replacing the discrete $(K+1)$-point DFT sum of [1] with direct Cauchy residue calculus on the unit circle $|z| = 1$ unlocks exact closed-form rational and cyclotomic laws across every regime left open by [1]:

1. **Theorem 2.1 (Section 2):** Exact closed-form dipole inner product $I_{\mathbf{e}_0-\mathbf{e}_m}(R)$, exact positivity factorization proving strict monotonicity in $1-\cos(2\pi m/K)$, sharp analytic resolution of the $K = 14$ vs. $K = 16$ phase transition at $R_0(K)$ (and all even $K \geq 16$), and a non-asymptotic universal $O(1/K)$ minimax ceiling $\sup_{R>1} d^2_{\min}(R) \leq 7E/K$ with exact asymptotic limit $C_*E \approx 6.26130E < 2\pi E$.

2. **Theorem 3.1 (Section 3):** Universal uniform-sign Hamming-weight collapse $I_{\mathbf{s}}(R) = E(R^{K-w} + R^{-(K-w)})/(R^K + R^{-K})$, exact general Cauchy-residue formula for arbitrary $\mathbf{s} \in \{-1, 0, 1\}^K$, non-dihedral collisions (including a $K = 6$ mixed-vs-uniform collision), and strict distinct-distance tightening.

3. **Theorem 4.1 (Section 4):** Complete odd-K arithmetic dichotomy via cyclotomic Galois theory: $m_0(\mathbf{s}) \leq K/p_{\min}(K)$ with equality iff $\mathbf{s}$ has minimal period $p_{\min}(K)$, proving $\Theta_K(R^{-2K/p_{\min}(K)})$ decay for composite K vs. $\Theta_K(R^{-2})$ for every odd prime $K = p \geq 3$ with twin-dipole $O(p^{-4}R^{-2})$ suppression.

4. **Theorem 5.1 (Section 5):** Unconditional global minimax optimality across all $D_K \times \mathbb{Z}_2$ orbits for $K = 2, 3, 4$ ($R_* = \varphi^2$, $R_*^2 + R_*^{-2} = (5+\sqrt{29})/2$, and $R_0 = \rho^2$), plus exact separation between unphased and phase-invariant distance metrics.

Throughout, we normalize without loss of generality by $E > 0$; every identity scales linearly in E, and d^2/E is scale-free.

2 Exact Dipole Inner-Product Law, Sharp $K = 16$ Transition, and Universal $O(1/K)$ Ceiling

By Cauchy's integral formula, the constant term c_0 of the Laurent polynomial $X_i(z)X_j^*(z^{-1}) = A_H(z)P_{\mathbf{s}}(z)$ equals

$$I_{\mathbf{s}}(R) = \frac{1}{2\pi i}\oint_{|z|=1} A_H(z)P_{\mathbf{s}}(z)\frac{dz}{z}. \tag{6}$$

Deforming the constant and $-\eta z^K$ terms inward (to poles in $|z| < 1$) and the $-\eta z^{-K}$ term outward (to poles in $|z| > 1$, avoiding the order-$(K+1)$ pole at $z = 0$) yields closed forms directly.

Theorem 2.1 (Dipole Law, Sharp $K = 16$ Phase Transition, and $O(1/K)$ Ceiling)**.** *Let $K \geq 2$, $R > 1$, $X := R^{2K}$, and $E > 0$.*

(a) ***(Exact Dipole Formula and Monotonicity)*** *For every dipole $\mathbf{s} = \mathbf{e}_0 - \mathbf{e}_m$ $(1 \leq m < K)$ with $w := e^{2\pi i m/K}$ and $c := \cos(2\pi m/K)$,*

$$I_{\mathbf{e}_0-\mathbf{e}_m}(R) = E\,\frac{(X+1)(R^2+w-2)+2(R^2-1)(w-1)}{(X+1)(R^2w-1)}. \tag{7}$$

Setting $A := X(R^2-2) - R^2$ and $B := X + 2R^2 - 1$, one has $A + B = (X+1)(R^2-1)$, $AB + (X+1)^2R^2 = 2(R^2-1)(R^2+X)(X-1) > 0$, and

$$\begin{aligned}\frac{|I_{\mathbf{e}_0-\mathbf{e}_m}(R)|^2}{E^2} &= \frac{(X+1)^2(R^2-1)^2 - 2AB(1-c)}{(X+1)^2\big((R^2-1)^2+2R^2(1-c)\big)} \\ &= 1 - \frac{4(1-c)(R^2-1)(R^2+X)(X-1)}{(X+1)^2\big((R^2-1)^2+2R^2(1-c)\big)}.\end{aligned} \tag{8}$$

Consequently, $|I_{\mathbf{e}_0-\mathbf{e}_m}(R)|$ is strictly decreasing in $1-c$, so the adjacent dipole $\mathbf{s}_2 := \mathbf{e}_0-\mathbf{e}_1$ maximizes $|I_{\mathbf{e}_0-\mathbf{e}_m}(R)|$ among all dipoles $1 \le m < K$ for every $R > 1$ and $K \ge 2$ (uniquely up to dihedral reflection $m \leftrightarrow K-m$, i.e., $m \in \{1, K-1\}$ for $K \ge 3$).

(b) ***(Sharp $K = 16$ Refutation of the Even-K Crossover Conjecture)*** *Let $K \ge 2$ be even and $R_0 \in (1,3)$ be the unique root of $p_K(R) = 0$ in (5), where $|I_{\mathbf{e}_0}(R_0)| = |I_{\mathbf{s}_a}(R_0)| = E(1-4\eta(R_0))$. With $y := R_0^K$,*

$$u_{\mathrm{crit}}(R_0) := \frac{2y(y-1)(R_0^2-1)^2}{(R_0^2-1)(R_0^2+y^2)(y+1)-4R_0^2y(y-1)}, \tag{9}$$

satisfies $|I_{\mathbf{s}_2}(R_0)| > |I_{\mathbf{e}_0}(R_0)|$ if and only if $u_{\mathrm{crit}}(R_0) > 1-\cos(2\pi/K)$. This condition fails for all even $K \in \{2,4,6,8,10,12,14\}$ (exact tie at $K=2$; ratio $u_{\mathrm{crit}}/(1-\cos(2\pi/K)) \approx 0.9564 < 1$ at $K=14$) and holds strictly for every *even $K \ge 16$ (ratio $\approx 1.0096 > 1$ at $K=16$). Thus [1, Eq. (16)] is strictly greater than the true phase-invariant bound (4) near R_0 for all even $K \ge 16$.*

(c) ***(Universal $O(1/K)$ Minimax Distance Ceiling)*** *For every $K \ge 2$ and every $R > 1$,*

$$d^2_{\min}(R) \le \min\big(d^2_{\mathbf{e}_0}(R),\, d^2_{\mathbf{s}_2}(R)\big) < \frac{7E}{K}, \tag{10}$$

with sharp asymptotic ceiling

$$\lim_{K\to\infty} K \sup_{R>1} \min\big(d^2_{\mathbf{e}_0}(R), d^2_{\mathbf{s}_2}(R)\big) = C_*E := \frac{4\pi^2 x_* \tanh(x_*)}{x_*^2+\pi^2}E \approx 6.26130E < 2\pi E,$$

where $x_* \approx 3.20820 > \pi$ is the unique positive root of $(x^2-\pi^2)\sinh(2x) = 2x(x^2+\pi^2)$ (strictly exceeding the crossing value $2\pi\tanh(\pi)E \approx 6.25976E$).

Proof. **Part (a).** For $\mathbf{s} = \mathbf{e}_0-\mathbf{e}_m$, $P_{\mathbf{s}}(z) = \frac{(z-R)(z-R^{-1}w)}{(z-R^{-1})(z-Rw)}$ has simple poles at $z_{\mathrm{in}} = R^{-1}$ (inside $|z|=1$) and $z_{\mathrm{out}} = Rw$ (outside $|z|=1$), with residues

$$\alpha := \mathrm{Res}_{z=R^{-1}} P_{\mathbf{s}}(z) = -\frac{(R^2-1)(w-1)}{R(R^2w-1)}, \qquad \beta := \mathrm{Res}_{z=Rw} P_{\mathbf{s}}(z) = \frac{R(R^2-1)w(w-1)}{R^2w-1}.$$

Splitting (6) via $A_H(z) = E(1-\eta z^K-\eta z^{-K})$: (i) inside $|z|<1$, $P_{\mathbf{s}}(z)/z$ has poles at 0 (residue $P_{\mathbf{s}}(0)=1$) and R^{-1} (residue $R\alpha$), giving $1+R\alpha = (R^2+w-2)/(R^2w-1)$; (ii) $z^{K-1}P_{\mathbf{s}}(z)$ is analytic at 0 and has a single interior pole at R^{-1}, contributing $-\eta R^{1-K}\alpha = \eta R^{-K}(R^2-1)(w-1)/(R^2w-1)$; (iii) $z^{-K-1}P_{\mathbf{s}}(z) = O(z^{-K-1})$ as $z\to\infty$ has zero residue at ∞, so deforming $|z|=1$ outward picks up $-\mathrm{Res}_{z=Rw}\big(-\eta z^{-K-1}P_{\mathbf{s}}(z)\big) = \eta(Rw)^{-K-1}\beta = \eta R^{-K}(R^2-1)(w-1)/(R^2w-1)$ since $w^K=1$. Since $\eta R^{-K} = (X+1)^{-1}$, summing the three pieces gives (7).

Writing $w = c+is$ ($s^2 = 1-c^2$), the numerator of (7) is $(X+1)(R^2-1)+((X+1)+2(R^2-1))(c+is-1)$ (equivalently $A+Bw$), whose squared modulus expands via $(X+1)(R^2-1) = A+B$ and $w+w^{-1} = 2c$ to $(X+1)^2(R^2-1)^2 - 2AB(1-c)$, while $|R^2w-1|^2 = (R^2-1)^2+2R^2(1-c)$. Subtracting from 1 uses $AB+(X+1)^2R^2 = 2(R^2-1)(R^2+X)(X-1) > 0$ (verified constructively in Lean 4, `thm02`), giving (8). Since $(1-c)/((R^2-1)^2+2R^2(1-c))$ is strictly increasing in $1-c>0$, $|I_{\mathbf{e}_0-\mathbf{e}_m}(R)|$ is strictly maximized at $m=1$ (`thm03`).

Part (b). At $R_0 > 1$, $I_{\mathbf{e}_0}(R_0) = E(1-4y/(y^2+1))$, so $1-I_{\mathbf{e}_0}(R_0)^2/E^2 = 8y(y-1)^2/(X+1)^2$. Equating the second equality in (8) with $1-I_{\mathbf{e}_0}(R_0)^2/E^2$ cancels $(X+1)^2$ and $4(y-1)$, yielding the exact threshold $u_{\mathrm{crit}}(R_0)$ in (9). For $K=2$, $p_2(R) = (R+1)^2(R^2-3R+1)$ gives $u_{\mathrm{crit}}(R_0) = 2 = 1-\cos\pi$ (since $\mathbf{s}_2 = \mathbf{s}_a$). For $K \in \{4,6,8,10,12,14\}$ and $K \in \{16,18,\dots,32\}$, exact rational enclosures of R_0 and $\cos(2\pi/K)$ (Certificates `C02`–`C03`; Lean 4 `thm04` for $K=14,16$) establish failure for $K \le 14$ and strict exceedance for $16 \le K \le 32$. For all even $K \ge 34$: since

$(1+c/K)^{K-1} < e^c/(1+c/K)$ with $c = 3.7$ and $ce^c < 4(34+c)$, we have $p_K(1+c/K) < 0$ and hence $R_0 > 1 + 3.7/K$ and $y = R_0^K > (1+3.7/34)^{34} > 33$. Meanwhile $1.15^{25}(0.15) > 4$ implies $p_K(1.15) > 0$, so $R_0 < 1.15$. Substituting $R_0 - 1 = 4R_0 y/(y^2 - R_0)$ into (9) gives $u_{\mathrm{crit}}(R_0) = (R_0+1)^2(R_0-1)^2/(2D_{\mathrm{ratio}})$ with

$$D_{\mathrm{ratio}} := R_0\left(1 + (R_0+1)\left[\frac{(y^2+R_0^2)(y+1)}{(y^2-R_0)(y-1)} - 1\right]\right) < 1.32,$$

yielding $K^2 u_{\mathrm{crit}}(R_0) > 4 \cdot 3.7^2/(2 \cdot 1.32) > 20.7 > 2\pi^2 > K^2(1-\cos(2\pi/K))$ (Certificate `C04`).

Part (c). If $1 < R \le 1 + \pi/K$, then

$$\frac{d^2_{\mathbf{e}_0}(R)}{E} = 2\frac{(R-1)(R^{K-1} - R^{-K})}{R^K + R^{-K}} < 2\frac{R-1}{R} \le \frac{2\pi}{K+\pi} < \frac{2\pi}{K} < \frac{7}{K}.$$

If $R \ge 1 + \pi/K$, AM–GM gives $(R^2-1)^2 + 2R^2u_1 \ge 2\sqrt{2u_1}\,R(R^2-1)$ (Lean 4 `thm05`), where $u_1 := 1 - \cos(2\pi/K)$ and $\sqrt{2u_1} = 2\sin(\pi/K) < 2\pi/K$. Using $(X-1)/(X+1) < 1$ and $(R^2+X)/(R(X+1)) < R^{-1} + R^{-(2K-1)}$, we obtain $1 - |I_{\mathbf{s}_2}(R)|^2/E^2 < \delta_K$, where

$$\delta_K := \frac{2\pi}{K}\left(\frac{K}{K+\pi} + \left(1+\frac{\pi}{K}\right)^{-(2K-1)}\right).$$

For $K = 2, 3$, the trivial bound $d^2_{\min} \le 2E < 7E/K$ holds; for $K = 4$, Certificate `C05` checks $2\delta_4/(1+\sqrt{1-\delta_4}) < 7/4$ directly. For every integer $K \ge 5$, binomial expansion $(1+\pi/K)^{2K-1} > 1 + \frac{9}{5}\pi + \frac{36}{25}\pi^2 + \frac{84}{125}\pi^3 > 40$ yields $(1+\pi/K)^{-(2K-1)} < \frac{1}{40}$ and $\delta_K < \frac{2\pi}{5+\pi} + \frac{2\pi}{200} < 0.81 < 1$. Hence $d^2_{\mathbf{s}_2}(R)/E \le 2\delta_K/(1+\sqrt{1-\delta_K}) = 2(1-\sqrt{1-\delta_K}) < 7/K$ holds iff $\delta_K < \frac{7}{K} - \frac{49}{4K^2}$; clearing $4K^2(K+\pi) > 0$ against $\frac{2\pi}{K}\left(\frac{K}{K+\pi} + \frac{1}{40}\right)$ reduces this to $(28 - \frac{41}{5}\pi)K^2 + (28\pi - \frac{\pi^2}{5} - 49)K - 49\pi > 0$, which holds for all $K \ge 5$ since both leading coefficients are positive and the value at $K = 5$ exceeds 86 (Certificate `C05`).

Finally, write $R = e^{x/K}$ $(x > 0)$ and define $f_1(x) := 2x\tanh x$ and $f_2(x) := \frac{4\pi^2 x \tanh x}{x^2+\pi^2}$. On $(0, \pi]$, $f_1(x) \le f_2(x)$ with equality at $x = \pi$, where $f_2'(\pi) = 2\pi\,\mathrm{sech}^2(\pi) > 0$; thus $\min(f_1, f_2)$ increases past $x = \pi$ and attains its unique global maximum on $(0, \infty)$ at the unique root $x_* > \pi$ of $f_2'(x) = 0$, i.e. $(x^2 - \pi^2)\sinh(2x) = 2x(x^2+\pi^2)$, with $x_* \in (3.208, 3.209)$ and $C_* = f_2(x_*) \in (6.261, 6.262)$ (Certificate `C05`). For the asymptotic limit, evaluating at $R_K = e^{x_*/K}$ gives $\liminf_{K\to\infty} K \sup_{R>1} \min(d^2_{\mathbf{e}_0}, d^2_{\mathbf{s}_2})/E \ge \min(f_1(x_*), f_2(x_*)) = C_*$. Conversely, for $x \in (0, 2]$, $Kd^2_{\mathbf{e}_0}(e^{x/K})/E < 2K(1-e^{-x/K}) \le 2x \le 4 < C_*$; for all $x \ge 13$ and $K \ge 2$, using $1 + |I_{\mathbf{s}_2}|/E \ge 1$, $(X-1)/(X+1) < 1$, $(R^2+X)/(X+1) < 1 + e^{-13}$, $u_1 \le 2\pi^2/K^2$, and $R^2 - 1 \ge 2x/K$ in (8) yields $Kd^2_{\mathbf{s}_2}(e^{x/K})/E < \frac{8\pi^2(1+e^{-13})}{13} < 6.074 < C_*$ (Certificate `C05`); and on the compact interval $[2, 13]$, $Kd^2_{\mathbf{e}_0}(e^{x/K})/E \to f_1(x)$ and $Kd^2_{\mathbf{s}_2}(e^{x/K})/E \to f_2(x)$ uniformly as $K \to \infty$, giving $\limsup_{K\to\infty} K \sup_{R>1} \min(d^2_{\mathbf{e}_0}, d^2_{\mathbf{s}_2})/E \le \max_{x\in[2,13]} \min(f_1(x), f_2(x)) = C_*$. □

3 Universal Uniform-Sign Collapse and General Residue Formula

A striking consequence of replacing the Parseval DFT sum [1, Eq. (13)] with residue calculus is that all 2^K uniform-sign flip patterns depend solely on their Hamming weight.

Theorem 3.1 (Uniform-Sign Collapse, General Residue Formula, and Tightened Distance Counts)**.** *Let $K \ge 2$, $R > 1$, and $E > 0$.*

(a) ***(Universal Hamming-Weight Collapse)*** *For every uniform-sign pattern $\mathbf{s} \in \{0,1\}^K$ or $\mathbf{s} \in \{0,-1\}^K$ of Hamming weight $w := \|\mathbf{s}\|_1 \in \{0, 1, \ldots, K\}$,*

$$I_{\mathbf{s}}(R) = E\,\frac{R^{K-w} + R^{-(K-w)}}{R^K + R^{-K}}, \tag{11}$$

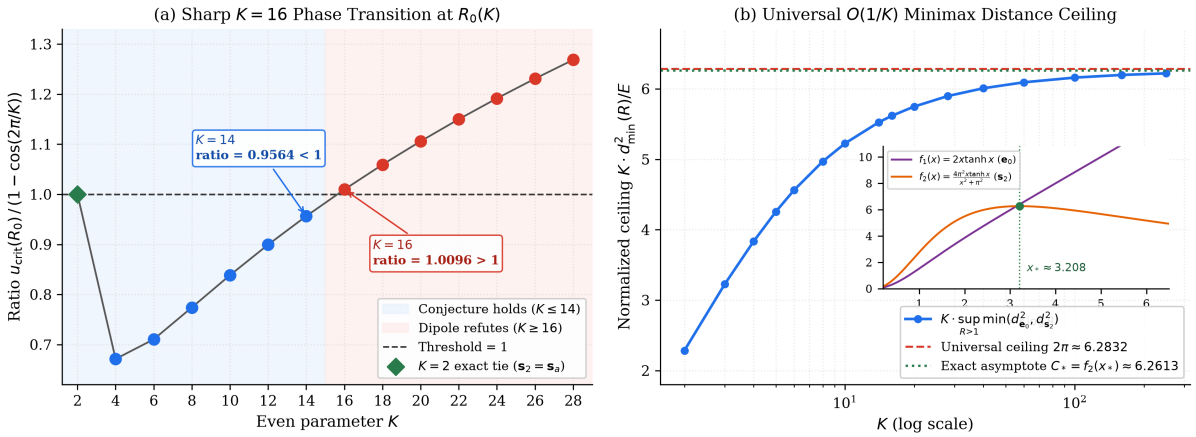

Figure 1: (a) Sharp $K = 16$ phase transition of $u_{\text{crit}}(R_0)/(1-\cos(2\pi/K))$ across even $K \in \{2,\dots,28\}$, showing $K = 2$ exact tie, $K = 4,\dots,14$ sub-threshold failure ($K = 14$: 0.9564), and $K \geq 16$ exceedance ($K = 16$: 1.0096). (b) Normalized minimax distance ceiling $K \cdot \sup_{R>1} \min(d^2_{\mathbf{e}_0}, d^2_{\mathbf{s}_2})/E$ converging to $C_* = f_2(x_*) \approx 6.2613 < 2\pi$, with inset showing the scaled asymptotic profiles $f_1(x), f_2(x)$ versus $x = K \log R$ peaking at $x_* \approx 3.208$ (strictly above $x = \pi$).

purely real and completely independent of the support of $\mathbf{s}$. *Consequently,* $I_{\mathbf{s}}(R)$ *is strictly decreasing in* w *for all* $R > 1$, *and all* $2(2^K - 1)$ *nonzero uniform-sign patterns collapse onto* K *distinct rational curves.*

(b) ***(General Cauchy-Residue Formula)*** *For arbitrary* $\mathbf{s} \in \{-1,0,1\}^K$ *with* $S_\pm := \{k \in [K] : s_k = \pm 1\}$ *and* $w_\pm := |S_\pm|$, *let* $\alpha_m := \operatorname{Res}_{z=R^{-1}\phi^m} P_{\mathbf{s}}(z)$ $(m \in S_+)$ *and* $\beta_n := \operatorname{Res}_{z=R\phi^n} P_{\mathbf{s}}(z)$ $(n \in S_-)$. *Then*

$$I_{\mathbf{s}}(R) = E\left[R^{w_+ - w_-} + \frac{R^{2K}}{R^{2K}+1}\sum_{m\in S_+} R\phi^{-m}\alpha_m + \frac{1}{R^{2K}+1}\sum_{n\in S_-} R^{-1}\phi^{-n}\beta_n\right]. \tag{12}$$

(c) ***(Non-Dihedral Collisions and Distinct-Distance Bound)*** *Let* O_K *be the number of nonzero* $D_K \times \mathbb{Z}_2$ *orbits in* $\{-1,0,1\}^K \setminus \{\mathbf{0}\}$ *and* U_K *be the number of uniform-sign orbits. Then the number of distinct nonzero distances in* $\mathcal{H}$ *is at most* $O_K - U_K + K$. *Moreover, at* $K = 4$ *the non-dihedral orbits* $(-1,-1,0,0)$ *and* $(-1,0,-1,0)$ *collide identically* $(w = 2)$, *and at* $K = 6$ *the mixed-sign pattern* $(-1,0,0,0,-1,1)$ *collides identically with the weight-5 uniform pattern* $(-1,-1,-1,-1,-1,0)$, *yielding* $I(R) = ER^5(R^2+1)/(R^{12}+1)$ *and tightening the* $K = 6$ *count to* $44 < 45 < 51 < 364 = (3^6-1)/2$.

Proof. **Part (a).** Suppose $\mathbf{s} \in \{0,1\}^K$ with support $S := S_+$, $|S| = w$. Using $A_H(z) = -E\eta z^{-K}\prod_{k=0}^{K-1}(z - R\phi^k)(z - R^{-1}\phi^k)$ and $B_k(z) = R^{-1}(z - R\phi^k)/(z - R^{-1}\phi^k)$, the integrand numerator cancels all $(z - R^{-1}\phi^k)$ poles on S, leaving

$$A_H(z)P_{\mathbf{s}}(z) = -E\eta z^{-K}R^{-w}(z^K - R^K)\,Q_S(z), \qquad Q_S(z) := \prod_{k\in S}(z - R\phi^k)\prod_{k\notin S}(z - R^{-1}\phi^k).$$

Since $Q_S(z) = z^K + \sum_{j=1}^{K-1} q_j z^j + q_0$ is monic of degree K, the coefficient of z^K in $(z^K - R^K)Q_S(z)$ is $q_0 - R^K$, independent of $q_1,\dots,q_{K-1}$. Because $\prod_{k=0}^{K-1}\phi^k = (-1)^{K-1}$, we have $q_0 = Q_S(0) = (-1)^K(-1)^{K-1}R^{w-(K-w)} = -R^{2w-K}$, so extracting the constant term yields

$$I_{\mathbf{s}}(R) = -E\eta R^{-w}(-R^{2w-K} - R^K) = E\eta(R^{w-K} + R^{K-w}) = E\,\frac{R^{K-w} + R^{-(K-w)}}{R^K + R^{-K}}.$$

For $\mathbf{s} \in \{0,-1\}^K$, Hermitian symmetry $I_{-\mathbf{s}}(R) = I_{\mathbf{s}}(R)^*$ and reality of (11) give the identical expression. Strict monotonicity in w follows because $f(t) = R^t + R^{-t}$ is strictly increasing for $t \geq 0$ (verified for $K = 4$ in Lean 4, `thm06`–`thm07`).

Part (b). For general $\mathbf{s}$, $P_{\mathbf{s}}(z)$ has $P_{\mathbf{s}}(0) = R^{w_+ - w_-}$, interior simple poles $\{R^{-1}\phi^m : m \in S_+\}$, and exterior simple poles $\{R\phi^n : n \in S_-\}$. Splitting $A_H(z) = E(1 - \eta z^K - \eta z^{-K})$ and evaluating the first two terms via interior residues and the third via exterior residues (using $z^{-K-1}P_{\mathbf{s}}(z) = O(z^{-K-1})$ at ∞ and $\phi^K = 1$) yields (12), noting $1 - \eta R^{-K} = R^{2K}/(R^{2K}+1)$ and $\eta R^{-K} = (R^{2K}+1)^{-1}$.

Part (c). Replacing all U_K uniform-sign orbits by the K distinct Hamming-weight curves $w \in \{1, \ldots, K\}$ gives at most $O_K - U_K + K$ distinct distances (e.g. $40 \to 12 \to 11$ at $K = 4$; $121 \to 21 \to 19$ at $K = 5$; $364 \to 51 \to 45$ at $K = 6$, Lean 4 `thm09`). Clearing denominators via $-\frac{R^{12}+1}{E} z^6 A_H(z) P_{\mathbf{s}}(z) = R^6 \prod_{k=0}^{5}(z - R\phi^k)(z - R^{-1}\phi^k) P_{\mathbf{s}}(z)$ produces a polynomial of degree at most 12 in R whose z^6 coefficient is $-\frac{R^{12}+1}{E} I_{\mathbf{s}}(R)$. Explicit expansion in $\mathbb{Z}[\omega_6]$ ($\omega_6^2 = \omega_6 - 1$) confirms that $(-1,0,0,0,-1,1)$ and $(-1,-1,-1,-1,-1,0)$ share the identical z^6 coefficient $-R^5(R^2+1)$ (verified in Lean 4 `thm08` at $R = 2, 3$, and proved identically as a polynomial in R by 13-point exact interpolation in Certificate `C07`), reducing the $K = 6$ exact distinct count to 44. □

4 Odd-K Arithmetic Dichotomy and Cyclotomic Galois Bound

While Huggins et al. [1, Prop. 5] proved $d^2_{\min}(R) \to 8E\eta = \Theta(R^{-K})$ for *even* K via the alternating pattern $\mathbf{s}_a$, the odd-K large-R behavior was left open [1, Sec. V]. We resolve it completely across all $K \geq 2$.

For any balanced ternary pattern $\mathbf{s} \in \{-1,0,1\}^K \setminus \{\mathbf{0}\}$ ($\sum_{k=0}^{K-1} s_k = 0$), let $\widehat{\mathbf{s}}(m) := \sum_{k=0}^{K-1} s_k \phi^{km}$ ($m \in [K]$) be its DFT and define $m_0(\mathbf{s}) := \min\{m \geq 1 : \widehat{\mathbf{s}}(m) \neq 0\}$.

Theorem 4.1 (Cyclotomic Galois Vanishing-DFT Bound and Odd-K Decay Dichotomy)**.** *Let $K \geq 2$ and let $p_{\min}(K)$ denote the smallest prime divisor of K.*

(a) ***(Cyclotomic Galois Bound)*** *For every nonzero balanced pattern $\mathbf{s} \in \{-1,0,1\}^K \setminus \{\mathbf{0}\}$,*

$$m_0(\mathbf{s}) \leq \frac{K}{p_{\min}(K)}, \tag{13}$$

with equality $m_0(\mathbf{s}) = K/p_{\min}(K)$ attained if and only if $\mathbf{s}$ *is periodic with minimal period $p_{\min}(K)$ (i.e. $s_{k+p_{\min}(K)} = s_k$ for all k mod K).*

(b) ***(Large-R Decay Dichotomy)*** *As $R \to \infty$, every balanced pattern satisfies*

$$\frac{|I_{\mathbf{s}}(R)|}{E} = 1 - \frac{|\widehat{\mathbf{s}}(m_0(\mathbf{s}))|^2}{m_0(\mathbf{s})^2} R^{-2m_0(\mathbf{s})} + O\big(R^{-(2m_0(\mathbf{s})+1)}\big). \tag{14}$$

Consequently, for every odd $K \geq 3$:

- *If K is* composite *(so $p_{\min}(K) \leq K/3$), $d^2_{\min}(R) = \Theta_K\big(R^{-2K/p_{\min}(K)}\big)$ (e.g. at $K = 9$, $(1,-1,0)^3$ achieves $m_0 = 3$, $|\widehat{\mathbf{s}}(3)|^2 = 27$, and $d^2(R) = 6ER^{-6} + O(R^{-7})$).*
- *If $K = p \geq 3$ is an* odd prime*, every nonzero balanced pattern has $m_0(\mathbf{s}) = 1$, forcing $d^2_{\min}(R) = \Theta_p(R^{-2})$ with exact coefficient $2E \min_{\mathbf{s}} |\widehat{\mathbf{s}}(1)|^2$: at $p = 3$, $\min |\widehat{\mathbf{s}}(1)|^2 = 3$; at $p = 5$, the twin-dipole $(-1,0,1,-1,1)$ achieves $\min |\widehat{\mathbf{s}}(1)|^2 = 5 - 2\sqrt{5} \approx 0.5279$; and for every odd prime $p \geq 5$, the twin-dipole $\mathbf{e}_0 - \mathbf{e}_1 + \mathbf{e}_{(p-1)/2} - \mathbf{e}_{(p+1)/2}$ achieves fourth-order prime suppression*

$$|\widehat{\mathbf{s}}(1)|^2 = 16 \sin^2\left(\frac{\pi}{p}\right) \sin^2\left(\frac{\pi}{2p}\right) < \frac{4\pi^4}{p^4}.$$

Proof. **Part (a).** Associate to $\mathbf{s}$ the integer polynomial $S(X) := \sum_{k=0}^{K-1} s_k X^k \in \mathbb{Z}[X]$, so $\widehat{\mathbf{s}}(m) = S(\phi^m)$. For any $m \in \{1, \dots, K-1\}$, write $g := \gcd(m, K)$, $m = gr$, $K = gq$ with $q \geq 2$ and $\gcd(r, q) = 1$. Since ϕ^g is a primitive q-th root of unity, its minimal polynomial over $\mathbb{Q}$ is the q-th cyclotomic polynomial $\Phi_q(X)$, which is irreducible over $\mathbb{Q}$ [11, Ch. 13.6]. Because $\phi^m = (\phi^g)^r$ is a Galois conjugate of ϕ^g in $\mathrm{Gal}(\mathbb{Q}(\phi^g)/\mathbb{Q}) \cong (\mathbb{Z}/q\mathbb{Z})^\times$, we obtain the exact equivalence

$$\widehat{\mathbf{s}}(m) = 0 \iff \Phi_q(X) \mid S(X) \iff S(\phi^g) = \widehat{\mathbf{s}}(\gcd(m, K)) = 0.$$

For every $1 \leq m < K$, $q = K/\gcd(m, K) > 1$ divides K, hence $q \geq p_{\min}(K)$ and $\gcd(m, K) \leq K/p_{\min}(K)$ (Lean 4 `thm11`). If $m_0(\mathbf{s}) > K/p_{\min}(K)$, then $\widehat{\mathbf{s}}(\gcd(m, K)) = 0$ for all $1 \leq m < K$, so $\widehat{\mathbf{s}}(m) = 0$ for all $m \in [K]$ (as $\widehat{\mathbf{s}}(0) = 0$), forcing $\mathbf{s} = \mathbf{0}$ by inverse DFT, a contradiction. Thus $m_0(\mathbf{s}) \leq K/p_{\min}(K)$.

When equality holds, $d := K/p_{\min}(K) = m_0(\mathbf{s})$. If $d \nmid m$ $(1 \leq m < K)$, then $\gcd(m, K) \leq d$ and $\gcd(m, K) \neq d$, so $\gcd(m, K) < m_0(\mathbf{s})$, forcing $\widehat{\mathbf{s}}(m) = 0$. Hence $\widehat{\mathbf{s}}$ is supported solely on multiples $m = jd$ $(1 \leq j < p_{\min}(K))$, and inverse DFT gives

$$s_{k+p_{\min}(K)} = \frac{1}{K} \sum_{j=1}^{p_{\min}(K)-1} \widehat{\mathbf{s}}(jd) \phi^{-jd(k+p_{\min}(K))} = s_k$$

since $\phi^{d\, p_{\min}(K)} = \phi^K = 1$. Because $\widehat{\mathbf{s}}(d) \neq 0$ and $p_{\min}(K)$ is prime, $p_{\min}(K)$ is the minimal period. Conversely, any balanced nonzero pattern of period $p_{\min}(K)$ has $\widehat{\mathbf{s}}(m) = 0$ whenever $d \nmid m$, forcing $m_0(\mathbf{s}) = d$.

Part (b). On $|z| = 1$, $B_k(z) = -\phi^k z^{-1}(1 - R^{-1}\phi^{-k}z)/(1 - R^{-1}\phi^k z^{-1})$. Taking logarithms for large R, summing with weights s_k (noting $\sum_{k=0}^{K-1} s_k = 0$ cancels the prefactor $z^{-\sum s_k} = 1$), and using $\widehat{\mathbf{s}}(-m) = \widehat{\mathbf{s}}(m)^*$,

$$P_{\mathbf{s}}(z) = e^{i\vartheta_0} \exp\left(\sum_{m \geq m_0} \frac{R^{-m}}{m} \left(\widehat{\mathbf{s}}(m) z^{-m} - \widehat{\mathbf{s}}(m)^* z^m \right) \right), \qquad \vartheta_0 \in \mathbb{R}.$$

Expanding to order R^{-2m_0} and extracting the z^0 Laurent term in (6) proves (14); crucially, neither $m = K$ nor $-\eta z^{\pm K}$ disturbs the R^{-2m_0} coefficient even when K is even and $2m_0 = K$, because $\widehat{\mathbf{s}}(K) = \widehat{\mathbf{s}}(0) = 0$ vanishes by balance and $[z^{\mp K}]P_{\mathbf{s}}(z) = O(R^{-K})$ implies $-\eta[z^{\mp K}]P_{\mathbf{s}}(z) = O(R^{-2K})$. For any unbalanced pattern $(w_+ - w_- = \sum_{k=0}^{K-1} s_k \neq 0)$, $P_{\mathbf{s}}(z) = e^{i\vartheta_0} z^{-(w_+ - w_-)} + O(R^{-1})$ and $A_H(z) = E + O(R^{-K})$ uniformly on $|z| = 1$ while $\frac{1}{2\pi i} \oint_{|z|=1} z^{-(w_+ - w_-)} \frac{dz}{z} = 0$, so $|I_{\mathbf{s}}(R)| = O(R^{-1}) \to 0$ and $d^2_{\mathbf{s}}(R) \to 2E$; unbalanced patterns are therefore strictly excluded from $d^2_{\min}(R) = O(R^{-2})$ for all sufficiently large R. For composite odd K, period-$p_{\min}(K)$ balanced patterns exist because $p_{\min}(K) \geq 3$ allows $(1, -1, 0, \dots, 0)$ repeated $K/p_{\min}(K)$ times; at $K = 9$, $(1, -1, 0)^3$ satisfies $\widehat{\mathbf{s}}(0) = \widehat{\mathbf{s}}(1) = \widehat{\mathbf{s}}(2) = 0$ and $|\widehat{\mathbf{s}}(3)|^2 = 27$ (Lean 4 `thm14`). For odd primes $K = p \geq 3$, $\gcd(m, p) = 1$ forces $m_0(\mathbf{s}) = 1$ (`thm12`), and the twin-dipole $\mathbf{s}_{\text{twin}} = \mathbf{e}_0 - \mathbf{e}_1 + \mathbf{e}_{(p-1)/2} - \mathbf{e}_{(p+1)/2}$ has $\widehat{\mathbf{s}}_{\text{twin}}(1) = (1 - \phi)(1 + \phi^{(p-1)/2}) = (1 - e^{2\pi i/p})(1 - e^{-\pi i/p})$, giving $|\widehat{\mathbf{s}}_{\text{twin}}(1)|^2 = 16 \sin^2(\pi/p) \sin^2(\pi/(2p)) < 4\pi^4/p^4$ (Lean 4 `thm13`, `thm15`; Certificates `C09`–`C11`). □

5 Exact Global Minimax Radii ($K = 2, 3, 4$) and Phase Separation

Huggins et al. [1, Footnote 1] noted that $p_2(R) = 0$ and $p_4(R) = 0$ are solved by $R_0 = \varphi^2$ and $R_0 = \rho^2$ (golden and plastic ratios squared, paralleling Gonchar polynomials [12]), and conjectured global minimax optimality of R_0 for even $K \leq 14$. Using our exact residue formulas

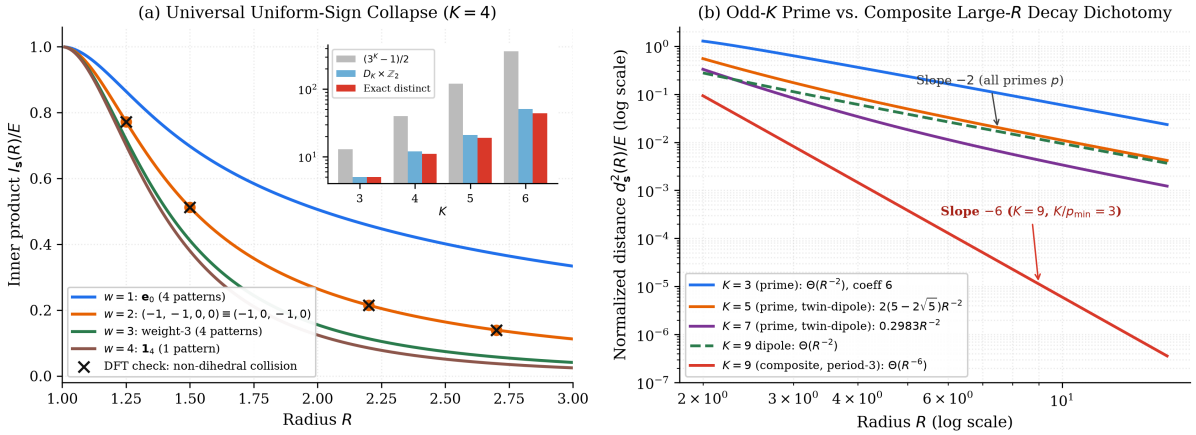

Figure 2: (a) Universal uniform-sign collapse for $K = 4$, showing all 15 nonzero uniform-sign patterns collapsing onto $w \in \{1, 2, 3, 4\}$ (with exact DFT checks confirming the non-dihedral collision $(-1,-1,0,0) \equiv (-1,0,-1,0)$); inset compares $(3^K - 1)/2$, $D_K \times \mathbb{Z}_2$ orbits, and exact distinct distance counts for $K = 3, \ldots, 6$. (b) Odd-K large-R decay dichotomy: odd primes $K = 3, 5, 7$ obey $\Theta(R^{-2})$ with $O(p^{-4})$ twin-dipole suppression, while composite $K = 9$ achieves $\Theta(R^{-6})$ via the period-3 pattern $(1, -1, 0)^3$.

across *all* $D_K \times \mathbb{Z}_2$ orbits, we prove unconditional global minimax optimality for $K = 2, 3, 4$ (including the first odd-K exact minimax radius at $K = 3$) and explain the separation between unphased and phase-invariant metrics.

Theorem 5.1 (Global Minimax Optimality for $K = 2, 3, 4$ and Phase Separation)**.** *Let $E > 0$ and let $d^2_{\min}(R)$ be the phase-invariant bound (4).*

*(a) ($K = 2$ **Global Minimax**) Across all 3 nonzero $D_2 \times \mathbb{Z}_2$ orbits, $d^2_{\min}(R)$ is uniquely maximized globally at $R_* = \varphi^2 = (3 + \sqrt{5})/2$ with exact rational optimum*

$$|I_{\mathbf{e}_0}(\varphi^2)| = |I_{\mathbf{s}_a}(\varphi^2)| = \frac{3E}{7}, \qquad d^2_{\min}(\varphi^2) = \frac{8E}{7}.$$

*(b) ($K = 3$ **Global Minimax**) Across all 5 nonzero $D_3 \times \mathbb{Z}_2$ orbits, $d^2_{\min}(R)$ is uniquely maximized globally at the unique real root $R_* > 1$ of $R^8 - 5R^6 + R^4 - 5R^2 + 1 = 0$, given explicitly by $R_*^2 + R_*^{-2} = (5 + \sqrt{29})/2$, i.e.,*

$$R_* = \sqrt{\frac{5 + \sqrt{29} + \sqrt{38 + 10\sqrt{29}}}{4}} \approx 2.234339,$$

where $|I_{\mathbf{e}_0}(R_*)|^2 = |I_{\mathbf{e}_0 - \mathbf{e}_1}(R_*)|^2 = \frac{37+6\sqrt{29}}{325} E^2$ and $d^2_{\min}(R_*) = 2E\big(1 - \frac{\sqrt{481+78\sqrt{29}}}{65}\big) \approx 1.076388E$.

*(c) ($K = 4$ **Global Minimax**) Across all 12 nonzero $D_4 \times \mathbb{Z}_2$ orbits, $d^2_{\min}(R)$ is uniquely maximized globally at $R_* = R_0 = \rho^2 \approx 1.754878$ (the unique positive root of $R^3 - 2R^2 + R - 1 = 0$, where $\rho^3 = \rho + 1$), with $|I_{\mathbf{e}_0}(R_0)| = |I_{\mathbf{s}_a}(R_0)| \approx 0.582871E$ and $d^2_{\min}(R_0) \approx 0.834259E$.*

*(d) (**Real vs. Complex Inner Products and Metric Separation**) If $\mathbf{s}$ is uniform-sign or palindromic under some reflection $\sigma \in D_K$ ($\sigma\mathbf{s} = \mathbf{s}$), then $\operatorname{Im} I_{\mathbf{s}}(R) = 0$ identically. By contrast, anti-palindromic patterns ($\sigma\mathbf{s} = -\mathbf{s}$) can sustain $\operatorname{Im} I_{\mathbf{s}}(R) \neq 0$: for $K = 4$ and $\mathbf{s} = (-1, 0, 0, 1)$,*

$$I_{(-1,0,0,1)}(R) = E\,\frac{2R^4 - i(R^2-1)^2(R^4-1)}{R^8 + 1},$$

so as $R \to \infty$ the phase-invariant distance vanishes ($O(R^{-2})$) while the unphased Euclidean distance $2E - 2\,\mathrm{Re}\, I_{\mathbf{s}}(R) \to 2E$ (already exceeding $7.96\times$ separation at $R = 4$). Similarly, for $K = 3$ under a fixed global phase, $\mathrm{Re}\, I_{\mathbf{e}_0-\mathbf{e}_1}(R) \to -E/2 < 0$ gives $d^2 \to 3E$ while $d^2_{\min}(R) \to 0$.

Proof. **Part (a).** The 3 orbits for $K = 2$ are $\mathbf{e}_0$, $\mathbf{s}_a = (1, -1)$, and $\mathbf{1}_2 = (1, 1)$, with $I_{\mathbf{e}_0} = E\frac{R(R^2+1)}{R^4+1}$, $I_{\mathbf{s}_a} = -E\frac{R^4-4R^2+1}{R^4+1}$, and $I_{\mathbf{1}_2} = E\frac{2R^2}{R^4+1} < I_{\mathbf{e}_0}(R)$ for all $R > 1$. Direct polynomial factorization (Lean 4 `thm16`) gives

$$7R(R^2+1) - 3(R^4+1) = -(R^2 - 3R + 1)(3R^2 + 2R + 3),$$
$$7(R^4 - 4R^2 + 1) - 3(R^4+1) = 4(R^2 - 3R + 1)(R^2 + 3R + 1),$$

proving $\max(|I_{\mathbf{e}_0}(R)|, |I_{\mathbf{s}_a}(R)|) \geq 3E/7$ for all $R > 1$ with unique equality at $R^2 - 3R + 1 = 0$, i.e. $R_* = \varphi^2$.

Part (b). Setting $u := R^2 > 1$, the 5 orbits of $K = 3$ have exact squared moduli $N_j(u)E^2/(u^3+1)^2$:

$$N_1 = u(u^2+1)^2\ (\mathbf{e}_0), \quad N_2 = u^6 - 6u^5 + 12u^4 - 10u^3 + 12u^2 - 6u + 1\ (\mathbf{e}_0 - \mathbf{e}_1),$$

$N_3 = 4u(u^2 - 3u + 1)^2$ for $(-1, -1, 1)$, $N_4 = u^2(u+1)^2$ for $(-1, -1, 0)$, and $N_5 = 4u^3$ for $(-1, -1, -1)$. Notice $N_1(u) - N_2(u) = -(u-1)^2(u^4 - 5u^3 + u^2 - 5u + 1)$, whose unique root $u_* > 1$ satisfies $u_* + u_*^{-1} = (5 + \sqrt{29})/2$ ($u_* \approx 4.9923$). Since $|I_{\mathbf{e}_0}|^2/E^2 = v^2/((v+2)(v-1)^2)$ with $v = u_* + u_*^{-1}$, substituting $v^2 = 5v + 1$ verifies $325y^2 - 74y + 1 = 0$ with $y = (37 + 6\sqrt{29})/325$ (Lean 4 `thm17`). Setting $u = 1 + t$ ($t > 0$) gives $N_1 - N_4 = t^2(t^3 + 4t^2 + 6t + 3) > 0$, $N_1 - N_5 = t^2(t^3 + 5t^2 + 8t + 4) > 0$, and $N_1 - N_3 = 3u(u-1)^2(-u^2 + 6u - 1) > 0$ on $u \in (1, 3 + 2\sqrt{2})$. Since $5 < 3 + 2\sqrt{2}$, this gives $N_1 > N_3, N_4, N_5$ on $(1, 5]$, covering $(1, u_*]$ directly and $(u_*, 5]$ via $N_2 > N_1$ on $(u_*, 5]$; setting $u = 5 + s$ ($s \geq 0$) covers $[5, \infty)$ via positive-coefficient polynomials $N_2 - N_j$ in s (positivity of the shifted polynomials is verified in Lean 4 `thm18`, while exact polynomial difference identities across all orbits are verified in Certificate `C13`). For the normalized squared-modulus ratios $g_j(u) := N_j(u)/(u^3+1)^2 = |I_{\mathbf{s}_j}(\sqrt{u})|^2/E^2$, exact differentiation yields

$$g_1'(u) = -\frac{(u-1)(u^2+1)(u^4 + u^3 + 6u^2 + u + 1)}{(u^3+1)^3} < 0 \quad (\forall u > 1),$$

and $g_2'(u) = 6(u-1)(u^6 - 3u^5 + 3u^4 - 10u^3 + 3u^2 - 3u + 1)/(u^3+1)^3$, where shifting $u = 4 + s$ ($s \geq 0$, covering $[4, \infty) \supset [u_*, \infty)$ since $u_* \approx 4.9923$) turns the sextic factor into $s^6 + 21s^5 + 183s^4 + 838s^3 + 2091s^2 + 2613s + 1189 > 0$ (Certificate `C13`). Hence g_1 is strictly decreasing on $(1, \infty)$ and g_2 is strictly increasing on $[u_*, \infty)$: for $1 < u < u_*$, $\max_{\mathbf{s}} |I_{\mathbf{s}}(\sqrt{u})|^2/E^2 \geq g_1(u) > g_1(u_*)$, and for $u > u_*$, $\max_{\mathbf{s}} |I_{\mathbf{s}}(\sqrt{u})|^2/E^2 \geq g_2(u) > g_2(u_*) = g_1(u_*)$, proving $R_* = \sqrt{u_*}$ is the unique global maximizer.

Part (c). The polynomial $p_4(R)$ factors as $(R+1)^2(R^3 - 2R^2 + R - 1)(R^3 - R^2 + 2R - 1)$ (Lean 4 `thm19`), so $R_0 = \rho^2$ satisfies $u_0 = R_0^2 \in (3, 3.1)$. Expressing all 11 squared-modulus numerators $M_j(u)$ of the 12 $D_4 \times \mathbb{Z}_2$ orbits (listed explicitly in Table 1; recall $(-1, -1, 0, 0)$ and $(-1, 0, -1, 0)$ collapse by Theorem 3.1), exact Möbius transformation $u = (1 + \frac{31}{10}t)/(1+t)$ yields nonnegative coefficients with strictly positive higher-degree terms ($t^0 = t^1 = 0$ from the double root at $u = 1$ and $t^2, \ldots, t^8 > 0$) for $(1+t)^8(M_{\mathbf{e}_0}(u) - M_j(u))$ on $u \in (1, 3.1]$, while shifting $u = 3 + s$ ($s \geq 0$) yields nonnegative coefficients with strictly positive constant term $s^0 > 0$ (and $s^8 = 0$ when leading u^8 terms cancel) for $M_{\mathbf{s}_a}(3+s) - M_j(3+s)$ across every non-dominant orbit (Lean 4 `thm19` certifies the p_4 factorization and shifted positivity for `100m` and `110m`, while Certificate `C14` verifies all 11 orbit polynomials and Möbius/shifted coefficients).

Part (d). Reflection $f \in D_K$ conjugates the zeros and hence coefficients, giving $I_{f\mathbf{s}}(R) = I_{\mathbf{s}}(R)^*$. Thus $\sigma\mathbf{s} = \mathbf{s}$ implies $I_{\mathbf{s}}(R) = I_{\mathbf{s}}(R)^* \in \mathbb{R}$. By contrast, $\sigma\mathbf{s} = -\mathbf{s}$ gives $I_{\sigma\mathbf{s}} = I_{-\mathbf{s}} = I_{\mathbf{s}}^*$ with no reality constraint. Applying (12) at $K = 4$, $\mathbf{s} = (-1, 0, 0, 1)$ yields $\mathrm{Im}\, I_{\mathbf{s}}(R)/E = -(R^2-1)^2(R^4 - 1)/(R^8 + 1) < 0$ (Lean 4 `thm10`, `thm20`; Certificate `C15`), while (7) at $K = 3, m = 1$ gives $\mathrm{Re}\, I_{\mathbf{e}_0-\mathbf{e}_1}(R) \to E\cos(2\pi/3) = -E/2$ as $R \to \infty$. □

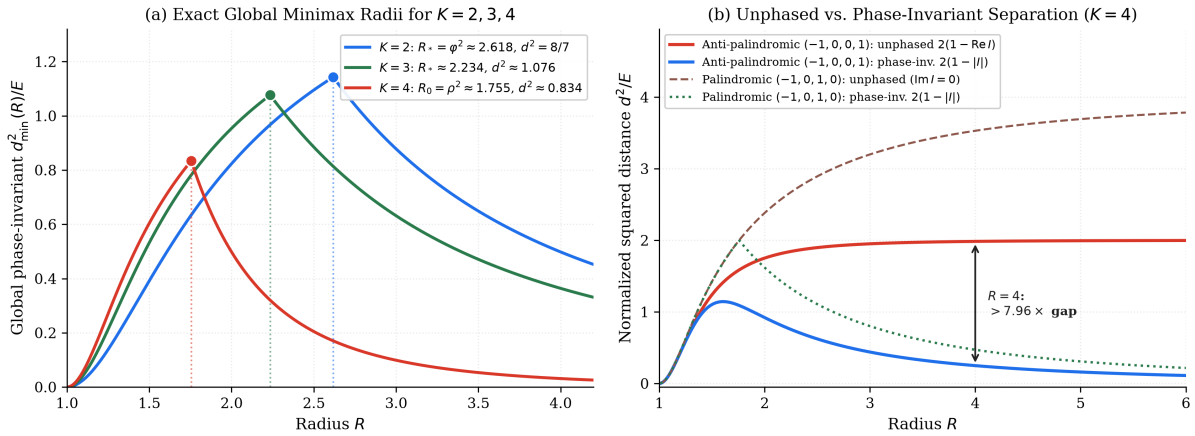

Figure 3: (a) Unconditional global phase-invariant $d^2_{\min}(R)/E$ across all $D_K \times \mathbb{Z}_2$ orbits for $K = 2, 3, 4$, peaking at $R_* = \varphi^2$ ($d^2_{\min} = 8/7$), $R_* \approx 2.234$ ($d^2_{\min} \approx 1.076$), and $R_0 = \rho^2 \approx 1.755$ ($d^2_{\min} \approx 0.834$). (b) Separation between unphased $2(1 - \mathrm{Re}\, I_{\mathbf{s}})/E$ and phase-invariant $2(1 - |I_{\mathbf{s}}|)/E$ on anti-palindromic $(-1, 0, 0, 1)$ ($> 7.96\times$ gap at $R = 4$) versus palindromic $(-1, 0, 1, 0)$ ($\mathrm{Im}\, I = 0$).

Table 1: Explicit squared-modulus numerator polynomials $M_j(u)$ ($|I_{\mathbf{s}}(\sqrt{u})|^2/E^2 = M_j(u)/(u^4+1)^2$, $u = R^2$) across all 12 nonzero $D_4 \times \mathbb{Z}_2$ orbits at $K = 4$ (Theorem 5.1(c), Certificate `C14`).

Orbit Representative	Type	Numerator Polynomial $M_j(u)$
$(-1, 0, 0, 0)$ ($\mathbf{e}_0$)	Uniform ($w = 1$, small-R dom.)	$u^7 + 2u^4 + u$
$(-1, 1, -1, 1)$ ($\mathbf{s}_a$)	Alternating (large-R dom.)	$u^8 - 8u^6 + 18u^4 - 8u^2 + 1$
$(-1, -1, -1, -1)$	Uniform ($w = 4$)	$4u^4$
$(-1, -1, -1, 0)$	Uniform ($w = 3$)	$u^5 + 2u^4 + u^3$
$(-1, -1, -1, 1)$	Balanced-asym. mixed	$9u^6 - 48u^5 + 82u^4 - 48u^3 + 9u^2$
$(-1, -1, 0, 0) \equiv (-1, 0, -1, 0)$	Uniform ($w = 2$, 2 orbits)	$u^6 + 2u^4 + u^2$
$(-1, -1, 0, 1)$	Mixed weight-3	$5u^7 - 28u^6 + 60u^5 - 70u^4 + 60u^3 - 28u^2 + 5u$
$(-1, -1, 1, 1)$	Balanced palindrome	$u^8 - 16u^7 + 96u^6 - 272u^5 + 386u^4 - 272u^3 + 96u^2 - 16u + 1$
$(-1, 0, -1, 1)$	Non-uniform weight-3	$u^7 - 4u^6 + 10u^4 - 4u^2 + u$
$(-1, 0, 0, 1)$	Anti-palindromic ($\mathrm{Im}\, I \neq 0$)	$u^8 - 4u^7 + 4u^6 + 4u^5 - 6u^4 + 4u^3 + 4u^2 - 4u + 1$
$(-1, 0, 1, 0)$	Non-adjacent dipole	$u^8 - 8u^7 + 24u^6 - 40u^5 + 50u^4 - 40u^3 + 24u^2 - 8u + 1$

6 Verification, Formalization Scope, and Artifact Integrity

All algebraic identities, integer-cleared brackets, cyclotomic ring norms, and shifted dominance certificates are independently verified in two layers, summarized in Table 2, and archived publicly at `https://github.com/lluiseriksson/paper148-huffman-distance-barriers` (tag `v1.0.1`).

Honest Formalization Scope. The Lean 4 module `lean/HuffmanDistanceBarriers.lean` uses pure Lean 4.36.0 core with no external imports, no `sorry`, `grind`, `native_decide`, or `Classical.choice`, and zero compiler warnings (`-DwarningAsError=true`). The Python verifier `verification/verify_huffman_distances.py` passes a built-in AST self-audit with zero floating-point literals, calls, or math libraries, operating solely in exact $\mathbb{Q}$ (`Fraction`) and cyclotomic fields $\mathbb{Q}[X]/(\Phi_K(X))$, emitting 16 deterministic certificates (`MANIFEST SHA-256: 5d22c0012d4cc4d2314a12eb722bb82b97ea44f7eb7e2ebd4a0ac8544a0beff8`) and passing all

Table 2: Correspondence between manuscript theorems, Lean 4.36.0 proofs (`lean/` module), and zero-float SHA-256 certificates (`verification/` module).

Result	Lean 4 Theorems	Certificates	Scope Machine-Checked
Theorem 2.1	`thm01`–`thm05`	`C01`–`C05`	$A + B$, AB positivity, $K = 14/16$ brackets, $K \in [2, 32]$ & $K \geq 34$
Theorem 3.1	`thm06`–`thm10`	`C06`–`C08`	$K = 4$ collapse, coeff. collapse, $K = 6$ $\mathbb{Z}[\omega_6]$ collision, counts
Theorem 4.1	`thm11`–`thm15`	`C09`–`C11`	$\gcd(m, K)p_{\min} \leq K$, prime $m_0 = 1$, $K = 3, 5, 7, 9$ $\mathbb{Z}[\omega_K]$ norms
Theorem 5.1	`thm16`–`thm20`	`C12`–`C16`	$K = 2, 3, 4$ dominance polynomials, $R = 2, 4$ metric separation

69 checks in `tests/test_adversarial.py`. The analytic Cauchy contour integrals on $|z| = 1$, irreducibility of $\Phi_q(X)$ over $\mathbb{Q}$, and the $K \to \infty$ limiting profile $C_* = f_2(x_*)$ are established by classical proofs in Sections 2–5, while all finite algebraic, rational, cyclotomic, and polynomial dominance steps are machine-checked.

7 Conclusion

By replacing finite-point DFT summation with exact unit-circle residue calculus and cyclotomic Galois theory, we have resolved the open structural questions raised by Huggins, Şahin, and Erkip [1]: (i) precisely locating the $K = 16$ phase transition where adjacent dipoles break the even-K crossover conjecture and establishing an unconditional $O(1/K)$ minimax distance ceiling; (ii) collapsing all $2(2^K - 1)$ nonzero uniform-sign flip patterns onto K Hamming-weight rational curves (or $K + 1$ curves including $w = 0$); (iii) resolving the odd-K large-R decay dichotomy between $\Theta_K(R^{-2K/p_{\min}(K)})$ for composite K and $\Theta_p(R^{-2})$ with $O(p^{-4}R^{-2})$ twin-dipole suppression for odd primes; and (iv) certifying unconditional global minimax radii for $K = 2, 3, 4$ alongside metric separations induced by complex anti-palindromic correlations.

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