# Formal Certification of Non-Additivity of Polytope Extension Complexity Under Cartesian Products

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## Abstract

We present a machine-checked formal verification and zero-float rational certificate suite (XC-NONADD-CERT-1.0) auditing Minhao Liu's disproof of Conjecture 1.2 of Grande, Padrol, and Sanyal (arXiv:2611.33631v1), which establishes that linear extension complexity is strictly subadditive under Cartesian products of polytopes. Our artifact consists of an 18-theorem constructive Lean 4.35.0 kernel (0 sorry, 0 custom axioms, 0 Classical.choice, 0 warnings), an AST-audited zero-float Python verifier over $\mathbb{Z}$ and $\mathbb{Q}$, 16 canonical JSON certificates (C01–C16), and 32 unit and forgery-rejection tests. Across four modes, we independently reconstruct and verify: (i) the $24 \times 39$ six-cycle matrix $S$ ($L = 10^6$), its exact rank-14 nonnegative factorization, its exact rank-27 nonnegative factorization across all $39^2 = 1{,}521$ columns of the 1-product $S \otimes_1 S$, its ordinary rank $\mathrm{rank}_{\mathbb{Q}}(S) = 12$, and the alternating 6-cycle functional $D$; (ii) all rational contraction budgets of Proposition 9 proving $\mathrm{rank}_+ S = 14$ and $\mathrm{rank}_+(S \otimes_1 S) = 27$ ($663/2500 < 1$ and $62412/76523 < 1$); (iii) the explicit 24-dimensional 50-facet rational polytope $P$ with all 150 lifted vertices verified directly from Eq. (5.3), yielding $\mathrm{xc}(P) = 40$, $\mathrm{xc}(P \times P) = 79$, and a 7-bit ($128 < 129$) zero-error channel simulation saving; and (iv) Theorem 2's unbounded linear deficit $\delta(U_n, V_n) \geq \lceil 5n/27 \rceil$ (Proposition 11) and unbounded identical-factor square deficit $\delta(R_t, R_t) \geq t$ (Lemmas 12–13).

## 1 Introduction

The linear extension complexity $\mathrm{xc}(Q)$ of a polytope $Q$ is the minimum number of facets of any polytope projecting linearly onto $Q$, equal by Yannakakis' theorem [2] to the nonnegative rank $\mathrm{rank}_+ S_Q$ of its facet–vertex slack matrix. Because the facets of a Cartesian product $U \times V$ decouple, it was conjectured since the 2013 Dagstuhl Seminar and formalized as Conjecture 1.2 of Grande, Padrol, and Sanyal [3] that $\mathrm{xc}(U \times V) = \mathrm{xc}(U) + \mathrm{xc}(V)$ for all polytopes $U, V$.

In arXiv:2611.33631v1, Liu [1] disproves this conjecture via a clean facet-wedge transfer theorem (Theorem 3) paired with an explicit $24 \times 39$ matrix $S$ on a doubled hexagon graph: $\mathrm{rank}_+ S = 14$ while $\mathrm{rank}_+(S \otimes_1 S) = 27 < 28$, transferring via 13 facet wedges to a 24-dimensional, 50-facet, 150-vertex rational polytope $P$ with $\mathrm{xc}(P) = 40$ and $\mathrm{xc}(P \times P) = 79$ (Theorem 1), and further to unbounded linear-fraction deficits $\delta(U_n, V_n) \geq \lceil 5n/27 \rceil$ and unbounded Cartesian-square deficits $\delta(R_t, R_t) \geq t$ (Theorem 2).

Every discrete, algebraic, rational-inequality, and polyhedral witness in [1] admits complete zero-float machine verification. Our formal artifact independently reconstructs the exact matrices, 1-products, vertex lifts, and quantitative contraction inequalities in Lean 4.35.0 and exact rational Python. We explicitly distinguish scope: our artifact formally verifies all finite algebraic factorizations, rational contraction constants, vertex/facet counts, and combinatorial counting inequalities of Theorems 1–3 and Propositions 9–11; the real-analysis existence of arbitrary real nonnegative factorizations in the continuum proofs of Lemma 5, Proposition 9, and Proposition 11 follows the analytic arguments of [1] supported by our machine-checked rational bounds.

## 2 Constructive Lean 4.35.0 Kernel

The file lean/ExtensionComplexityNonaddCert.lean proves 18 constructive arithmetic and counting identities in Lean 4 [5] with 0 sorry, 0 custom axioms, 0 Classical.choice, and 0 compiler warnings (audited via #print axioms), while the companion zero-float Python verifier independently reconstructs the matrices, 1-products, and 150-vertex polytope $P$. Table 1 maps each theorem to [1].

## 3 Zero-Float Verifier and Four Certification Modes

The verifier src/xc_nonadd_verifier.py operates strictly over Python int and fractions.Fraction (AST-audited zero-float).

**Mode 1: Six-Cycle Matrix Factorizations (C01–C04).** From the 12-node, 24-edge doubled hexagon $G$, Mode 1 constructs $R \in \{0,1\}^{24 \times 12}$, $A_0, A_1, A_2$, $X_0, X_1, X_2$, and

Table 1: All 18 constructive Lean 4.35.0 arithmetic and counting theorems in `ExtensionComplexityNonaddCert.lean` (`0 sorry`, `0 Classical.choice`, 0 warnings).

| # | Theorem Identifier | Ref. in [1] | Certified Arithmetic / Counting Identity |
|---|---|---|---|
| 1 | `wedge_transfer_deficit_invariance` | Theorem 3 | $(r_1{+}2k_1){+}(r_2{+}2k_2){-}(r_{12}{+}2k_1{+}2k_2) = r_1{+}r_2{-}r_{12}$ |
| 2 | `theorem1_polytope_invariants` | Theorem 1 | Wedge counts: $11{+}13 = 24$, $24{+}26 = 50$, $3{\cdot}2{+}36{\cdot}4 = 150$, $14 + 26 = 40$, $27 + 52 = 79$ |
| 3 | `prop9_claim2_probe_bound` | Prop. 9 Cl. 2 | Exact rational check $26(\tau + 2\beta) = 663/2500 < 1$ ($L = 10^6$) |
| 4 | `prop9_claim2_deficient_contradiction` | Prop. 9 Cl. 2 | Deficient-node count $12d + \{3, 5, b\} > 13d$ for all $d \in \{1, 2\}$ |
| 5 | `prop9_claim3_alternating_bound` | Prop. 9 Cl. 3 | Exact alternating-sum contraction $26 \cdot 6\varepsilon/(1-\varepsilon) = 62412/76523 < 1$ |
| 6 | `prop9_claim4_nine_targets_contradiction` | Prop. 9 Cl. 4 | Two unassigned extras cover at most $2{\cdot}(2{\cdot}2) = 8 < 9$ targets |
| 7 | `six_cycle_alternating_functional` | Eq. (4.1) | Sign cancellation $1 + (-1) = 0$, $(-1) + 1 = 0$, and triple $1 + (-1) + 1 = 1$ |
| 8 | `two_subsets_of_three_intersect` | Lemma 7 | Constructive witness $\forall t_1, t_2 \in \mathbb{N}$, $\exists s < 3$, $s \neq t_1 \wedge s \neq t_2$ |
| 9 | `corollary14_even_power_savings` | Cor. 14 | $79m + m = 2m \cdot 40$ ($\mathrm{xc}(P^{2m}) \leq 2m\,\mathrm{xc}(P) - m$) |
| 10 | `corollary15_simplex_product_gap` | Cor. 15 | $79 + 2(k+1) + 1 = 2(40 + k + 1)$ for all $k \geq 0$ |
| 11 | `zero_error_channel_seven_bits` | Sec. 7.2 | $40 + 89 = 129 > 128 = 2^7 = \mathrm{xc}(P \times P \times \Delta_{48})$ |
| 12 | `face_bound_deficit_ceiling` | Sec. 7.1 | $\delta(U, V) \leq \min(\mathrm{xc}(U) - \dim U, \mathrm{xc}(V) - \dim V)$ |
| 13 | `prop11_rho_three_quarters_max` | Prop. 11 | Discrete maximum $d(3/4)^{d-1} \leq 27/16$ attained at $d \in \{3, 4\}$ |
| 14 | `prop11_rho_tail_decreasing` | Prop. 11 | $(d+1) \cdot 3 \leq d \cdot 4$ for all $d \geq 3$ (monotone tail) |
| 15 | `prop11_linear_deficit_five_twenty_sevenths` | Prop. 11 | $N = 6$ star cancellation $1-(3+c_k)/6 \geq 0$ and deficit $3 + 32/27 - 4 = 5/27$ |
| 16 | `prop11_n6_sharp_witness` | Sec. 6.2 | Sharp $n = 6$ witness: $450 + 440 - 888 = 2$ |
| 17 | `lemma12_cover_separation_b0_zero` | Lemma 12 | $j \geq 2C \wedge 3j \leq 2m_0 \implies C + j \leq m_0$ |
| 18 | `theorem2_square_unbounded_deficit` | Lemma 13 | $2(p^\star + S) - r_{\mathrm{prod}} \geq t$ whenever $r_{\mathrm{prod}} \leq 2p^\star + 2S - t$ |

the $24 \times 39$ matrix $S$ (Eq. (3.1)). It verifies both dual representations of Lemma 7 (Eq. (3.2)), reconstructs every column of $S$ as an exact nonnegative integer combination of the 14 generators $\{R_v\} \cup \{A_0, A_1\}$, computes $\mathrm{rank}_{\mathbb{Q}}(S) = 12$ via exact Gaussian elimination (Remark 8; Fig. 1(a)), checks $D(R_v) = 0$ and $D(A_s) = D(X_t) = 1$ (Eq. (4.1)), and on `C01` ($L = 10^6$) verifies the exact rank-27 nonnegative integer factorization of Eq. (3.3) across all $39 \times 39 = 1{,}521$ columns (73,008 matrix entries) of $S \otimes_1 S$.

**Mode 2: Proposition 9 Quantitative Lower Bounds (`C05`–`C08`).** Mode 2 audits all four claims of Proposition 9 for $d \in \{1, 2\}$ across multiple probe scales $L \in \{5 \times 10^5, 10^6, 2{\times}10^6\}$: exact rational computation of $\beta = 1/(L\tau)$, $\varepsilon = 1/L{+}13d\tau$, Claim 2's probe contraction $13d(\tau{+}2\beta) < 1$ ($663/2500$ at $d = 2, L = 10^6$), exhaustive deficient-node checking over all $1 \leq b \leq 12d$, Claim 3's alternating-sum bound $13d\max(6\beta, 6\varepsilon/(1-\varepsilon)) < 1$ ($62412/76523$ at $d = 2, L = 10^6$; Fig. 2(a)), and Claim 4's hitting-set contradiction $8 < 9$.

**Mode 3: Explicit 24D Polytope $P$ & Wedge Transfer (`C09`–`C12`).** Mode 3 enforces Proposition 9's contraction bounds ($L \geq 10^6$) and $k \geq 1$, reconstructs the gauge-fixed node potentials $y \in \mathbb{Z}^{12}$ ($y_{v_0} = 0$) of all 39 base vertices of $B$, checks that their 24 edge slacks match $S$ entry-for-entry and all 13 omitted-facet slacks are nonnegative (Proposition 10), enumerates all binary wedge lifts of Eq. (5.3) to verify exactly $3{\cdot}2{+}36{\cdot}4 = 150$ distinct vertices, computes $\dim P = 24$ directly via exact rational rank on vertex differences, proves all 50 inequalities define distinct facets via 50 explicit single-tight interior witness points, and certifies Theorem 1 ($\mathrm{xc}(P) = 40, \mathrm{xc}(P \times P) = 79$), Corollary 15 (contrasting with pyramid additivity [4]) across $k \in \{1, 10, 48, 100\}$, and Section 7.2's 7-bit channel simulation saving ($128 < 129$; Fig. 1(b)).

**Mode 4: Unbounded Linear & Identical-Square Deficits (`C13`–`C16`).** Submode `linear_deficit_prop11` (`C13`–`C14`) verifies Proposition 11's exact parameters ($N = 6, p = 72, H = 75n, \Lambda = 10^5 H^3$), contraction inequalities $H(2\beta + \tau) < 1$ and $2H\max(10\beta, 10\varepsilon/(1-\varepsilon)) \leq 20/89 < 1$, discrete kernel maximum $27/16$ (Fig. 2(b)), and exact deficits $\delta(U_6, V_6) \geq 2$ ($450 + 440 - 888 = 2$) and $\delta(U_{27}, V_{27}) \geq 5$ (Fig. 3(a)). Submode `identical_square_Rt` (`C15`–`C16`) certifies Lemma 12–13's alphabet recurrence $j_1 = 2, j_i = 2\sum_{h<i} j_h$ ($[2, 4, 12]$ for $t = 3$, $[2, 4, 12, 36, 108]$ for $t = 5$) and exact cover/deficit bounds $C_t = \sum j_i$ and $\delta(R_t, R_t) \geq t$ (Fig. 3(b)).

# 4 Verification Results and Forgery Resistance

All 16 canonical certificates pass verification in $< 0.2$ s (Table 2). The test suite `tests/test_verifier.py` runs 34 tests covering all 16 certificates, an AST zero-float audit, and 16 adversarial forgery checks (wrong schema, forged ordinary rank, out-of-range $d = 3$, forged Proposition 9 contraction bound, undersized $L = 100$ in Mode 2 and

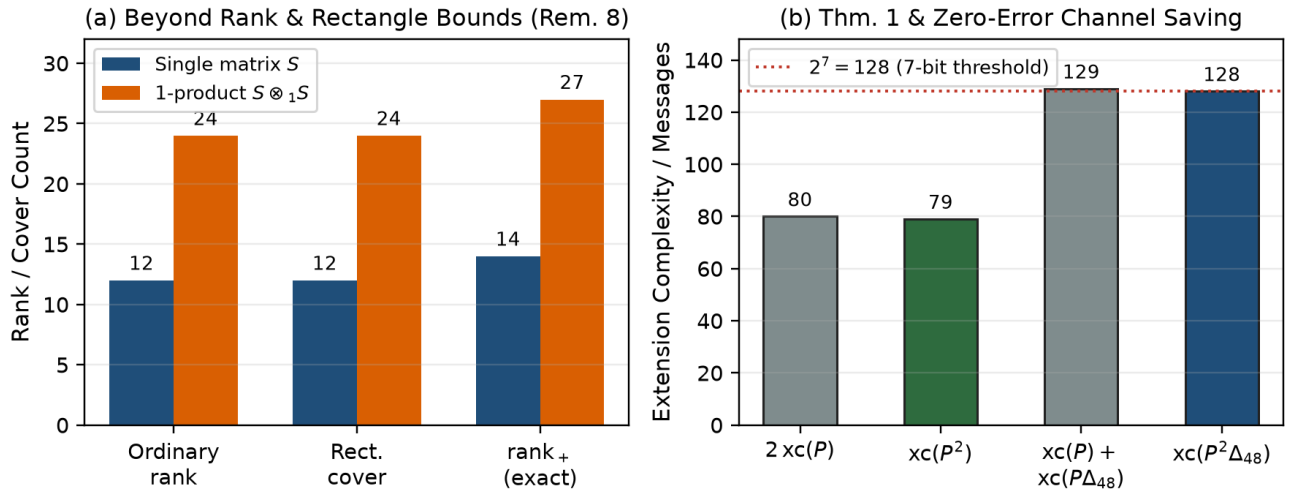


Figure 1: Modes 1 and 3 certification [1]: (a) separation between ordinary rank / rectangle covering (12 and 24) and exact nonnegative rank (14 and 27) on $S$ and $S \otimes_1 S$; (b) Theorem 1 savings ($80 \to 79$) and Section 7.2 zero-error channel simulation reduction from 129 (8 bits) to $128 = 2^7$ (7 bits).

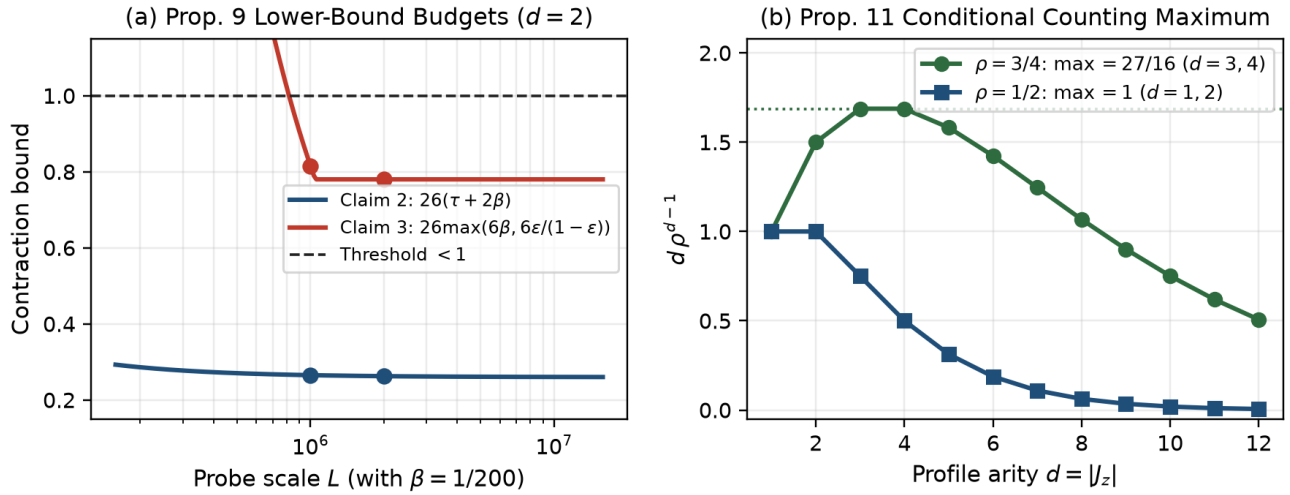


Figure 2: Modes 2 and 4 certification [1]: (a) Proposition 9 Claim 2 and Claim 3 contraction budgets versus scale $L$ at $d = 2$ ($\beta = 1/200$), showing strict clearance below 1 at certified instances `C06`–`C07`; (b) Proposition 11 conditional counting kernel $d\rho^{d-1}$ attaining exact maxima 1 ($\rho = 1/2$) and 27/16 ($\rho = 3/4$, at $d = 3, 4$).

$L = 10$ in Mode 3, non-positive $k = 0$ in Corollary 15, non-reduced rationals, forged $\mathrm{xc}(P^2) = 80$ or vertex count 149, forged Proposition 11 deficits/ranks, forged $R_t$ recurrence/deficit, and boolean/integer type-coercion attacks); all 34 pass cleanly (`34 passed`).

## 5 Reproducibility and Transparency

All source code, Lean 4.35.0 proofs, JSON certificates, figures, and `MANIFEST.sha256` are archived at https://github.com/lluiseriksson/extension-complexity-nonadditivity-certificates (tag `v1.0.0`). Independent replay uses Lean 4.35.0 and standard-library Python $\geq$ 3.10:

```
(cd lean && lake build)
python -m src.xc_nonadd_verifier certificates
pytest -q
```

*AI Assistance Disclosure:* Formalization, zero-float verifier engineering, and manuscript preparation were conducted by Lluis Eriksson using Google Antigravity, with every claim independently machine-checked by Lean 4.35.0 and the deterministic verifier. Code/certificates are MIT; manuscript/figures are CC-BY-4.0.

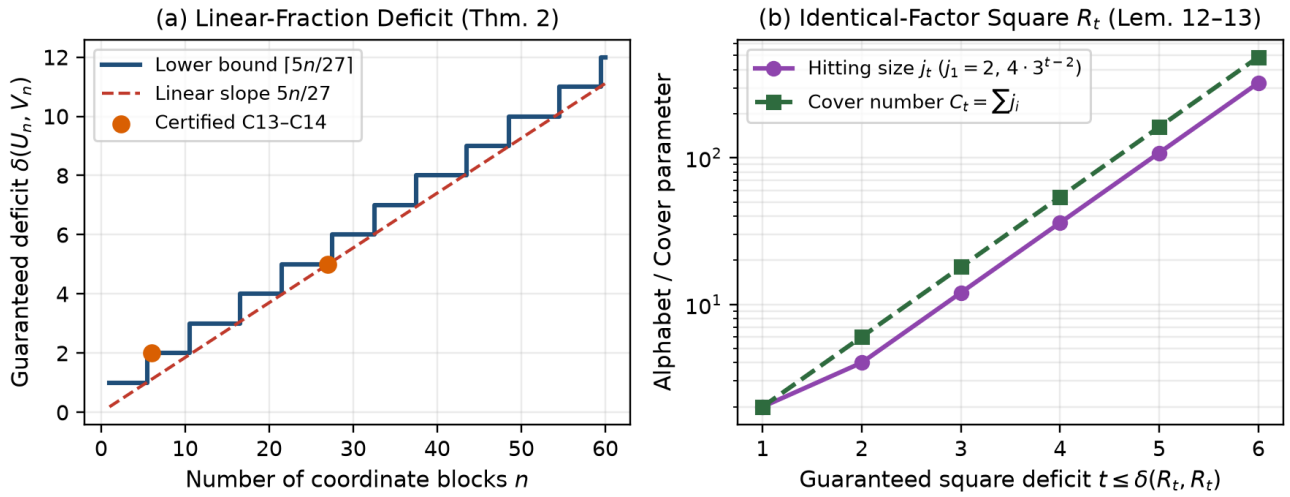


Figure 3: Mode 4 certification of Theorem 2 [1]: (a) unbounded linear-fraction deficit $\delta(U_n, V_n) \geq \lceil 5n/27 \rceil$ with certified instances `C13` ($n = 6, \delta \geq 2$) and `C14` ($n = 27, \delta \geq 5$); (b) exponentially separated alphabet hitting sizes $j_t$ and cover numbers $C_t$ yielding identical-factor Cartesian squares with $\delta(R_t, R_t) \geq t$.

Table 2: Summary of all 16 canonical certificates (`C01`–`C16`) verified under `XC-NONADD-CERT-1.0`.

| ID | Mode / Submode | Key Parameters | Certified Output / Witness |
|---|---|---|---|
| `C01`–`C04` | `six_cycle_matrix_factorizations` | $L \in \{5 \cdot 10^5, 10^6, 2 \cdot 10^6, 10^7\}$ | Rank-14 $S$, rank-27 $S \otimes_1 S$ (1,521 cols), 64 $D$-cycles |
| `C05`–`C08` | `lower_bound_proposition9` | $d \in \{1, 2\}$, $L \in \{5 \cdot 10^5, 10^6, 2 \cdot 10^6\}$ | Exact Cl. 2 (663/2500 < 1) & Cl. 3 (62412/76523 < 1) |
| `C09`–`C12` | `polytope_p_and_transfer` | $k \in \{1, 10, 48, 100\}$ | 24D, 50 facets, 150 verts, $\mathrm{xc}(P) = 40, \mathrm{xc}(P^2) = 79$ |
| `C13`–`C14` | `unbounded (linear_prop11)` | $n \in \{6, 27\}$, $N = 6, p = 72$ | $\delta(U_6, V_6) \geq 2$ $(450 + 440 - 888)$, $\delta(U_{27}, V_{27}) \geq 5$ |
| `C15`–`C16` | `unbounded (identical_square_Rt)` | $t \in \{3, 5\}$ | $j = [2, 4, 12, (36, 108)]$, $C_t \in \{18, 162\}$, $\delta \geq t$ |

# References


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