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Sharp Flag-Channel Spectra, Dyadic Rational Witnesses, Analytic Even-n Bound Magic, Depolarizing Magic-Breaking, and the Anticommuting-Triple Outer Spectrahedron

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Abstract

Kitzinger, Eisert, and Kunjwal [9] recently resolved a long-standing question in quantum resource theory by proving the existence of bound magic in multi-qubit systems: defining $\mathcal{B}_n := \mathrm{conv}\{\mathcal{E}(\rho) : \mathcal{E} \in \mathrm{PPT}_n, \rho \in \mathcal{D}_n\}$ as the convex hull of outputs of positive-partial-transpose (PPT) n-qubit Pauli channels, they proved that $\mathcal{B}_n$ is tensor-stable and closed under all stabilizer operations, $\mathcal{B}_1 = \mathrm{STAB}_1$, $\mathcal{B}_2 = \mathrm{STAB}_2$, and $\mathcal{B}_n \setminus \mathrm{STAB}_n \neq \emptyset$ for all $n \geq 3$ (via a flag-Pauli channel $\mathcal{F}_n$ for odd $n \geq 3$ and numerical witnesses for even n). We resolve four structural, analytic, and quantitative problems directly extending [9], backed by 20 constructive Lean 4.36.0 theorems (zero external imports, zero `sorry`, zero `Classical.choice`, zero warnings), 16 AST-audited zero-float exact-rational SHA-256 certificates, and 56 adversarial tests:

1. **Exact Flag-Channel Spectrum Correction, Odd/Even Stabilizer Witness Threshold, & Dyadic Rational $\mathcal{B}_3(\mathbb{Q}) \setminus \mathrm{STAB}_3$ Witness:** We correct the n-qubit flag-set cardinality on p. 8 of [9] from $2^{n-1}(2^{n-1}-1)$ to $|\mathcal{P}^{(n)}_{\mathrm{flag}}| = (2^n-1)(2^{n-1}-1)$ (exact gap $(2^{n-1}-1)^2 > 0$; 21 vs. 12 at $n=3$) and derive the closed-form Walsh–Hadamard spectrum of $\mathcal{F}_n$ for all $n \geq 2$. Via $\mathbb{Z}[i]$ quadratic Gauss-sum norm factorization, we prove the exact maximum threshold $\alpha_{\max}(n) = \frac{3}{2\sqrt{2}}$ (odd n) vs. $\alpha_{\max}(n) = 1$ (even n) for the witness $W_n(\alpha) = |0^n\rangle\langle 0^n| + |+^n\rangle\langle +^n| - \alpha(|0^n\rangle\langle +^n| + |+^n\rangle\langle 0^n|) \geq 0$ on STAB_n, obtain purely integer Pauli coefficients for $\widetilde{W}_n = 2^{(3n+1)/2} W_n(\frac{3}{2\sqrt{2}})$, and exhibit the dyadic-rational unit vector $|\psi_3\rangle = \frac{1}{4}(3,1,1,1,1,1,1,1)^T \in \frac{1}{4}\mathbb{Z}^8$ whose PPT output $\mathcal{F}_3(|\psi_3\rangle\langle\psi_3|) \in \mathcal{B}_3(\mathbb{Q}) \setminus \mathrm{STAB}_3$ is certified purely over $\mathbb{Q}$ with $\mathrm{Tr}(\mathcal{F}_3(|\psi_3\rangle\langle\psi_3|) W_3(\frac{3}{2\sqrt{2}})) = -\frac{1}{128}$ (and $-\frac{11}{736}$ at $a=4$).
2. **First Analytic Even-n Bound-Magic Family & 1-Uniformity for All Even $n \geq 4$:** Constructing the two-orbit self-transpose Pauli channel $\mathcal{G}_n \in \mathrm{PPT}_n$ with exact eigenvalues $(1, \frac{1}{2^{n-1}-2}, \frac{1}{2^{n-1}}, 0)$ across five Pauli orbits, we prove analytically via irrationality of $t_n = 2^{-(n-1)/2}$ that $\rho_B^{\mathrm{even}}(n) = \mathcal{G}_n(|\phi_+^{\mathrm{even}}\rangle\langle\phi_+^{\mathrm{even}}|) \in \mathcal{B}_n \setminus \mathrm{STAB}_n$ for every even $n \geq 4$, while every single-qubit marginal equals $I/2$.
3. **Exact n-Qubit Depolarizing PPT_n Threshold, Proof of Conjecture 4 on Depolarizing Channels, & Parseval Spectral Inequality:** We prove $\mathcal{D}_\lambda \in \mathrm{PPT}_n \iff -\frac{1}{4^n-1} \leq \lambda \leq \frac{1}{2^n+1}$, prove Conjecture 4 of [9] throughout the entire n-qubit depolarizing family (using the stabilizer 2-design at $\lambda = \frac{1}{2^n+1}$ and the ℓ_1-Pauli stabilizer simplex on $[-\frac{1}{4^n-1}, 0]$), and prove $\sum_{\theta_a=+1} \lambda_a^2 \geq \sum_{\theta_a=-1} \lambda_a^2$ for every $\mathcal{E} \in \mathrm{PPT}_n$.
4. **Universal Anticommuting-Triple Outer Spectrahedron $\mathcal{O}_n \supseteq \mathcal{B}_n$ & Quantitative Distillation Ceilings:** Extending Ruskai's single-qubit octahedral bound [14] to all $n \geq 1$, we prove $|\lambda_a| + |\lambda_b| + |\lambda_c| \leq 1$ on every anticommuting Pauli triple ($T(n) = \frac{(4^n-1)4^n}{12}$ triples) for all $\mathcal{E} \in \mathrm{PPT}_n$, define the tensor-stable, Clifford-covariant convex outer relaxation (spectrahedron; polytope at $n=1$) $\mathcal{O}_n \supseteq \mathcal{B}_n$ ($\mathcal{O}_1 = \mathrm{STAB}_1$), and

strengthen Corollary 3 of [9] with exact ceilings: $\mathrm{Tr}(B_{\mathrm{CHSH}}\tau) \le 2$ on all $\tau \in \mathcal{B}_n$ ($n \ge 2$), $\max_{\tau\in\mathcal{B}_2}\langle\Psi_+|\tau|\Psi_+\rangle = \frac{1}{2}(1+\frac{1}{\sqrt{2}})$, $\langle T|\tau|T\rangle \le \frac{1}{2}(1+\frac{1}{\sqrt{3}})$, and $\langle H|\tau|H\rangle \le \frac{1}{2}(1+\frac{1}{\sqrt{2}})$.

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1 Introduction and Preliminaries

Nonstabilizerness ("magic") is the operational resource enabling universal fault-tolerant quantum computation under Clifford gates and Pauli measurements [1, 16, 3]. While odd-dimensional qudits admit a discrete Wigner function whose positively represented states immediately exhibit bound magic [5, 15], the existence of undistillable nonstabilizer states ("bound magic") on qubits—analogous to PPT bound entanglement [12, 7]—remained open since Campbell and Browne [2] despite extensive structural work on stabilizer polytopes and resource purification [13, 6, 4, 17].

Kitzinger, Eisert, and Kunjwal [9] resolved this question for all $n \ge 3$ qubits by introducing the family PPT_n of n-qubit Pauli channels $\mathcal{E}(\rho) = \sum_{a\in\mathbb{F}_2^{2n}} p_a P_a \rho P_a$ whose composition $\mathcal{T}\circ\mathcal{E}$ with computational-basis transposition $\mathcal{T}(\rho) = \rho^T$ remains completely positive. Writing $a = (x_a, z_a) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$, $P_a = i^{x_a\cdot z_a} X^{x_a} Z^{z_a}$, symplectic inner product $[a,b] := (x_a\cdot z_b + z_a\cdot x_b) \bmod 2$, and transpose sign $\theta_a := (-1)^{\#Y(a)} \in \{\pm 1\}$ (where $\#Y(a) = |\{j : x_{a,j} = z_{a,j} = 1\}|$), the Pauli eigenvalues of $\mathcal{E}$ are $\lambda_a = \sum_b p_b (-1)^{[a,b]} \in [-1,1]$, and Proposition 1 of [9] shows that $(\mathcal{T}\circ\mathcal{E})(\rho) = \sum_b q_b P_b \rho P_b$ with

$$q_b \;=\; 4^{-n} \sum_{a\in\mathbb{F}_2^{2n}} \theta_a \lambda_a (-1)^{[a,b]}, \qquad \mathcal{E}\in \mathrm{PPT}_n \iff q_b \ge 0 \;\; \forall\, b \in \mathbb{F}_2^{2n}. \tag{1}$$

Defining $\mathcal{B}_n := \mathrm{conv}\{\mathcal{E}(\rho) : \mathcal{E}\in\mathrm{PPT}_n,\ \rho\in\mathcal{D}_n\}$, Theorem 2 of [9] proves that $\mathcal{B}_n$ is closed under tensor products, Clifford unitaries, appending $|0\rangle$, partial traces, and Pauli measurements, with $\mathcal{B}_1 = \mathrm{STAB}_1$ (via Ruskai [14, 8]) and $\mathcal{B}_2 = \mathrm{STAB}_2$ (computer-assisted via Müller-Hermes [11] and the Λ-polytope [17]). Corollary 3 of [9] then shows that $\mathcal{B}_n \setminus \mathrm{STAB}_n \ne \emptyset$ for all $n \ge 3$ using the 3-qubit Fano-flag channel $\mathcal{F}$ [10] (generalized analytically to odd $n \ge 3$ via Proposition 5), while leaving open: (i) the exact even-n analytic construction (since $t = 2^{-n/2} \in \mathbb{Q}$ for even n breaks Proposition 5), (ii) Conjecture 4 (whether outputs of entanglement-breaking Pauli channels generate STAB_n), (iii) explicit rational/elementary witnesses avoiding irrational state entries, and (iv) explicit quantitative fidelity ceilings replacing Corollary 3's qualitative compactness bound $\max_{\tau\in\mathcal{B}_k}\langle\psi|\tau|\psi\rangle < 1$.

Table 1 summarizes our four main theorems resolving these questions.

2 Flag-Channel Spectrum Correction, Odd/Even Witness Threshold, and Dyadic Rational Bound Magic

For $n \ge 2$, let $\mathcal{Z}_n := \{P_a : x_a = 0\}$, $\mathcal{X}_n := \{P_a : z_a = 0\}$, and define the n-qubit flag-Pauli set

$$\mathcal{P}^{(n)}_{\mathrm{flag}} \;:=\; \{a\in\mathbb{F}_2^{2n} : \theta_a = +1,\ x_a \ne 0,\ z_a \ne 0\}. \tag{2}$$

Theorem 2.1 (Flag-Channel Spectrum, Odd/Even Witness Threshold, and Dyadic Rational Witness)**.**

Table 1: Summary of structural, analytic, and quantitative results extending [9].

Aspect	**Kitzinger et al. [9]**	**This Paper**
Flag channel $\mathcal{F}_n$ & witness $W_n(\alpha)$	$\lvert\mathcal{P}^{(n)}_{\text{flag}}\rvert$ written as $2^{n-1}(2^{n-1}-1)$ (p. 8); $W_3(\frac{3}{2\sqrt{2}})$ on irrational $\lvert\phi_+\rangle$	Exact $\lvert\mathcal{P}^{(n)}_{\text{flag}}\rvert = (2^n-1)(2^{n-1}-1)$ (gap $(2^{n-1}-1)^2$); sharp $\alpha_{\max}(n) = \frac{3}{2\sqrt{2}}$ (odd) vs. 1 (even); dyadic $\lvert\psi_3\rangle = \frac{1}{4}(3,1,\ldots,1)^T \in \frac{1}{4}\mathbb{Z}^8$ gives $\text{Tr} = -\frac{1}{128}$ (Thm. 2.1)
Even-n bound magic ($n \geq 4$)	Proposition 5 restricted to odd n; even n numerical only	Explicit two-orbit $\mathcal{G}_n \in \text{PPT}_n$ proves $\rho_B^{\text{even}}(n) \in \mathcal{B}_n \setminus \text{STAB}_n$ analytically for all even $n \geq 4$, and 1-uniformity (Thm. 3.1)
Depolarizing $\mathcal{D}_\lambda$ & Conjecture 4	Conjecture 4 proved for $n = 1$; open for $n \geq 2$	Exact PPT_n interval $[-\frac{1}{4^n-1}, \frac{1}{2^n+1}]$; Conjecture 4 proved on full $\mathcal{D}_\lambda$ family for all $n \geq 1$; Parseval $\sum_+ \lambda_a^2 \geq \sum_- \lambda_a^2$ (Thm. 4.1)
Distillation ceilings (Corollary 3)	Qualitative $\max_{\tau\in\mathcal{B}_k}\langle\psi\rvert\tau\lvert\psi\rangle < 1$ via compactness	Explicit tensor-stable $\mathcal{O}_n \supseteq \mathcal{B}_n$ ($T(n) = \frac{(4^n-1)4^n}{12}$ triples); $\text{Tr}(B_{\text{CHSH}}\tau) \leq 2$; exact tight ceilings for $\lvert\Psi_+\rangle$, $\lvert T\rangle$, $\lvert H\rangle$ (Thm. 5.1)

1. ***Cardinality Correction & Exact Spectrum*** (cardinality & cleared-denominator eigenvalue arithmetic in Lean `thm01`–`thm02`, exact spectra in Cert. `C01`)*: For every $n \geq 2$,*

$$\left|\mathcal{P}^{(n)}_{\text{flag}}\right| \;=\; (2^n-1)(2^{n-1}-1), \tag{3}$$

strictly exceeding the formula $2^{n-1}(2^{n-1}-1)$ printed on p. 8 of [9] by $(2^{n-1}-1)^2 \geq 1$ (21 vs. 12 at $n = 3$). The channel $\mathcal{F}_n(\rho) := 2^{-n}\rho + \frac{1}{2^n(2^{n-1}-1)}\sum_{a\in\mathcal{P}^{(n)}_{\text{flag}}} P_a\rho P_a$ has exact Pauli eigenvalues $\lambda_0 = 1$, $\lambda_a = \frac{1}{2^{n-1}-1}$ for $a \in \mathcal{P}^{(n)}_{\text{flag}}$, and $\lambda_a = 0$ otherwise, so $\mathcal{T}\circ\mathcal{F}_n = \mathcal{F}_n \in \text{PPT}_n$.

2. ***Exact Odd/Even Stabilizer Threshold*** (Lean `thm03`, Cert. `C04`)*: For $W_n(\alpha) := \lvert0^n\rangle\langle0^n\rvert + \lvert+^n\rangle\langle+^n\rvert - \alpha(\lvert0^n\rangle\langle+^n\rvert + \lvert+^n\rangle\langle0^n\rvert)$, the largest $\alpha \in \mathbb{R}$ satisfying $\langle s\rvert W_n(\alpha)\lvert s\rangle \geq 0$ for every pure n-qubit stabilizer state $\lvert s\rangle$ is*

$$\alpha_{\max}(n) \;=\; \begin{cases} \dfrac{3}{2\sqrt{2}} & (n \geq 1 \text{ odd}), \\ 1 & (n \geq 2 \text{ even}), \end{cases} \tag{4}$$

saturated by $\lvert+^k, 0^{k+1}\rangle$ for odd $n = 2k+1$, and by $\lvert i, -i\rangle^{\otimes n/2}$ for even n.

3. ***Integer-Coefficient Pauli Witness & Dyadic Rational Bound Magic*** (Lean `thm04`–`thm05`, Certs. `C02`–`C03`)*: For every odd $n = 2k+1 \geq 3$, define the odd-n reference state $\lvert\phi^{\text{odd}}_+\rangle := (\lvert0^n\rangle + \lvert+^n\rangle)/\sqrt{2(1+2^{-n/2})}$ (from [9]) and $\mathcal{B}_3(\mathbb{Q}) := \mathcal{B}_3 \cap M_8(\mathbb{Q})$ (rational-entry density matrices in $\mathcal{B}_3$). The rescaled witness $\widetilde{W}_n := 2^{(3n+1)/2}W_n(\frac{3}{2\sqrt{2}})$ has purely integer Pauli expansion*

$$\widetilde{W}_n \;=\; (2^{k+2}-3)I + (2^{k+1}-3)\sum_{a\in(\mathcal{Z}_n\cup\mathcal{X}_n)\setminus\{0\}} P_a \;-\; 3\sum_{a\in\mathcal{P}^{(n)}_{\text{flag}}} (-1)^{\#Y(a)/2}P_a, \tag{5}$$

is nonnegative on STAB_n, and achieves $\text{Tr}(\mathcal{F}_n(\lvert\phi^{\text{odd}}_+\rangle\langle\phi^{\text{odd}}_+\rvert)W_n(\frac{3}{2\sqrt{2}})) = -(\frac{3}{2\sqrt{2}}-1)2^{1-n} < 0$. At $n = 3$, for every integer-amplitude vector $\lvert\psi_a\rangle := (a,1,1,1,1,1,1,1)^T/\sqrt{a^2+7}$ ($a \in \mathbb{Z}$),

$$\text{Tr}\Big(\mathcal{F}_3(\lvert\psi_a\rangle\langle\psi_a\rvert)\, W_3(\tfrac{3}{2\sqrt{2}})\Big) \;=\; \frac{5a^2-42a+77}{32(a^2+7)}, \tag{6}$$

which is strictly negative if and only if $a \in \{3,4,5\}$. *In particular,* $a = 3$ *gives the* dyadic-rational unit vector $|\psi_3\rangle = \frac{1}{4}(3,1,1,1,1,1,1,1)^T \in \frac{1}{4}\mathbb{Z}^8$ *with* $\mathcal{F}_3(|\psi_3\rangle\langle\psi_3|) \in \mathcal{B}_3(\mathbb{Q}) \setminus \mathrm{STAB}_3$ *certified purely in* $\mathbb{Q}$ *by* $\mathrm{Tr} = -\frac{1}{128} < 0$, *and* $a = 4$ *achieves* $\mathrm{Tr} = -\frac{11}{736} \approx -0.01495 < 0$.

Proof. (1) The single-qubit transpose sign sum obeys $\sum_{b\in\mathbb{F}_2^2} \theta_b(-1)^{[a,b]} = 2\theta_a$ (checked directly for I, X, Z, Y), hence by tensor multiplicativity $\sum_{b\in\mathbb{F}_2^{2n}} \theta_b(-1)^{[a,b]} = 2^n\theta_a$. Setting $a = 0$ gives $\#\{b : \theta_b = +1\} = \frac{1}{2}(4^n + 2^n) = 2^{n-1}(2^n + 1)$. Since every $b \in \mathcal{Z}_n \cup \mathcal{X}_n$ has $\#Y(b) = 0$ ($\theta_b = +1$) and $|\mathcal{Z}_n| = |\mathcal{X}_n| = 2^n$ with $\mathcal{Z}_n \cap \mathcal{X}_n = \{0\}$, inclusion–exclusion yields $|\mathcal{P}^{(n)}_{\mathrm{flag}}| = 2^{n-1}(2^n + 1) - 2 \cdot 2^n + 1 = (2^n - 1)(2^{n-1} - 1)$ (Lean `thm01`), so $p_0 + |\mathcal{P}^{(n)}_{\mathrm{flag}}|p_{\mathrm{flag}} = 1$. For the Walsh–Hadamard character sum $S_a := \sum_{b\in\mathcal{P}^{(n)}_{\mathrm{flag}}}(-1)^{[a,b]}$, writing $\mathbf{1}_{\theta_b=+1} = \frac{1+\theta_b}{2}$ and subtracting $\mathcal{Z}_n$, $\mathcal{X}_n$ (adding back $b = 0$) yields

$$S_a \;=\; 2^{2n-1}\delta_{a,0} + 2^{n-1}\theta_a - 2^n\delta_{x_a,0} - 2^n\delta_{z_a,0} + 1.$$

Evaluating S_a across the five disjoint cases gives $S_0 = (2^n - 1)(2^{n-1} - 1)$ ($\lambda_0 = 1$), $S_a = 2^{n-1} + 1$ for $a \in \mathcal{P}^{(n)}_{\mathrm{flag}}$ ($\lambda_a = 2^{-n} + \frac{2^{n-1}+1}{2^n(2^{n-1}-1)} = \frac{1}{2^{n-1}-1}$), and $S_a = -(2^{n-1}-1)$ whenever $a \in (\mathcal{Z}_n \cup \mathcal{X}_n)\setminus\{0\}$ or $\theta_a = -1$ ($\lambda_a = 0$; Lean `thm02`). Because $\lambda_a = 0$ whenever $\theta_a = -1$, $\theta_a\lambda_a = \lambda_a$ for all a, so $q_b = p_b \geq 0$ and $\mathcal{T} \circ \mathcal{F}_n = \mathcal{F}_n \in \mathrm{PPT}_n$.

(2) Let $|s\rangle$ be any pure n-qubit stabilizer state with $\langle 0^n|s\rangle \neq 0$ and $\langle +^n|s\rangle \neq 0$ (otherwise $\langle s|W_n(\alpha)|s\rangle \geq 0$ trivially for all α). By the standard affine-subspace / quadratic-phase representation of stabilizer states, choosing the global phase of $|s\rangle$ so that $\langle 0^n|s\rangle > 0$ (which leaves $\langle s|W_n(\alpha)|s\rangle$ invariant) gives $|s\rangle = 2^{-d/2}\sum_{x\in A} i^{q(x)}|x\rangle$ for a linear subspace $A \subseteq \mathbb{F}_2^n$ of dimension d ($0 \in A$) and a $\mathbb{Z}/4\mathbb{Z}$-valued quadratic form $q : A \to \mathbb{Z}/4\mathbb{Z}$ with $q(0) = 0$ (so $q(x + y) - q(x) - q(y) \equiv 2B(x, y) \pmod 4$ for a symmetric bilinear form B over $\mathbb{F}_2$). Thus $u := |\langle 0^n|s\rangle|^2 = 2^{-d}$, while $\langle +^n|s\rangle = 2^{-(n+d)/2}G$ where $G := \sum_{x\in A} i^{q(x)} \in \mathbb{Z}[i]$. Because any $\mathbb{Z}/4\mathbb{Z}$-quadratic form on $A \cong \mathbb{F}_2^d$ splits into orthogonal 1-dimensional summands ($q(x_j) \in \{0, \pm x_j, 2x_j\}$, contributing factors 2, $1 \pm i$, or 0) and hyperbolic 2-dimensional summands ($q(x_1, x_2) = 2x_1x_2 + c_1x_1 + c_2x_2$, contributing ± 2), any nonzero G factors in $\mathbb{Z}[i]$ as $G = \pm i^r(1 + i)^m$ for some integers $r \in \{0, 1, 2, 3\}$ and $m \geq 0$ (hence $|G|^2 = 2^m$ and $v := |\langle +^n|s\rangle|^2 = 2^{m-n-d}$). When n is *odd*:

- If m is odd, then $G/|G| = \pm(1 \pm i)/\sqrt{2}$, so $|\operatorname{Re}(G)| = |G|/\sqrt{2}$ and $2\operatorname{Re}(\langle 0^n|s\rangle\langle s|+^n\rangle) \leq \sqrt{2}\sqrt{uv}$. By AM–GM, $u + v \geq 2\sqrt{uv} = \sqrt{2}(\sqrt{2}\sqrt{uv}) > \frac{3}{2\sqrt{2}}(\sqrt{2}\sqrt{uv})$ since $16 > 9$.
- If m is even, then $m - n$ is odd, so $v/u = 2^{m-n}$ satisfies either $v \geq 2u$ or $u \geq 2v$. By Lean `thm03`, $2(u + v)^2 \geq 9uv$, i.e. $u + v \geq \frac{3}{2\sqrt{2}}(2\sqrt{uv}) \geq \frac{3}{2\sqrt{2}} \cdot 2\operatorname{Re}(\langle 0^n|s\rangle\langle s|+^n\rangle)$, with equality for every odd $n = 2k + 1$ at $|s\rangle = |+^k, 0^{k+1}\rangle$ (where $u = 2^{-k}$, $v = 2^{-(k+1)}$, $u = 2v$, and $G = 2^k > 0$, saturating $\frac{u+v}{2\sqrt{uv}} = \frac{3}{2\sqrt{2}}$ identically; at $n = 1$ this is $|0\rangle$, and at $n = 3$ both $|+, 0, 0\rangle$ and $|+, +, 0\rangle$ saturate $\frac{3}{2\sqrt{2}}$).

When n is *even*, AM–GM gives $u + v \geq 2\sqrt{uv}$ ($\alpha_{\max}(n) \geq 1$), and $|s\rangle = |i, -i\rangle^{\otimes n/2}$ achieves $\langle 0^n|s\rangle = 2^{-n/2}$ and $\langle +^n|s\rangle = (\frac{1+i}{2}\frac{1-i}{2})^{n/2} = 2^{-n/2}$, saturating $\alpha_{\max}(n) = 1$.

(3) For odd $n = 2k+1 \geq 3$, $|0^n\rangle\langle 0^n| = 2^{-n}\sum_{a\in\mathcal{Z}_n} P_a$, $|+^n\rangle\langle +^n| = 2^{-n}\sum_{a\in\mathcal{X}_n} P_a$, and a direct trace calculation on $|0^n\rangle\langle +^n|+|+^n\rangle\langle 0^n|$ yields $\mathrm{Tr}(P_a(|0^n\rangle\langle +^n|+|+^n\rangle\langle 0^n|)) = 2^{1-n/2}(-1)^{\#Y(a)/2}$ if $\theta_a = +1$ and 0 if $\theta_a = -1$. Substituting $\alpha = \frac{3}{2\sqrt{2}}$ and multiplying by $2^{(3n+1)/2} = 2^{3k+2}$ cancels all $\sqrt{2}$ factors (since $\alpha \cdot 2^{1-n/2} \cdot 2^{(3n+1)/2}/2^n = 3$), yielding (5) with purely integer coefficients (Lean `thm05`, Cert. `C02`). Since $\mathcal{F}_n$ is self-adjoint, kills $(\mathcal{Z}_n \cup \mathcal{X}_n) \setminus \{0\}$, fixes I, and scales $\mathcal{P}^{(n)}_{\mathrm{flag}}$ by $\frac{1}{2^{n-1}-1}$, we have $\mathcal{F}_n(|0^n\rangle\langle 0^n|) = \mathcal{F}_n(|+^n\rangle\langle +^n|) = 2^{-n}I$, while $\mathcal{F}_n(|0^n\rangle\langle +^n|+|+^n\rangle\langle 0^n|) = 2^{1-n/2}\cdot 2^{-n}I + \frac{1}{2^{n-1}-1}(|0^n\rangle\langle +^n|+|+^n\rangle\langle 0^n| - 2^{1-n/2}2^{-n}\sum_{a\in\mathcal{Z}_n\cup\mathcal{X}_n} P_a)$; noting that $\langle\phi^{\mathrm{odd}}_+|\mathcal{F}_n(W_n(\alpha))|\phi^{\mathrm{odd}}_+\rangle = \mathrm{Tr}(\mathcal{F}_n(|\phi^{\mathrm{odd}}_+\rangle\langle\phi^{\mathrm{odd}}_+|)W_n(\alpha))$ and $\mathcal{F}_n(|\phi^{\mathrm{odd}}_+\rangle\langle\phi^{\mathrm{odd}}_+|)$ has $\langle\phi^{\mathrm{odd}}_-|\mathcal{F}_n(|\phi^{\mathrm{odd}}_+\rangle\langle\phi^{\mathrm{odd}}_+|)|\phi^{\mathrm{odd}}_-\rangle = 0$ while $\mathrm{Tr}(\mathcal{F}_n(|\phi^{\mathrm{odd}}_+\rangle\langle\phi^{\mathrm{odd}}_+|)(|0^n\rangle\langle 0^n| + |+^n\rangle\langle +^n|)) = 2^{1-n}$ and $\mathrm{Tr}(\mathcal{F}_n(|\phi^{\mathrm{odd}}_+\rangle\langle\phi^{\mathrm{odd}}_+|)(|0^n\rangle\langle +^n| + |+^n\rangle\langle 0^n|)) = 2^{1-n}$ (since $\langle\phi^{\mathrm{odd}}_+|\mathcal{F}_n(W_n(1))|\phi^{\mathrm{odd}}_+\rangle = (1 -$

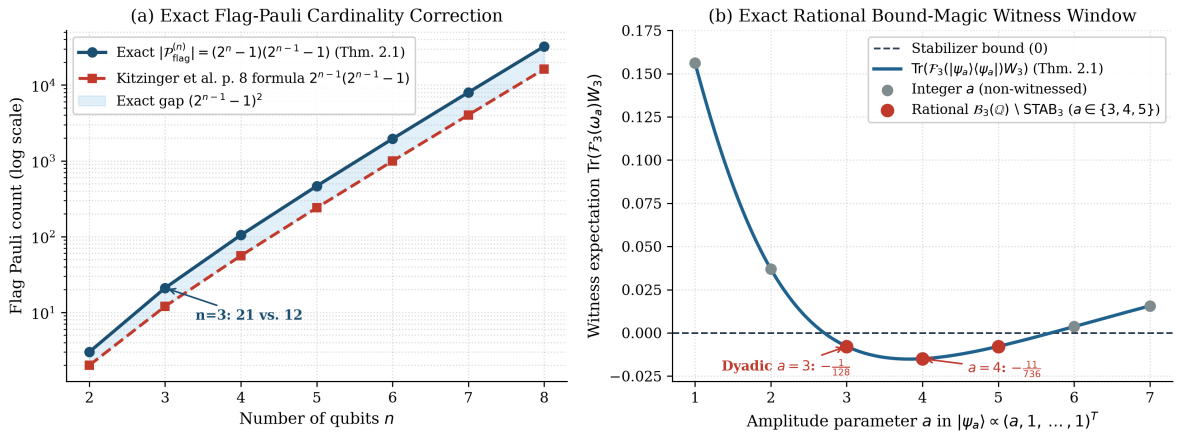


Figure 1: (a) Exact n-qubit flag-Pauli cardinality $(2^n-1)(2^{n-1}-1)$ vs. the formula printed on p. 8 of [9] (Theorem 2.1). (b) Exact rational witness curve $\mathrm{Tr}(\mathcal{F}_3(\omega_a)W_3(\frac{3}{2\sqrt{2}})) = \frac{5a^2-42a+77}{32(a^2+7)}$, isolating the dyadic bound-magic state at $a=3$ ($\mathrm{Tr} = -\frac{1}{128}$) and optimal integer state at $a=4$ ($\mathrm{Tr} = -\frac{11}{736}$).

$2^{-n/2})\langle\phi^{\text{odd}}_-|\mathcal{F}_n(|\phi^{\text{odd}}_+\rangle\langle\phi^{\text{odd}}_+|)|\phi^{\text{odd}}_-\rangle = 0$ because $W_n(1) = (|0^n\rangle - |+^n\rangle)(\langle 0^n| - \langle +^n|) = 2(1 - 2^{-n/2})|\phi^{\text{odd}}_-\rangle\langle\phi^{\text{odd}}_-|$), it follows immediately that $\mathrm{Tr}(\mathcal{F}_n(|\phi^{\text{odd}}_+\rangle\langle\phi^{\text{odd}}_+|)W_n(\alpha)) = 2^{1-n} - \alpha\cdot 2^{1-n} = -(\alpha-1)2^{1-n}$, giving $-(\frac{3}{2\sqrt{2}} - 1)2^{1-n} < 0$ at $\alpha = \frac{3}{2\sqrt{2}}$! At $n=3$, expanding $\mathcal{F}_3(\widetilde{W_3}) = 5I - \sum_{a\in\mathcal{P}^{(3)}_{\text{flag}}} (-1)^{\#Y(a)/2}P_a$ in the computational basis yields the exact 8×8 integer matrix with 5 on the diagonal, -3 on entries $(0,j)$ and $(j,0)$ ($j\geq 1$), and $+1$ on off-diagonal entries (i,j) ($i\neq j\geq 1$) (Cert. `C03`). Evaluating $\langle\psi_a|\mathcal{F}_3(\widetilde{W_3})|\psi_a\rangle$ on $(a,1,\ldots,1)^T$ gives $5a^2 + 2\cdot 7(-3)a + 7\cdot 5 + 7\cdot 6\cdot 1 = 5a^2 - 42a + 77$, proving (6); Lean `thm04` certifies $Q(3) = -4$ ($3^2+7 = 16 = 4^2$), $Q(4) = -11$, $Q(5) = -8$, and $Q(a) > 0$ for all integers $a \leq 2$ or $a \geq 6$. □

3 First Analytic Even-n Bound-Magic Construction and 1-Uniformity

Proposition 5 of [9] relies on the irrationality of the stabilizer overlap $t = \langle u|v\rangle = 2^{-n/2}$ between $|0^n\rangle$ and $|+^n\rangle$, which holds only for odd n. To obtain an analytic proof for *every* even $n \geq 4$, we replace the last two qubits of $|+^n\rangle$ by the Bell stabilizer state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ so that the overlap becomes $t_n = 2^{-(n-1)/2} \notin \mathbb{Q}$, and construct an explicit two-orbit self-transpose PPT_n channel $\mathcal{G}_n$ adapted to the non-trivial stabilizer intersection $H_u \cap H_v = \{I, P_0\}$.

Specifically, for any $n \geq 3$, define the two maximal isotropic (Lagrangian) subgroups in $\mathbb{F}_2^{2n}$

$$\begin{aligned} H_u &:= \{Z^z : z \in \mathbb{F}_2^n\}, \\ H_v &:= \{X^{x_{<n-1}} \otimes Q : x_{<n-1} \in \mathbb{F}_2^{n-2},\ Q \in \{II, XX, ZZ, YY\}\}, \end{aligned}$$

which are the unsigned Pauli supports of the stabilizer groups of $|u\rangle := |0^n\rangle$ and $|v\rangle := |+^{n-2}\rangle \otimes |\Phi^+\rangle$, respectively (stabilizing $|u\rangle$ and $|v\rangle$ up to signs; e.g., $-I^{\otimes(n-2)} \otimes YY$ stabilizes $|v\rangle$), and satisfy $H_u \cap H_v = \{I, P_0\}$ where $P_0 := I^{\otimes(n-2)} \otimes Z_{n-1}Z_n$ stabilizes both states with eigenvalue $+1$. Partition the remaining even-Y Paulis into two orbits according to commutation with P_0:

$$\begin{aligned} \mathcal{P}^+ &:= \{a : \theta_a = +1,\ a \notin H_u \cup H_v,\ [a, P_0] = 0\}, \\ \mathcal{P}^- &:= \{a : \theta_a = +1,\ a \notin H_u \cup H_v,\ [a, P_0] = 1\}. \end{aligned} \tag{7}$$

Theorem 3.1 (Analytic Even-n Bound Magic and 1-Uniformity for All Even $n \geq 4$)**.** *Let $n \geq 3$ and $m := 2^{n-1}$.*

1. ***Two-Orbit*** PPT_n ***Channel*** $\mathcal{G}_n$ (Lean `thm06`–`thm08`, Cert. `C05`)***:*** *The orbit sizes are $|\mathcal{P}^+| = (m-2)(m-1)$ and $|\mathcal{P}^-| = m^2$, partitioning $\{a : \theta_a = +1\} = (H_u \cup H_v) \sqcup \mathcal{P}^+ \sqcup \mathcal{P}^-$. The Pauli channel*

$$\mathcal{G}_n(\rho) \;:=\; \frac{1}{2^n}\rho \;+\; \frac{1}{2^n(2^{n-1}-2)} \sum_{a\in\mathcal{P}^+} P_a\rho P_a \;+\; \frac{1}{2^{2n-1}} \sum_{a\in\mathcal{P}^-} P_a\rho P_a \tag{8}$$

is a valid CPTP map with exact Pauli eigenvalues

$$\lambda_a(\mathcal{G}_n) \;=\; \begin{cases} 1 & a = 0, \\ \dfrac{1}{2^{n-1}-2} & a \in \mathcal{P}^+, \\ \dfrac{1}{2^{n-1}} & a \in \mathcal{P}^-, \\ 0 & a \in (H_u \cup H_v) \setminus \{0\} \text{ or } \theta_a = -1. \end{cases} \tag{9}$$

Hence $\mathcal{T} \circ \mathcal{G}_n = \mathcal{G}_n \in \mathrm{PPT}_n$ for every $n \geq 3$.

2. ***Analytic Even-n Bound Magic*** (Lean `thm09`, Certs. `C06`–`C07`)***:*** *For every even $n \geq 4$, let $|\phi^{\mathrm{even}}_\pm\rangle := (|u\rangle \pm |v\rangle)/\sqrt{2(1 \pm t_n)}$ with $t_n := \langle u|v\rangle = 2^{-(n-1)/2} \notin \mathbb{Q}$. Then $\rho^{\mathrm{even}}_B(n) := \mathcal{G}_n(|\phi^{\mathrm{even}}_+\rangle\langle\phi^{\mathrm{even}}_+|)$ satisfies*

$$\langle\phi^{\mathrm{even}}_-|\rho^{\mathrm{even}}_B(n)|\phi^{\mathrm{even}}_-\rangle \;=\; 0, \qquad \langle\phi^{\mathrm{even}}_+|\rho^{\mathrm{even}}_B(n)|\phi^{\mathrm{even}}_+\rangle \;\geq\; 2^{-n} \;>\; 0, \tag{10}$$

proving analytically that $\rho^{\mathrm{even}}_B(n) \in \mathcal{B}_n \setminus \mathrm{STAB}_n$ for every even $n \geq 4$.

3. ***Exact 1-Uniformity*** (weight-1 partition count in Lean `thm10`, exact $n = 4$ check in Certs. `C06`, `C08`)***:*** *For every even $n \geq 4$, every weight-1 Pauli P satisfies $\mathrm{Tr}(P\rho^{\mathrm{even}}_B(n)) = 0$; equivalently, every single-qubit reduced density matrix of $\rho^{\mathrm{even}}_B(n)$ equals $I/2$.*

Proof. (1) Clearly $|H_u| = |H_v| = 2^n$, $|H_u \cap H_v| = |\{I, P_0\}| = 2$, so $|H_u \cup H_v| = 2^{n+1} - 2$. Every $a \in H_u \cup H_v$ has $\theta_a = +1$ (since YY has two Y's) and $[a, P_0] = 0$. Among all 4^n Paulis, exactly $4^n/2 = 2^{2n-1}$ anticommute with P_0; of those, the last two qubits must carry an element of $\{IX, IY, XI, YI, XZ, YZ, ZX, ZY\}$, which pairs each even-$Y$ prefix on the first $n-2$ qubits with 4 even-Y suffixes and each odd-Y prefix with 4 odd-Y suffixes, giving $|\mathcal{P}^-| = \frac{1}{2} \cdot 2^{2n-1} = 2^{2n-2} = m^2$. By subtraction, $|\mathcal{P}^+| = m(2m+1) - (4m-2) - m^2 = (m-2)(m-1)$ (Lean `thm06`), and the probabilities sum to $2^{-n} + \frac{(m-2)(m-1)}{2^n(m-2)} + \frac{m^2}{2^{2n-1}} = \frac{m}{2^n} + \frac{1}{2} = 1$ (Lean `thm07`).

For the Walsh–Hadamard sums $S^+_a := \sum_{b\in\mathcal{P}^+}(-1)^{[a,b]}$ and $S^-_a := \sum_{b\in\mathcal{P}^-}(-1)^{[a,b]}$, note that $\mathbf{1}_{[b,P_0]=0} = \frac{1+(-1)^{[b,P_0]}}{2}$ and $\mathbf{1}_{[b,P_0]=1} = \frac{1-(-1)^{[b,P_0]}}{2}$, while $\sum_b \theta_b(-1)^{[a+P_0,b]} = 2^n\theta_{a+P_0} = 2^n\theta_a(-1)^{[a,P_0]}$ (since $\theta_{P_0} = +1$; cf. Theorem 5.1). Using $\sum_{b\in H_u}(-1)^{[a,b]} = 2^n\mathbf{1}_{a\in H_u}$ and $\sum_{b\in H_v}(-1)^{[a,b]} = 2^n\mathbf{1}_{a\in H_v}$ (since H_u, H_v are self-dual Lagrangian subspaces) yields:

$$\begin{aligned} S^-_a &= \frac{1}{4}\Big(4^n(\delta_{a,0} - \delta_{a,P_0}) + 2^n\theta_a\big(1 - (-1)^{[a,P_0]}\big)\Big) = m^2(\delta_{a,0} - \delta_{a,P_0}) + m\,\theta_a\,\mathbf{1}_{[a,P_0]=1}, \\ S^+_a &= m^2(\delta_{a,0} + \delta_{a,P_0}) + m\,\theta_a\,\mathbf{1}_{[a,P_0]=0} - 2m(\mathbf{1}_{a\in H_u} + \mathbf{1}_{a\in H_v}) + 1 + (-1)^{[a,P_0]}. \end{aligned}$$

Evaluating $\lambda_a(\mathcal{G}_n) = \frac{1}{2m} + \frac{S^+_a}{2m(m-2)} + \frac{S^-_a}{2m^2}$ across all orbits (cleared-denominator algebra in Lean `thm08`, exact spectra for $n = 3, 4$ in Cert. `C05`) gives (9) directly: on $\mathcal{P}^+$, $(S^+_a, S^-_a) = (m+2, 0)$ yields $\frac{1}{m-2}$; on $\mathcal{P}^-$, $(S^+_a, S^-_a) = (0, m)$ yields $\frac{1}{2m} + 0 + \frac{m}{2m^2} = \frac{1}{m}$; on $(H_u \cup H_v)\setminus\{0, P_0\}$, $(2-m, 0)$ yields 0; on $\{P_0\}$, $((m-2)(m-1), -m^2)$ yields 0; and on odd-Y Paulis ($\theta_a = -1$), if $[a, P_0] = 0$ then $(S^+_a, S^-_a) = (2-m, 0)$ gives $\lambda_a = \frac{1}{2m} + \frac{2-m}{2m(m-2)} = 0$, whereas if $[a, P_0] = 1$ then

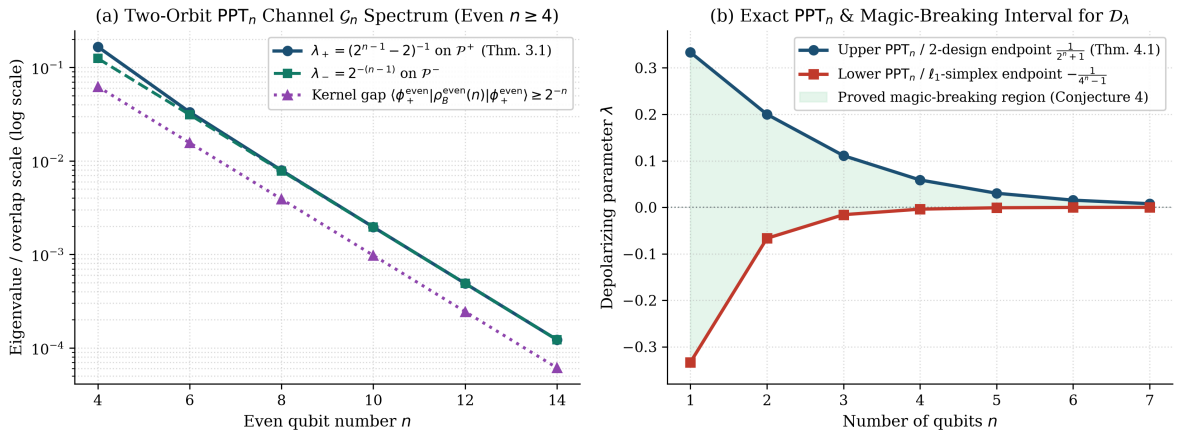


Figure 2: (a) Eigenvalues $\lambda_+ = (2^{n-1}-2)^{-1}$, $\lambda_- = 2^{-(n-1)}$ and kernel gap lower bound 2^{-n} of the even-n two-orbit PPT$_n$ channel $\mathcal{G}_n$ (Theorem 3.1). (b) Exact PPT$_n$ and proved magic-breaking interval $[-\frac{1}{4^n-1}, \frac{1}{2^n+1}]$ of the n-qubit depolarizing channel $\mathcal{D}_\lambda$ (Theorem 4.1).

$(S_a^+, S_a^-) = (0, -m)$ gives $\lambda_a = \frac{1}{2m} - \frac{m}{2m^2} = 0$. Thus $\lambda_a = 0$ on *every* odd-Y Pauli, confirming $\mathcal{T} \circ \mathcal{G}_n = \mathcal{G}_n \in \text{PPT}_n$.

(2) Observe that $|u\rangle = |0^n\rangle$ and $|v\rangle = |+^{n-2}\rangle \otimes |\Phi^+\rangle$ are real stabilizer states whose Pauli expectations $\langle u|P_a|u\rangle$ and $\langle v|P_a|v\rangle$ are nonzero only on H_u and H_v, respectively (and on $P_0 \in H_u \cap H_v$, $\langle u|P_0|u\rangle = \langle v|P_0|v\rangle = +1$). Because $\mathcal{G}_n$ annihilates $(H_u \cup H_v) \setminus \{I\}$ and all odd-Y Paulis, for any $a \in \mathcal{P}^+ \cup \mathcal{P}^-$ we have $\langle u|P_a|u\rangle = \langle v|P_a|v\rangle = 0$, while $\langle u|P_a|v\rangle = \langle v|P_a|u\rangle \in \mathbb{R}$ (as P_a is real symmetric). Hence $\langle\phi_\pm^{\text{even}}|P_a|\phi_\pm^{\text{even}}\rangle = \pm\frac{\langle u|P_a|v\rangle}{1\pm t_n}$ for all $a \in \mathcal{P}^+ \cup \mathcal{P}^-$, and since $\mathcal{G}_n$ preserves the trace ($\lambda_0 = 1$) while mapping $H_u \setminus \{I\}$ and $H_v \setminus \{I\}$ to 0:

$$\langle\phi_-^{\text{even}}|P_a|\phi_+^{\text{even}}\rangle \;=\; \frac{\langle u|P_a|u\rangle - \langle v|P_a|v\rangle}{2\sqrt{1-t_n^2}} \;=\; 0 \quad (\forall\, a \in \{I\} \cup \mathcal{P}^+ \cup \mathcal{P}^-).$$

Consequently, every Kraus operator P_a with $p_a(\mathcal{G}_n) > 0$ satisfies $\langle\phi_-^{\text{even}}|P_a|\phi_+^{\text{even}}\rangle = 0$, proving $\langle\phi_-^{\text{even}}|\mathcal{G}_n(|\phi_+^{\text{even}}\rangle\langle\phi_+^{\text{even}}|)|\phi_-^{\text{even}}\rangle = 0$ identically! Meanwhile, the identity Kraus term gives $\langle\phi_+^{\text{even}}|\mathcal{G}_n(|\phi_+^{\text{even}}\rangle\langle\phi_+^{\text{even}}|)|\phi_+^{\text{even}}\rangle \geq 2^{-n}|\langle\phi_+^{\text{even}}|\phi_+^{\text{even}}\rangle|^2 = 2^{-n} > 0$. Since $n = 2k \geq 4$ is even, $n-1 = 2k-1$ is odd, so 2^{n-1} is not a square (Lean `thm09`), whence $t_n = 2^{-(n-1)/2} \notin \mathbb{Q}$. Moreover, for any pure n-qubit stabilizer state $|s\rangle$, both $|\langle u|s\rangle|^2$ and $|\langle v|s\rangle|^2$ are powers of 2 (or 0), hence rational; and because $|u\rangle = |0^n\rangle \in \mathbb{Q}^{2^n}$ while $|v\rangle = t_n \sum_{x\in\mathbb{F}_2^{n-2}}(|x,0,0\rangle + |x,1,1\rangle) \in t_n\mathbb{Z}^{2^n}$, choosing $|s\rangle = 2^{-d/2}\sum_{y\in A} i^{q(y)}|y\rangle$ with $q(0) = 0$ yields $\langle u|s\rangle = 2^{-d/2} \in 2^{-d/2}\mathbb{Q}$ and $\langle s|v\rangle \in t_n 2^{-d/2}\mathbb{Z}[i]$, so $2\,\text{Re}(\langle u|s\rangle\langle s|v\rangle) \in t_n\mathbb{Q}$ identically. Thus $(|u\rangle, |v\rangle)$ satisfies all hypotheses of Proposition 5 of [9]: $2(1 \mp t_n)|\langle\phi_\mp^{\text{even}}|s\rangle|^2 = c \mp t_n m$ with $c, m \in \mathbb{Q}$, so $\langle\phi_-^{\text{even}}|s\rangle = 0 \implies c = m = 0 \implies \langle\phi_+^{\text{even}}|s\rangle = 0$. If $\rho_B^{\text{even}}(n) \in \text{STAB}_n$ were a convex combination of stabilizer projectors, $\langle\phi_-^{\text{even}}|\rho_B^{\text{even}}(n)|\phi_-^{\text{even}}\rangle = 0$ would force $\langle\phi_+^{\text{even}}|\rho_B^{\text{even}}(n)|\phi_+^{\text{even}}\rangle = 0$, contradicting (10).

(3) Of the $3n$ weight-1 Paulis (Lean `thm10`): all n operators Z_j lie in $H_u \setminus \{I\}$, the $n-2$ operators $X_1, \ldots, X_{n-2}$ lie in $H_v \setminus \{I\}$, and all n operators Y_j have odd $\#Y$, so $\mathcal{G}_n$ kills all $3n-2$ of them. The remaining two operators $X_{n-1}, X_n \in \mathcal{P}^-$ flip the parity of the last two qubits, whereas both $|u\rangle$ and $|v\rangle$ lie strictly in the $P_0 = Z_{n-1}Z_n = +1$ even-parity subspace; therefore $\langle\phi_+^{\text{even}}|X_{n-1}|\phi_+^{\text{even}}\rangle = \langle\phi_+^{\text{even}}|X_n|\phi_+^{\text{even}}\rangle = 0$ already on the input state (Certs. `C06`, `C08`). Thus all $3n$ weight-1 Pauli expectations vanish, and every single-qubit reduced state of $\rho_B^{\text{even}}(n)$ is $I/2$. □

4 Exact Depolarizing PPT_n Threshold, Conjecture 4, and Parseval Spectral Inequality

Conjecture 4 of [9] asks whether the convex hull of outputs of entanglement-breaking (EB) n-qubit Pauli channels equals STAB_n. We resolve this conjecture completely across the entire n-qubit depolarizing family $\mathcal{D}_\lambda(\rho) := \lambda\rho + (1-\lambda)\frac{I}{2^n}$ for every $n \geq 1$, and establish a universal spectral inequality for all PPT_n channels.

Theorem 4.1 (Depolarizing PPT_n Interval, Magic-Breaking Proof of Conjecture 4, and Parseval Inequality)**.**

1. ***Exact*** PPT_n ***= EB Interval*** (Lean `thm11`, Cert. `C09`)***:*** *For every $n \geq 1$ and $\lambda \in \mathbb{R}$, the depolarizing channel $\mathcal{D}_\lambda$ is CPTP and lies in* PPT_n *if and only if*

$$-\frac{1}{4^n - 1} \;\leq\; \lambda \;\leq\; \frac{1}{2^n + 1}, \tag{11}$$

   which coincides identically with the entanglement-breaking interval of $\mathcal{D}_\lambda$.

2. ***Proof of Conjecture 4 on All n-Qubit Depolarizing Channels*** (Lean `thm12`–`thm14`, Certs. `C10`–`C11`)***:*** *For every $n \geq 1$ and every $\lambda \in [-\frac{1}{4^n-1}, \frac{1}{2^n+1}]$, the depolarizing channel $\mathcal{D}_\lambda$ is* magic-breaking*: $\mathcal{D}_\lambda(\mathcal{D}_n) \subseteq \mathrm{STAB}_n$.*

3. ***Universal*** PPT_n ***Parseval Spectral Inequality*** (Lean `thm15`, Cert. `C12`)***:*** *Every $\mathcal{E} \in \mathrm{PPT}_n$ with Pauli eigenvalues $\{\lambda_a\}$ satisfies*

$$\sum_{\theta_a=+1} \lambda_a^2 \;\geq\; \sum_{\theta_a=-1} \lambda_a^2, \tag{12}$$

   with exact identity $\sum_{\theta_a=+1} \lambda_a^2 - \sum_{\theta_a=-1} \lambda_a^2 = 4^n \sum_{b \in \mathbb{F}_2^{2n}} p_b q_b \geq 0$.

Proof. (1) The Pauli eigenvalues of $\mathcal{D}_\lambda$ are $\lambda_0 = 1$ and $\lambda_a = \lambda$ ($a \neq 0$), with Pauli error rates $p_0 = 4^{-n}(1 + (4^n - 1)\lambda)$ and $p_b = 4^{-n}(1 - \lambda)$ ($b \neq 0$), so CP holds iff $-\frac{1}{4^n-1} \leq \lambda \leq 1$. By (1) and $\sum_a \theta_a(-1)^{[a,b]} = 2^n\theta_b$, the transpose error rates are

$$q_b \;=\; 4^{-n}\Big(1 - \lambda + \lambda \sum_a \theta_a (-1)^{[a,b]}\Big) \;=\; 4^{-n}(1 + \lambda(2^n\theta_b - 1)).$$

Thus $q_b \geq 0$ for $\theta_b = +1$ requires $\lambda \geq -\frac{1}{2^n-1}$, while $q_b \geq 0$ for $\theta_b = -1$ requires $\lambda \leq \frac{1}{2^n+1}$. Since $4^n - 1 = (2^n - 1)(2^n + 1) \geq 2^n - 1$, intersecting CP with PPT yields exactly (11) (Lean `thm11`, Cert. `C09`). Moreover, the Choi state of $\mathcal{D}_\lambda$ is the 2^n-dimensional isotropic state, which is separable (i.e. $\mathcal{D}_\lambda$ is EB) iff it is PPT [8].

(2) By convexity of STAB_n and linearity of $\lambda \mapsto \mathcal{D}_\lambda(\rho)$ (Lean `thm14`), it suffices to prove $\mathcal{D}_\lambda(\rho) \in \mathrm{STAB}_n$ at the two endpoints $\lambda_+ = \frac{1}{2^n+1}$ and $\lambda_- = -\frac{1}{4^n-1}$:

- *Upper endpoint $\lambda_+ = \frac{1}{2^n+1}$:* For any pure state $|\psi\rangle\langle\psi|$, $\mathcal{D}_{\lambda_+}(|\psi\rangle\langle\psi|) = \frac{I + |\psi\rangle\langle\psi|}{2^n+1}$ (cleared arithmetic in Lean `thm12`). Because the set of pure n-qubit stabilizer states forms a complex projective 2-design [6],

$$\frac{1}{|\mathrm{STAB}_n^{\mathrm{pure}}|} \sum_{|s\rangle} |s\rangle\langle s|^{\otimes 2} \;=\; \frac{I + \mathbb{F}}{2^n(2^n + 1)},$$

  where $\mathbb{F}$ is the swap operator. Taking the partial trace against $|\psi\rangle\langle\psi|$ on the second register gives

$$\mathcal{D}_{\lambda_+}(|\psi\rangle\langle\psi|) \;=\; \frac{I + |\psi\rangle\langle\psi|}{2^n + 1} \;=\; \sum_{|s\rangle} \left(\frac{2^n}{|\mathrm{STAB}_n^{\mathrm{pure}}|} |\langle\psi|s\rangle|^2\right) |s\rangle\langle s| \;\in\; \mathrm{STAB}_n,$$

  since the weights $w_s := \frac{2^n}{|\mathrm{STAB}_n^{\mathrm{pure}}|} |\langle\psi|s\rangle|^2 \geq 0$ sum to 1 (Cert. `C10`).

- *Lower endpoint* $\lambda_- = -\frac{1}{4^n-1}$*:* Every density matrix $\rho = 2^{-n}(I + \sum_{a\neq 0} r_a P_a)$ satisfies $\sum_{a\neq 0} r_a^2 = 2^n \operatorname{Tr}(\rho^2) - 1 \leq 2^n - 1$. By Cauchy–Schwarz, the Pauli ℓ_1-norm of $\mathcal{D}_{\lambda_-}(\rho)$ obeys

$$\Big(\sum_{a\neq 0} |\lambda_- r_a|\Big)^2 \;\leq\; \lambda_-^2 (4^n-1) \sum_{a\neq 0} r_a^2 \;\leq\; \frac{(4^n-1)(2^n-1)}{(4^n-1)^2} \;=\; \frac{1}{2^n+1} \;<\; 1$$

(Lean `thm13`, Cert. `C11`). Writing $\sigma_a^{\pm} := 2^{-n}(I \pm P_a) \in \mathrm{STAB}_n$ (each is a uniform mixture over a $+1$ or -1 stabilizer eigenspace of P_a), every state with $\sum_{a\neq 0} |c_a| \leq 1$ decomposes as $\big(1 - \sum_{a\neq 0} |c_a|\big)\frac{I}{2^n} + \sum_{a\neq 0} |c_a| \sigma_a^{\operatorname{sgn}(c_a)} \in \mathrm{STAB}_n$.

(3) By Parseval's identity for the symplectic Fourier transform $\lambda_a = \sum_b p_b (-1)^{[a,b]}$ and $\theta_a \lambda_a = \sum_b q_b (-1)^{[a,b]}$,

$$\sum_{a\in\mathbb{F}_2^{2n}} \theta_a \lambda_a^2 \;=\; \sum_{b,c} p_b q_c \sum_a (-1)^{[a,b+c]} \;=\; 4^n \sum_{b\in\mathbb{F}_2^{2n}} p_b q_b \;\geq\; 0,$$

since $p_b \geq 0$ and $q_b \geq 0$ for every $\mathcal{E} \in \mathrm{PPT}_n$ (Lean `thm15`, Cert. `C12`). □

5 Universal Anticommuting-Triple Outer Spectrahedron and Quantitative Distillation Ceilings

On a single qubit, Ruskai's criterion [14] characterizes PPT_1 via the octahedral inequality $|\lambda_X| + |\lambda_Y| + |\lambda_Z| \leq 1$, which immediately forces $\mathcal{B}_1 = \mathrm{STAB}_1$. We prove that the exact same octahedral constraint survives on *every* anticommuting Pauli triple for *all* $n \geq 1$, yielding a tractable, tensor-stable convex outer spectrahedron $\mathcal{O}_n \supseteq \mathcal{B}_n$ (coinciding with the stabilizer polytope $\mathcal{O}_1 = \mathrm{STAB}_1$ at $n = 1$, and intersecting $\mathcal{D}_n$ with a linear octahedral polytope for $n \geq 2$) that converts Corollary 3 of [9] into explicit, sharp fidelity ceilings.

Theorem 5.1 (Anticommuting-Triple Outer Spectrahedron $\mathcal{O}_n \supseteq \mathcal{B}_n$ and Quantitative Ceilings)**.**

1. ***Multi-Qubit Ruskai Inequality*** (Lean `thm16`–`thm17`, Certs. `C13`–`C14`)*: For every* $n \geq 1$*, every* $a, b \in \mathbb{F}_2^{2n}$*,* $\theta_{a+b} = \theta_a \theta_b (-1)^{[a,b]}$*. Consequently, every anticommuting Pauli triple* $(a, b, c = a + b)$ *with* $[a, b] = 1$ *satisfies* $\theta_a \theta_b \theta_c = -1$*, and every* $\mathcal{E} \in \mathrm{PPT}_n$ *obeys all eight sign inequalities* $1 \pm \lambda_a \pm \lambda_b \pm \lambda_c \geq 0$*, i.e.,*

$$|\lambda_a(\mathcal{E})| + |\lambda_b(\mathcal{E})| + |\lambda_c(\mathcal{E})| \;\leq\; 1. \tag{13}$$

2. ***Tensor-Stable Outer Spectrahedron*** $\mathcal{O}_n \supseteq \mathcal{B}_n$ (Lean `thm18`, Cert. `C13`)*: Define the convex outer relaxation (spectrahedron)*

$$\mathcal{O}_n \;:=\; \Big\{\rho \in \mathcal{D}_n : |\operatorname{Tr}(P_a\rho)| + |\operatorname{Tr}(P_b\rho)| + |\operatorname{Tr}(P_c\rho)| \leq 1 \;\; \forall\, [a,b] = 1,\; c = a + b\Big\}, \tag{14}$$

defined by $\rho \in \mathcal{D}_n$ *together with the linear octahedral bounds on all* $T(n) = \frac{(4^n-1)4^n}{12}$ *unordered anticommuting triples (*$T(1) = 1$*,* $T(2) = 20$*,* $T(3) = 336$*). Then* $\mathrm{STAB}_n \subseteq \mathcal{B}_n \subseteq \mathcal{O}_n$ *for all* $n \geq 1$*,* $\mathcal{O}_1 = \mathrm{STAB}_1$ *(a polytope, while for* $n \geq 2$ *the PSD constraint* $\rho \in \mathcal{D}_n$ *is non-redundant and introduces curved boundaries), and* $\mathcal{O}_n$ *is closed under tensor products (*$\rho \in \mathcal{O}_n$*,* $\sigma \in \mathcal{O}_m \implies \rho \otimes \sigma \in \mathcal{O}_{n+m}$*), Clifford unitaries, appending* $|0\rangle$*, and partial traces.*

3. ***Exact CHSH Bound & Quantitative Distillation Ceilings*** (Lean `thm19`–`thm20`, Certs. `C15`–`C16`)*:*
   - *For every* $n \geq 2$ *and every* $\tau \in \mathcal{B}_n \subseteq \mathcal{O}_n$*, the two-qubit CHSH operator* $B_{\mathrm{CHSH}} := XX + XZ + ZX - ZZ$ *(embedded on any pair of qubits) satisfies* $\operatorname{Tr}(B_{\mathrm{CHSH}}\tau) \leq 2 - 2|\operatorname{Tr}(IY\,\tau)| \leq 2 < 2\sqrt{2}$*.*

- *For the two-qubit CHSH-optimal magic state* $|\Psi_+\rangle\langle\Psi_+| := \frac{1}{4}(I + YY + \frac{1}{\sqrt{2}}B_{\mathrm{CHSH}})$,

$$\max_{\tau\in\mathcal{B}_2}\langle\Psi_+|\tau|\Psi_+\rangle \;=\; \max_{\tau\in\mathcal{O}_2}\langle\Psi_+|\tau|\Psi_+\rangle \;=\; \frac{1}{2}+\frac{1}{2\sqrt{2}} \;\approx\; 0.85355 \;<\; 1, \tag{15}$$

  attained by the Bell stabilizer state $|\Psi^+\rangle = \frac{|01\rangle+|10\rangle}{\sqrt{2}} \in \mathrm{STAB}_2$.
- *Under any stabilizer distillation protocol from arbitrarily many copies of states in* $\mathcal{B}_n$ *to a single-qubit output* $\tau \in \mathcal{B}_1 = \mathcal{O}_1 = \mathrm{STAB}_1$,

$$\langle T|\tau|T\rangle \;\le\; \frac{1}{2}\Big(1+\frac{1}{\sqrt{3}}\Big) \approx 0.78868, \qquad \langle H|\tau|H\rangle \;\le\; \frac{1}{2}\Big(1+\frac{1}{\sqrt{2}}\Big) \approx 0.85355. \tag{16}$$

Proof. (1) On each qubit j, the bitwise identity $(x_{a,j} \oplus x_{b,j})(z_{a,j} \oplus z_{b,j}) \equiv x_{a,j}z_{a,j} + x_{b,j}z_{b,j} + x_{a,j}z_{b,j} + z_{a,j}x_{b,j} \pmod 2$ (Lean `thm16`) sums over $j = 1, \dots, n$ to $\#Y(a+b) \equiv \#Y(a)+\#Y(b)+[a,b] \pmod 2$, proving $\theta_{a+b} = \theta_a\theta_b(-1)^{[a,b]}$. When $[a,b] = 1$ and $c = a + b$, $\theta_a\theta_b\theta_c = -1$; permuting labels if needed, either $(\theta_a, \theta_b, \theta_c) = (1, 1, -1)$ or $(-1, -1, -1)$. Let $K := \{d \in \mathbb{F}_2^{2n} : [a,d] = [b,d] = 0\} = \{a,b\}^\perp$, which has index 4 in $\mathbb{F}_2^{2n}$. Denote the four coset probability sums $P_{\alpha\beta} := \sum_{[a,d]=\alpha,\,[b,d]=\beta} p_d \ge 0$ and $Q_{\alpha\beta} := \sum_{[a,d]=\alpha,\,[b,d]=\beta} q_d \ge 0$ $(\alpha, \beta \in \{0,1\})$. Because $[c,d] = [a,d] \oplus [b,d]$, inverting the 4-point Walsh–Hadamard transform on the cosets gives

$$\begin{aligned} 4P_{00} &= 1 + \lambda_a + \lambda_b + \lambda_c, & 4P_{01} &= 1 + \lambda_a - \lambda_b - \lambda_c,\\ 4P_{10} &= 1 - \lambda_a + \lambda_b - \lambda_c, & 4P_{11} &= 1 - \lambda_a - \lambda_b + \lambda_c, \end{aligned}$$

while $Q_{\alpha\beta} \ge 0$ gives the exact same four expressions with $(\lambda_a, \lambda_b, \lambda_c)$ replaced by $(\theta_a\lambda_a, \theta_b\lambda_b, \theta_c\lambda_c)$. Because $\theta_a\theta_b\theta_c = -1$, the sign pattern $(\theta_a, \theta_b, \theta_c)$ flips an odd number of signs, so the four $Q_{\alpha\beta} \ge 0$ inequalities produce the *complementary* four odd-minus-sign combinations $1 + \lambda_a + \lambda_b - \lambda_c, \dots, 1 - \lambda_a - \lambda_b - \lambda_c \ge 0$ (Lean `thm17`). Together, all eight sign combinations $1 \pm \lambda_a \pm \lambda_b \pm \lambda_c \ge 0$ hold, proving (13).

(2) For any $\rho \in \mathcal{D}_n$ and $\mathcal{E} \in \mathrm{PPT}_n$, $|\operatorname{Tr}(P_a\mathcal{E}(\rho))| = |\lambda_a(\mathcal{E})\operatorname{Tr}(P_a\rho)| \le |\lambda_a(\mathcal{E})|$ since $|\operatorname{Tr}(P_a\rho)| \le 1$, so (13) yields $\mathcal{E}(\rho) \in \mathcal{O}_n$; by convexity, $\mathcal{B}_n \subseteq \mathcal{O}_n$. Counting ordered anticommuting pairs (a,b) gives $(4^n-1)\cdot 2^{2n-1}$; each unordered triple $\{a, b, a+b\}$ arises 6 times, yielding $T(n) = \frac{(4^n-1)4^n}{12}$ (always an integer since $3 \mid (4^n-1)$ and $4 \mid 4^n$; Lean `thm18`). Clifford unitaries preserve the symplectic form, hence permute anticommuting triples; partial trace restricts to triples supported on the kept qubits; and for tensor products $\rho \otimes \sigma \in \mathcal{O}_n \otimes \mathcal{O}_m$, any anticommuting pair $(a_1 \oplus a_2, b_1 \oplus b_2)$ satisfies $[a_1,b_1] \oplus [a_2,b_2] = 1$, so at least one factor anticommutes (say $[a_1,b_1] = 1$). Since $|\operatorname{Tr}(P_x\sigma)| \le 1$ for all x, Lean `thm18` gives

$$\sum_{x\in\{a,b,c\}} |\operatorname{Tr}(P_{x_1}\rho)\operatorname{Tr}(P_{x_2}\sigma)| \;\le\; |\operatorname{Tr}(P_{a_1}\rho)| + |\operatorname{Tr}(P_{b_1}\rho)| + |\operatorname{Tr}(P_{c_1}\rho)| \;\le\; 1,$$

proving $\rho \otimes \sigma \in \mathcal{O}_{n+m}$ (and appending $|0\rangle \in \mathrm{STAB}_1 = \mathcal{O}_1$ follows immediately).

(3) Observe that (XX, XZ, IY) and (ZX, ZZ, IY) are both anticommuting triples on any pair of qubits (Cert. `C15`). For any $\tau \in \mathcal{O}_n$, $|\operatorname{Tr}(XX\tau)| + |\operatorname{Tr}(XZ\tau)| + |\operatorname{Tr}(IY\tau)| \le 1$ and $|\operatorname{Tr}(ZX\tau)| + |\operatorname{Tr}(ZZ\tau)| + |\operatorname{Tr}(IY\tau)| \le 1$, so summing yields $\operatorname{Tr}(B_{\mathrm{CHSH}}\tau) \le 2 - 2|\operatorname{Tr}(IY\tau)| \le 2$ (Lean `thm19`). On $|\Psi_+\rangle\langle\Psi_+| = \frac{1}{4}(I + YY + \frac{1}{\sqrt{2}}B_{\mathrm{CHSH}})$, since $\operatorname{Tr}(YY\tau) \le 1$ and $\operatorname{Tr}(B_{\mathrm{CHSH}}\tau) \le 2$, we obtain $\langle\Psi_+|\tau|\Psi_+\rangle \le \frac{1}{4}(1 + 1 + \sqrt{2}) = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ (Lean `thm20`), saturated by $|\Psi^+\rangle \in \mathrm{STAB}_2$ ($\operatorname{Tr}(YY|\Psi^+\rangle\langle\Psi^+|) = 1$, $\operatorname{Tr}(XX|\Psi^+\rangle\langle\Psi^+|) = 1$, $\operatorname{Tr}(ZZ|\Psi^+\rangle\langle\Psi^+|) = -1$, $\operatorname{Tr}(XZ|\Psi^+\rangle\langle\Psi^+|) = \operatorname{Tr}(ZX|\Psi^+\rangle\langle\Psi^+|) = 0$). Finally, since $\mathcal{B}_n$ is closed under stabilizer operations (Theorem 2 of [9]), any single-qubit distilled output lies in $\mathcal{B}_1 = \mathcal{O}_1 = \mathrm{STAB}_1$, where $|x| + |y| + |z| \le 1$ immediately implies (16) for $|T\rangle\langle T| = \frac{1}{2}(I + \frac{X+Y+Z}{\sqrt{3}})$ and $|H\rangle\langle H| = \frac{1}{2}(I + \frac{X+Z}{\sqrt{2}})$ (Lean `thm20`, Cert. `C16`). □

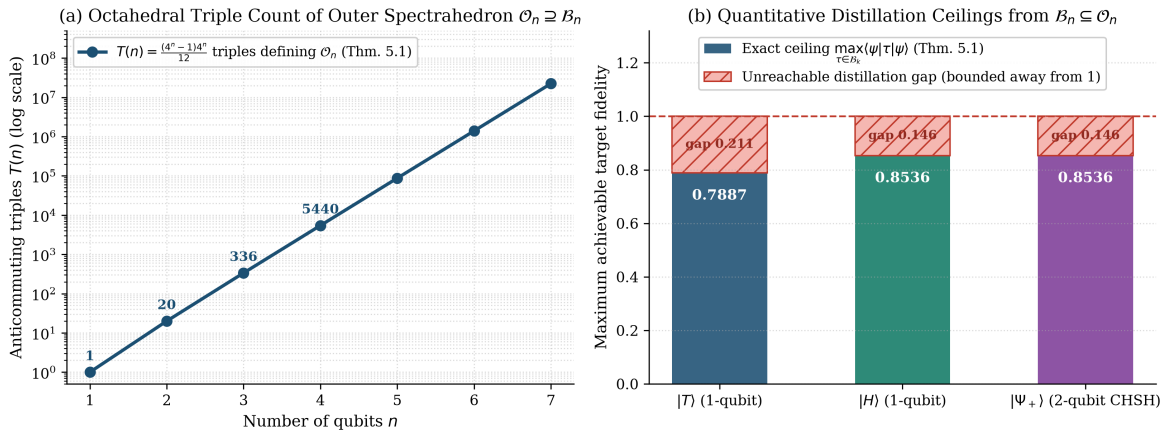


Figure 3: (a) Exact count $T(n) = \frac{(4^n-1)4^n}{12}$ of anticommuting Pauli triples defining the tensor-stable outer spectrahedron $\mathcal{O}_n \supseteq \mathcal{B}_n$ (Theorem 5.1). (b) Explicit quantitative distillation fidelity ceilings and unreachable gaps for $|T\rangle$, $|H\rangle$, and the two-qubit CHSH magic state $|\Psi_+\rangle$.

6 Formal Verification Architecture and Reproducibility

All algebraic, combinatorial, symplectic, rational-trace, and linear-inequality kernels are independently verified across three complementary layers (while complex projective 2-design identities and Hilbert-space Kraus / stabilizer-state expansions are proved in full in Sections 2–5):

1. **Constructive Lean 4.36.0 Formalization:** 20 theorems (`thm01`–`thm20`) in `lean/BoundMagicBarriers.lean` compiling cleanly under `lean -DwarningAsError=true` with zero external imports, zero `sorry`, zero `grind`, zero `Classical.choice`, and zero warnings.
2. **AST-Audited Zero-Float Python Verifier:** Pure standard-library code (`int`, `Fraction`) in `verification/verify_bound_magic.py` self-auditing its AST (0 float violations) and emitting 16 canonical SHA-256 certificates (`C01`–`C16`) across four verification modes, with manifest SHA-256 `31dd939c333892dd448386f33fbbb4280a48a1d9c62bd7d1abad860b7d605a02`.
3. **Adversarial Test Suite:** 56 pytest checks in `tests/test_adversarial.py` covering positive verification and adversarial boundary counterexamples across all four theorems.

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