All-Field Morse–Bott Stability at Critical Homogeneous Orbits

Half-Grassmann No-Spinodal Rigidity and Exact Jacobi Metastability

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Abstract

Let a compact group act orthogonally on a sphere and let μ be the invariant law of a proper antipodal orbit. We study the orbital potential $U_t(a) = \int \exp(t\langle a, u\rangle)\, d\mu(u)$. At a point of the source orbit, every spherical-harmonic coefficient of the orbit average is a squared norm. Combining this positivity with the strictly positive Gegenbauer–Bessel coefficients of the exponential kernel gives an exact negative spherical-Laplacian identity at every field $t > 0$. Provided the source orbit is critical—automatically so when the normal isotropy representation has no fixed vector—irreducibility, or a transitive symmetry of its irreducible normal blocks, upgrades this trace identity to strict Morse–Bott maximality for every field. We give an explicit torus-orbit counterexample showing why the criticality hypothesis cannot be omitted.

Applied to the centered projector embeddings of the real, complex, and quaternionic half-Grassmannians, the theorem proves all-field local rigidity of the balanced matrix–Bingham branch and excludes any radial spinodal. In the complex case we go further. For every two-block external spectrum we derive exact constrained-Hessian eigenvalues from the reversible Jacobi generator. Their low-field factors show a sharp metastability threshold: unbalanced strata with positive multiplicity $k > 2r/3$ are genuine local maxima at weak field, whereas those with $k \le 2r/3$ have an unstable block split. A Stein inequality proves the required Hessian sign in all sufficiently high Taylor degrees and leaves an explicit finite low-degree gate. Thus a global competing phase, if it exists, must arise by nonlocal coexistence rather than loss of stability of the balanced branch. We do not claim global spectral optimality or the resulting all-distortion rate–distortion function.

1 Introduction

Matrix–Bingham normalizers on Grassmannians are orbital Laplace transforms and hypergeometric functions of matrix argument. Their evaluation, asymptotics, and information geometry are well developed [10, 13, 11, 1, 2]. A different problem asks which external spectrum maximizes such a normalizer under trace and Frobenius constraints. The constraints do not define a majorization chain, so standard Schur or orbital-integral monotonicity does not decide the question [9, 15, 12].

The rank-two case $\mathrm{Gr}_{\mathbb{C}}(2, 4)$ admits an exact all-field solution [6]; arbitrary rank has previously been controlled at high field, yielding an exact high-fidelity rate–distortion segment [5]. The missing finite-field issue is global: can an intermediate external spectrum overtake the balanced two-block field? This paper does not resolve that global ordering. It instead closes its local part at every field and proves that a seemingly natural saddle-based shortcut is false.

The first observation is representation-free. Classical Schoenberg and Gangolli theory characterizes positive-definite zonal kernels by nonnegative harmonic coefficients [14, 8]. If the center of the orbital potential lies on the same orbit as the integration measure, the orbital harmonic coefficient is itself a squared mean harmonic. The spherical Laplacian is consequently strictly negative. Under an explicit criticality condition and a normal-isotropy hypothesis this trace sign becomes a complete normal-Hessian sign, giving an all-field Morse–Bott theorem. We also exhibit a proper antipodal orbit for which irreducibility alone does not force criticality.

Group-invariant kernel energies on compact homogeneous manifolds have also been used for quadrature and discrepancy [3]. Those results optimize an energy over measures or point sets on the homogeneous space. Our variable is instead an ambient external direction on a fixed Frobenius sphere, and the conclusion is a transverse Hessian theorem at the embedded source orbit.

For half-Grassmannians this proves that the balanced field never becomes spinodally unstable over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. In the complex case a second, independent Jacobi calculation classifies weak-field block splitting at every two-level spectrum. It reveals locally stable unbalanced branches near the balanced multiplicity. Hence local Hessians cannot eliminate all competitors: a complete global proof must compare separated values or prove a stronger Laplace order.

Our contributions are:

(i) a positive-mode spherical-Laplacian identity, a necessary criticality guard, and an all-field Morse–Bott criterion for homogeneous orbital potentials;

(ii) strict balanced local rigidity for the real, complex, and quaternionic half-Grassmann matrix–Bingham models;

(iii) exact Jacobi–Stein identities and exact constrained-Hessian operators for every complex two-block multiplicity stratum;

(iv) the sharp weak-field metastability threshold $k = 2r/3$, including its degenerate sixth-moment boundary; and

(v) a coefficient-tail theorem that reduces one global sign problem to finitely many exact moments at each fixed (r, k).

The addition theorem, matrix-beta law, and Jacobi ensemble are established ingredients. Novelty is claimed only for the joined all-field stability and metastability conclusions, not for those classical tools.

2 Orbital harmonic positivity

Let $N \geq 3$, let G be a compact group acting orthogonally on $\mathbb{R}^N$, let $\mathcal{O} = Gv \subset S^{N-1}$ be an orbit, and let μ be its invariant probability measure. Write $\lambda = (N-2)/2$ and let

$$K_\ell(x,y) = \frac{C_\ell^\lambda(\langle x, y\rangle)}{C_\ell^\lambda(1)} \tag{2.1}$$

be the normalized degree-ℓ spherical-harmonic reproducing kernel. The circle $N = 2$ has the analogous Fourier/Chebyshev formulation, but it is not needed in any application below and is excluded from the stated Gegenbauer normalization. For an orthonormal real harmonic basis $\{Y_{\ell j}\}_j$, the addition theorem gives

$$K_\ell(x,y) = c_{N,\ell} \sum_j Y_{\ell j}(x) Y_{\ell j}(y), \qquad c_{N,\ell} > 0. \tag{2.2}$$

Lemma 2.1 (Squared orbital coefficient). *For every $\ell \geq 0$ and every $v \in \mathcal{O}$,*

$$q_\ell := \int_{\mathcal{O}} K_\ell(u,v)\, d\mu(u) = c_{N,\ell} \sum_j \left| \int_{\mathcal{O}} Y_{\ell j}(u)\, d\mu(u) \right|^2 \geq 0. \tag{2.3}$$

If $\mathcal{O}$ is antipodal, then $q_\ell = 0$ for odd ℓ.

Proof. Invariance makes the first integral independent of the selected $v \in \mathcal{O}$. Averaging it once more over v and using (2.2) gives the squared-norm formula. Antipodality annihilates every odd harmonic average. □

The evaluation point belonging to the source orbit is essential. Away from the orbit the corresponding coefficient is an inner product of two harmonic vectors and need not be nonnegative.

Let an analytic zonal kernel have the expansion

$$\Phi(x) = \sum_{\ell \geq 0} a_\ell \frac{C_\ell^\lambda(x)}{C_\ell^\lambda(1)}, \qquad a_\ell \geq 0, \tag{2.4}$$

with locally uniform convergence after two derivatives, and define

$$U_\Phi(a) = \int_{\mathcal{O}} \Phi(\langle a, u\rangle)\, d\mu(u), \qquad a \in S^{N-1}. \tag{2.5}$$

Proposition 2.2 (Negative orbital Laplacian). *At every $v \in \mathcal{O}$,*

$$\Delta_{S^{N-1}} U_\Phi(v) = -\sum_{\ell \geq 1} \ell(\ell + N - 2) a_\ell q_\ell \leq 0. \tag{2.6}$$

The inequality is strict if $a_\ell q_\ell > 0$ for at least one $\ell \geq 1$.

Proof. Integrate (2.4), apply theorem 2.1, and use $\Delta K_\ell = -\ell(\ell + N - 2) K_\ell$ in the first argument. □

For $\Phi_t(x) = e^{tx}$ all Gegenbauer coefficients are strictly positive for $t > 0$. If $\mathcal{O}$ is a proper orbit, its invariant measure is not uniform on the ambient sphere. Since spherical harmonics determine finite measures on the compact sphere, some nonconstant q_ℓ is positive. Thus theorem 2.2 is strict for every $t > 0$.

3 A critical all-field Morse–Bott theorem

The potential is G-invariant, hence constant along $\mathcal{O}$, but constancy on the orbit does not by itself make an orbit point critical: a normal gradient can remain. The following elementary guard isolates the missing condition.

Lemma 3.1 (Criticality guard). *For every $v \in \mathcal{O}$, the spherical gradient satisfies*

$$\nabla_S U_\Phi(v) \in (N_v \mathcal{O})^{G_v}. \tag{3.1}$$

Consequently $(N_v\mathcal{O})^{G_v} = \{0\}$ implies that the whole orbit is critical. At a critical orbit, the spherical Hessian annihilates $T_v\mathcal{O}$, including all mixed tangent–normal entries.

Proof. Invariance makes the gradient orthogonal to the orbit and fixed by the stabilizer, proving (3.1). For an infinitesimal orbit field $X^\#$ one has $dU_\Phi(X^\#) = 0$. Covariantly differentiating this identity in an arbitrary spherical direction Z gives

$$\operatorname{Hess}_S U_\Phi(Z, X^\#) + dU_\Phi(\nabla_Z X^\#) = 0.$$

The second term vanishes at a critical point. Orbit homogeneity transports criticality and the Hessian-kernel conclusion from v to every point of $\mathcal{O}$. □

Theorem 3.2 (Critical homogeneous orbital Morse–Bott stability)**.** *Assume that* $\mathcal{O} \subset S^{N-1}$ *is proper and antipodal, that* v *is a critical point of* U_Φ*—for example,* $(N_v\mathcal{O})^{G_v} = \{0\}$*—and that either*

(a) the real G_v*-representation on* $N_v\mathcal{O}$ *is irreducible; or*

(b) it is a direct sum of pairwise inequivalent irreducibles and a symmetry of the sphere fixing v *and preserving* U_Φ *permutes the summands transitively.*

If Φ *satisfies (2.4) with* $a_\ell > 0$ *for every* ℓ*, then* $\mathcal{O}$ *is a nondegenerate Morse–Bott manifold of strict local maxima of* U_Φ*. In particular this holds for* $\Phi_t(x) = e^{tx}$ *at every* $t > 0$*.*

Proof. By theorem 3.1, the tangent and mixed Hessian entries vanish. Schur's lemma makes the self-adjoint Hessian scalar on every indicated irreducible normal summand; the extra symmetry makes those scalars equal in case (b). Thus the spherical Laplacian is the dimension of the normal space times their common scalar. It is strictly negative by theorem 2.2; hence the normal Hessian is negative definite. Its kernel is exactly $T_v\mathcal{O}$, and the Morse–Bott lemma gives transverse strict local maximality. □

Remark 3.3 (Why criticality is essential)**.** *Let* $G = SO(2) \times SO(2)$ *act on* $\mathbb{R}^2 \oplus \mathbb{R}^2$ *and take* $v = (ae_1, be_1)$ *with* $a^2 + b^2 = 1$ *and* $a \neq b$*. The proper antipodal orbit* $aS^1 \times bS^1 \subset S^3$ *has a one-dimensional, hence irreducible, spherical normal representation, but that representation is trivial. For* $A(\theta) = (\cos\theta\, e_1, \sin\theta\, e_1)$*,*

$$U_t(A(\theta)) = I_0(ta\cos\theta)I_0(tb\sin\theta),$$

and at $\cos\theta_0 = a$*,* $\sin\theta_0 = b$*,*

$$\frac{d}{d\theta}\log U_t(A(\theta))\bigg|_{\theta_0} = tab\left[\frac{I_1(tb^2)}{I_0(tb^2)} - \frac{I_1(ta^2)}{I_0(ta^2)}\right]. \tag{3.2}$$

Its small-field expansion is $\frac{1}{2}t^2ab(b^2 - a^2) + O(t^4)$*, so the source point is not critical for all sufficiently small* $t > 0$*. This example satisfies the original irreducibility condition but violates the fixed-vector guard in theorem 3.1.*

Corollary 3.4 (Uniform tubular exclusion)**.** *Assume* $(N_v\mathcal{O})^{G_v} = \{0\}$ *and either normal-isotropy condition of theorem 3.2. For every compact* $[t_0, t_1] \subset (0, \infty)$ *there is a tubular neighborhood* $\mathcal{T}$ *of* $\mathcal{O}$ *such that*

$$U_t(a) < U_t(v), \qquad a \in \mathcal{T} \setminus \mathcal{O}, \quad v \in \mathcal{O}, \quad t \in [t_0, t_1]. \tag{3.3}$$

Proof. The smallest magnitude normal-Hessian eigenvalue is continuous on the compact set $\mathcal{O} \times [t_0, t_1]$ and is bounded away from zero. A uniform Taylor remainder and compactness give the claimed tube. □

4 Half-Grassmann no-spinodal rigidity

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, let P be a Haar rank-r orthogonal projector on $\mathbb{F}^{2r}$, and set

$$S_P = P - \frac{1}{2}I, \qquad \rho^2 = \|S_P\|_F^2 = \frac{r}{2}, \qquad u_P = S_P/\rho. \tag{4.1}$$

The orbit $\mathcal{O}_\mathbb{F} = \{u_P\}$ is antipodal because $I - P$ has the same law. Its real ambient dimensions are

$$N_\mathbb{R} = r(2r+1) - 1, \qquad N_\mathbb{C} = 4r^2 - 1, \qquad N_\mathbb{H} = 8r^2 - 2r - 1. \tag{4.2}$$

At

$$v_\star = \frac{\operatorname{diag}(I_r, -I_r)}{\sqrt{2r}}, \tag{4.3}$$

the off-diagonal block is tangent to the conjugacy orbit. The spherical normal space is the direct sum of the two block-diagonal traceless self-adjoint modules. Each is irreducible under its stabilizer factor; block exchange together with antipodality makes their Hessian scalars equal. More explicitly, if W swaps the two r-blocks, then $A \mapsto -WAW^*$ fixes $v_\star$, exchanges the normal modules, and preserves the potential because μ is both conjugation-invariant and antipodal. Moreover, a vector fixed by the block stabilizer is scalar on each block; membership in either traceless normal module forces both scalars to vanish. Hence $(N_{v_\star}\mathcal{O}_{\mathbb{F}})^{G_{v_\star}} = \{0\}$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and theorem 3.1 proves criticality rather than assuming it.

Theorem 4.1 (Balanced no-spinodal theorem)**.** *For every $r \geq 2$, every $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and every $t > 0$, the balanced field orbit (4.3) is a strict constrained local maximizer of*

$$A \longmapsto \mathbb{E}\exp\bigl(t\rho\langle A, u_P\rangle\bigr), \qquad \operatorname{tr} A = 0, \quad \|A\|_F = 1. \tag{4.4}$$

Its constrained-Hessian kernel consists exactly of conjugacy-orbit directions. Thus the balanced branch has no finite-field radial spinodal.

Proof. The fixed-vector calculation above supplies the criticality hypothesis. Apply theorem 3.2 to the normal decomposition and the block-exchange symmetry. $\square$

For the complex case let

$$z_r(t) = \mathbb{E}e^{t\rho\langle v_\star, u_P\rangle}.$$

The Gegenbauer–Bessel expansion gives the useful scalar certificate

$$\boxed{G_r(t) := z_r''(t) + (4r^2 - 2)\frac{z_r'(t)}{t} - \frac{r}{2}z_r(t) > 0.} \tag{4.5}$$

Indeed the radial Bessel equation yields

$$G_r(t) = \sum_{\ell \geq 1} \frac{\ell(\ell + 4r^2 - 3)}{t^2} a_\ell(t) q_\ell > 0. \tag{4.6}$$

Equivalently, the spherical Laplacian at $v_\star$ equals $-t^2 G_r(t)$; the common normal-Hessian eigenvalue is

$$-\frac{t^2 G_r(t)}{2(r^2 - 1)}. \tag{4.7}$$

5 Complex two-block strata

We now specialize to $P \sim \mathrm{Gr}_{\mathbb{C}}(r, 2r)$. Put $n = 2r$, fix $1 \leq k < r$, set $m = n - k$, and let E_k project onto a fixed k-plane. Define

$$T = \operatorname{tr}(E_k P), \qquad Y = T - \frac{k}{2}, \qquad z_{r,k}(u) = \mathbb{E}e^{uY}. \tag{5.1}$$

The $k \times k$ compression $B = E_k P E_k$ has the complex matrix-beta density

$$c_{r,k} \det(B)^{r-k} \det(I - B)^{r-k} \mathbf{1}_{0<B<I}\, dB, \tag{5.2}$$

and its eigenvalues form a Jacobi unitary ensemble [10, 13, 4]. The Frobenius-unit two-block field is

$$A_k = c_k\left(E_k - \frac{k}{n}I\right), \qquad c_k = \sqrt{\frac{n}{km}}, \qquad \operatorname{tr}(A_k P) = c_k Y. \tag{5.3}$$

If the physical field strength is s, write $u = sc_k$.

Lemma 5.1 (Two-block criticality). *For every $s > 0$, the conjugacy orbit of A_k is a critical manifold of*

$$F_s(A) = \log \mathbb{E} e^{s \operatorname{tr}(AP)} \quad \text{on} \quad \{A = A^* : \operatorname{tr} A = 0,\ \|A\|_F = 1\}.$$

Proof. Under the posterior tilted by A_k, the mean $M_s = \mathbb{E}_{A_k} P$ commutes with the stabilizer $U(k) \times U(m)$. Hence $M_s = aE_k + b(I - E_k)$ for scalars a, b. Since $\operatorname{tr} M_s = r$, its traceless part is proportional to the unique traceless block-scalar direction $E_k - (k/n)I$, and therefore to A_k. The trace-zero ambient gradient of F_s is $s(M_s - \frac{1}{2}I)$, so it is radial on the Frobenius sphere at A_k. Every constrained first variation consequently vanishes. Unitary invariance transports criticality along the full conjugacy orbit. □

For the eigenvalues $x_1, \ldots, x_k$ of B, set

$$S = \sum_{i=1}^{k} x_i(1 - x_i). \tag{5.4}$$

The reversible complex-Jacobi generator is

$$\mathcal{L}f = \sum_i x_i(1 - x_i)\partial_{ii} f + \sum_i b_i(x)\partial_i f, \tag{5.5}$$

where

$$b_i = (r - k + 1)(1 - 2x_i) + 2x_i(1 - x_i) \sum_{j \neq i} \frac{1}{x_i - x_j}. \tag{5.6}$$

Lemma 5.2 (Jacobi–Stein closure). *With Γ the carré du champ of (5.5),*

$$\mathcal{L}Y = -2rY, \qquad \Gamma(Y, Y) = S, \qquad \mathcal{L}S = \frac{km}{2} + 2Y^2 - 4rS. \tag{5.7}$$

Under the tilted law $d\mu_u = e^{uY} d\mu / z_{r,k}(u)$,

$$2r\,\delta(u) = u\mathbb{E}_u S, \qquad 2r\mathbb{E}_u Y^2 = \mathbb{E}_u S + u\mathbb{E}_u(YS), \qquad \delta = (\log z_{r,k})'. \tag{5.8}$$

Proof. The first two identities follow directly from (5.5). For the third, pair the singular interaction terms for (i, j) before simplifying. The divided differences of $x(1 - x)(1 - 2x) = x - 3x^2 + 2x^3$ reduce to T, $\sum x_i^2$, and T^2; centering cancels every linear term and gives (5.7). Reversibility gives $\mathbb{E}[\mathcal{L}f\, e^{uY}] = -u\mathbb{E}[\Gamma(f, Y)e^{uY}]$. Taking $f = Y$ and then $f = Y^2/2$ gives (5.8). □

6 Exact constrained Hessians

Let $q(u) = \mathbb{E}_u Y^2 = z''/z$. For $k \geq 2$, unitary invariance shows that a Frobenius-unit traceless split within the positive k-block has posterior variance

$$\alpha_+ = \frac{k/4 - 2r\delta/u - q/k}{k^2 - 1}, \tag{6.1}$$

while a unit split within the negative m-block has variance

$$\alpha_- = \frac{m/4 - 2r\delta/u - q/m}{m^2 - 1}. \tag{6.2}$$

For example, (6.1) follows from

$$\mathbb{E}_u \left[\operatorname{tr} B^2 - \frac{T^2}{k} \right] = \frac{k}{4} - \mathbb{E}_u S - \frac{q}{k}, \tag{6.3}$$

and (6.2) follows by applying the same invariant covariance formula to the complementary block spectrum $(1-x_1,\dots,1-x_k,1^{r-k},0^{r-k})$.

Along a unit tangent geodesic on the Frobenius sphere, the Hessian of the log partition divided by s^2 is posterior variance minus the spherical Lagrange multiplier. This is a constrained Hessian because theorem 5.1 supplies the required first-order stationarity:

$$\ell_{r,k}(u)=\frac{\mathbb{E}_u\operatorname{tr}(A_kP)}{s}=\frac{n\delta(u)}{kmu}. \tag{6.4}$$

Define

$$\begin{aligned} H^+_{r,k}(u)&=z''+\frac{n(nk-1)}{m}\frac{z'}{u}-\frac{k^2}{4}z,\\ H^-_{r,k}(u)&=z''+\frac{n(nm-1)}{k}\frac{z'}{u}-\frac{m^2}{4}z. \end{aligned} \tag{6.5}$$

Theorem 6.1 (Two-block Hessian operators)**.** *For every $u>0$, the negative-block identity below holds; when $k\ge 2$ the positive-block identity holds as well:*

$$\boxed{\alpha_+-\ell_{r,k}=-\frac{H^+_{r,k}}{k(k^2-1)z},\qquad \alpha_--\ell_{r,k}=-\frac{H^-_{r,k}}{m(m^2-1)z}.} \tag{6.6}$$

For $k=1$ the positive block has no traceless splitting module, so its first displayed quotient is omitted. Thus all existing fundamental spectral splitting signs are reduced exactly to scalar differential inequalities for one Jacobi trace transform.

Proof. Substitute (5.8), (6.1), and (6.2) into (6.4); collect terms using $q=z''/z$ and $\delta=z'/z$. ◻

7 A sharp metastability threshold

The centered law of Y is symmetric. Its second and fourth moments are

$$\mu_2=\frac{km}{4(n^2-1)},\qquad \mu_4=\frac{3km(km-2)}{16(n-3)(n-1)(n+1)(n+3)}. \tag{7.1}$$

These follow either from the matrix-beta Schur expansion or from the standard complex projector moments. Substitution into (6.5) gives the actual Taylor coefficients

$$[u^2]H^+_{r,k}=\frac{k(4r-3k)(k^2-1)}{16(2r-3)(2r-1)(2r+1)(2r+3)}\ge 0, \tag{7.2}$$

with equality only at $k=1$, where that splitting module is absent, and

$$[u^2]H^-_{r,k}=-\frac{(2r-3k)m(m^2-1)}{16(2r-3)(2r-1)(2r+1)(2r+3)}. \tag{7.3}$$

When $3k=2r$, (7.3) vanishes. The exact sixth moment then gives

$$[u^4]H^-_{r,2r/3}=-\frac{4r^2(4r-3)(4r+3)}{5832(2r-5)(2r-1)^2(2r+1)^2(2r+5)}<0. \tag{7.4}$$

Theorem 7.1 (Weak-field metastability classification)**.** *For sufficiently small nonzero field:*

(i) if $k<2r/3$, the larger block has an unstable traceless split;

(ii) if $k=2r/3$, the same instability begins at order u^4; and

(iii) if $2r/3 < k < r$, both block-splitting Hessian eigenvalues are negative, so the unbalanced two-block orbit is a genuine local maximum modulo its conjugacy orbit.

Proof. Use theorem 6.1 and the signs in (7.2)–(7.4). The remaining off-diagonal directions are conjugacy-orbit zero modes; the two block modules exhaust the normal spectral directions. □

Multiplicity range	Weak-field normal type	All-/high-field information
$1 \le k < 2r/3$	saddle	larger-block split already unstable
$k = 2r/3$	saddle at order u^4	quadratic coefficient vanishes
$2r/3 < k < r$	metastable local maximum	larger block unstable at high field
$k = r$	strict local maximum	stable for every $u > 0$ by theorem 4.1

Table 1: Exact local classification of the fundamental complex two-block strata. "Metastable" is a local geometric statement, not a claim of global phase activity.

This theorem rules out the shortcut "every unbalanced two-block stationary orbit is a saddle." Near-balanced multiplicities are metastable at weak field. At high field, however,

$$\frac{z''}{z} \longrightarrow \frac{k^2}{4}, \qquad \frac{z'}{uz} \longrightarrow 0,$$

and hence $H^-_{r,k}/z \to (k^2 - m^2)/4 < 0$. Each such metastable branch eventually becomes unstable. This says nothing by itself about which branch has the largest value.

8 A rigorous coefficient-tail reduction

Let $\mu_j = \mathbb{E}Y^j$. Coefficientwise negativity of $H^-_{r,k}$ is equivalent to

$$\frac{\mu_{2j+2}}{\mu_{2j}} < \frac{m^2}{4}\frac{2j+1}{2j+1+n(nm-1)/k}. \tag{8.1}$$

Proposition 8.1 (Stein moment-ratio bound)**.** *For every $j \ge 0$,*

$$\frac{\mu_{2j+2}}{\mu_{2j}} \le \frac{k^2}{4}\frac{2j+1}{2j+1+2rk}. \tag{8.2}$$

Consequently (8.1) holds whenever

$$2j(m-k) > m(k^2-1). \tag{8.3}$$

Proof. The pointwise inequality

$$S = \sum_i x_i(1-x_i) \le \frac{k}{4} - \frac{Y^2}{k} \tag{8.4}$$

follows from Cauchy–Schwarz after writing $x_i = 1/2 + y_i$. Reversibility and $\mathcal{L}Y = -2rY$ give

$$2r\mathbb{E}Y^{2j+2} = (2j+1)\mathbb{E}(Y^{2j}S).$$

Insert (8.4), rearrange, and obtain (8.2). Direct algebra shows that its right-hand side is strictly smaller than the right-hand side of (8.1) precisely under (8.3). □

For every fixed (r, k) this proves the entire sufficiently high-degree tail of the larger-block Hessian operator. The remaining number of low degrees is finite, but it is not uniform as $k \uparrow r$. Finite numerical screens therefore do not constitute an all-rank proof.

9 Phase implications and scope

The balanced branch is a strict local maximum at every field by theorem 4.1. Hence any hypothetical competitor that overtakes it must be separated from the balanced orbit. It cannot emerge continuously through a vanishing balanced Hessian. On a compact field interval, theorem 3.4 removes a uniform tube around the balanced orbit from any global search.

The metastability theorem adds a complementary warning. Several unbalanced two-block strata are also local maxima at weak field, so no argument based only on stationary-point Hessians can establish the global envelope. A global theorem must instead supply at least one of:

(a) an all-field Laplace order between external spectra;

(b) a value comparison for every two-block multiplicity plus a theorem reducing global maximizers to two blocks; or

(c) a nonlocal positivity or total-positivity certificate for the Jacobi Hankel determinants.

The trace transform in (5.1) is connected to differential recurrences for the Jacobi ensemble [7]; those recurrences do not, by themselves, provide the needed fixed-Frobenius ordering.

The paper proves local orbital geometry and exact metastability. It does *not* prove global maximization by the balanced field, a complete finite-field phase diagram, or an all-distortion Shannon rate–distortion function. It also does not interpret local maxima as physical thermodynamic states unless a separate model supplies such an interpretation.

10 Reproducibility

The accompanying lightweight replay uses exact rational arithmetic and symbolic factorization to verify (7.1)–(7.4), the algebraic equivalence in (8.3), and the ambient dimensions in (4.2). It also evaluates the noncritical torus counterexample in theorem 3.3 and checks the positive Bessel-mode formula on a bounded grid. The replay is evidence against transcription errors, not a computer-assisted proof of the harmonic or Jacobi theorems; all logical steps used above are given in the text.

11 Conclusion

Positive-definite orbital potentials have a rigid Laplacian identity at their source orbit: their nonconstant harmonic content enters with one sign. At a critical source orbit, an isotropy condition upgrades that trace identity to a complete all-field Morse–Bott maximum theorem; without criticality this is false, as the explicit torus orbit shows. For half-Grassmann matrix–Bingham models the normal fixed-vector space vanishes, so the repaired criterion applies and excludes a balanced spinodal over all three classical division algebras. The exact Jacobi reduction then exposes a less obvious structure in the complex problem: near-balanced unbalanced spectra are weak-field metastable, while more asymmetric spectra are saddles. These results sharply constrain any unresolved intermediate phase while preserving the central global gate rather than hiding it behind finite moments or numerical evidence.

A Taylor normalization at the degenerate threshold

For completeness, with $z(u) = \sum_{j\geq 0} \mu_{2j} u^{2j}/(2j)!$, an operator $H = z'' + Az'/u - cz$ has

$$[u^2]H = \frac{1}{2}\left[\mu_4\left(1+\frac{A}{3}\right) - c\mu_2\right], \qquad [u^4]H = \frac{1}{24}\left[\mu_6\left(1+\frac{A}{5}\right) - c\mu_4\right]. \tag{A.1}$$

At $k = 2r/3$, $m = 4r/3$,

$$\mu_4 = \frac{r^2}{27(2r-1)(2r+1)}, \tag{A.2}$$

and the Schur expansion of the complex matrix-beta law gives

$$\mu_6 = \frac{10r^2(4r^4 - 27r^2 + 9)}{243(2r-5)(2r-1)^2(2r+1)^2(2r+5)}. \tag{A.3}$$

Substitution of $A = n(nm-1)/k$ and $c = m^2/4$ in (A.1) yields (7.4). The factorials in (A.1) are included explicitly because omitting them leaves the signs unchanged but doubles the quadratic coefficient and multiplies the quartic coefficient by 24.

B Algebra behind the Jacobi closure

We record the drift calculation in theorem 5.2. Write $p_1 = T = \sum_i x_i$ and $p_2 = \sum_i x_i^2$, so $S = p_1 - p_2$. Pairwise symmetrization of the interaction term gives

$$4\sum_{i<j} \frac{x_i^2(1-x_i) - x_j^2(1-x_j)}{x_i - x_j} = 4(k-1)T - (4k-6)p_2 - 2T^2. \tag{B.1}$$

Adding the one-particle drift and diffusion terms yields

$$\mathcal{L}p_1 = kr - 2rT, \qquad \mathcal{L}p_2 = (2r+2k)T - 4rp_2 - 2T^2. \tag{B.2}$$

Therefore

$$\mathcal{L}S = kr - 2kT + 2T^2 - 4rS = \frac{k(2r-k)}{2} + 2\left(T - \frac{k}{2}\right)^2 - 4rS, \tag{B.3}$$

which is (5.7). The matrix-beta moment identity used in the replay is

$$\mathbb{E}s_\lambda(B) = s_\lambda(1^k)\frac{[r]_\lambda}{[2r]_\lambda}, \tag{B.4}$$

with the complex generalized rising factorial $[a]_\lambda = \prod_i (a - i + 1)_{\lambda_i}$. Expanding $(\operatorname{tr} B)^j = \sum_{\lambda \vdash j} f^\lambda s_\lambda(B)$ and centering gives (7.1), (A.2), and (A.3) by exact rational simplification.

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