Local retention and spherical reconstruction for Boolean Doob chains

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Abstract

We study transitions restricted to a Hamming ball for the noise Doob kernel of a positive potential. At noise correlation one half, a floor on the stationary density and a cap on its third moment force the killed operator norm on every ball of radius at most $L/96$ to decay exponentially in dimension L. The estimate is uniform in the potential and needs no stationary marginal matching. Its proof combines a bound on the reciprocal noise average using the entropy of a transition row with an exponentially weighted row estimate. Exact symmetrization of the stationary density can nevertheless strictly decrease the conditional retention of a ball: we construct examples with matching through order $C\sqrt{L}$, a fixed perturbation amplitude, and ball radius proportional to dimension. On the half-noise balls of the example the relative conductance difference is exponentially small. We also obtain control of projected stationary edge laws under low-degree matching. These results concern local escape and leave the unrestricted spectral gap open.

1 Model and observables

Write μ for uniform probability on $\{-1,1\}^L$. The noise operator $P = P_\rho$ flips each coordinate independently with probability $(1-\rho)/2$, where $0 < \rho < 1$. Fix $h > 0$ with $\mathbb{E}_\mu hPh = 1$ and set

$$g = Ph, \quad \pi = hg, \quad \nu = \pi\mu, \qquad K(x,y) = P(x,y)h(y)/g(x).$$

The stationary edge probability is $J(x,y) = \mu(x)P(x,y)h(x)h(y)$. It is symmetric, and both marginals are ν. For a nonempty proper cut A, define the retention $r_h(A) = J(A,A)/\nu(A)$ and the conductance $\Phi_h(A) = 1 - r_h(A)$. The killed operator K_{AA} discards trajectories on their first exit from A. Its norm on $L^2(\nu|_A)$ bounds $r_h(A)$ by the Rayleigh quotient of $\mathbf{1}_A$. A scalar rescaling of h leaves K unchanged. In constructions using an unnormalized potential we explicitly give its energy.

The standing moment hypotheses for the first two theorems are $\pi \geq b$ and $\mathbb{E}_\mu \pi^s \leq 2$, with $0 < b < 1$ and $s = (1+\rho)/(1-\rho)$. A separate matching assumption, when imposed, means $\mathbb{E}_\mu \pi \chi_T = 0$ for every nonconstant Walsh character χ_T of degree at most k. Write $|x|$ for the number of minus signs. The operator R averages a function over all coordinate permutations. Radial reconstruction replaces π with $R\pi$ and solves $uPu = R\pi$. This preserves every shell mass. It need not preserve the conditional transition probability of a shell.

2 Exponential escape from a ball at half noise

Theorem 2.1. *For $\rho = 1/2$ let A be any nonempty subset of a ball of radius at most $L/96$ about any vertex. Under the standing moment hypotheses,*

$$\|K_{AA}\| \le \min\{1, C_b \gamma^L\}, \qquad \gamma = 3^{6/5} 2^{-29/15} (101/100)^{2/3} 5^{1/144} < 1.$$

The estimate applies without any symmetry, matching, or assumed lower bound on g or h.

Proof. All norms of functions in this argument use μ. Forward Boolean hypercontractivity gives $\|P_t f\|_v \le \|f\|_u$ for $1 < u \le v$ and $t^2 \le (u-1)/(v-1)$ [1, 2]. With $Q = P_{\sqrt{\rho}}$, energy normalization yields $\|Qh\|_2 = 1$. The floor and Jensen give $g \ge bP(1/g) \ge b/(Pg)$. Now $Pg = P_{\rho^{3/2}} Qh$. At half noise this implies $\|g^{-1}\|_9 \le b^{-1}$. Since $h = \pi/g$, Hölder gives $\|h\|_{9/4} \le 2^{1/3}/b$. The identity $Pg = P_{1/4} h$ and another use of the floor give $\|g^{-1}\|_{21} \le 2^{1/3}/b^2$. Consequently

$$\|h\|_{21/8} \le 2^{2/3}/b^2, \qquad \|g\|_{15/2} \le \|h\|_{21/8} \le M_b := 2^{2/3}/b^2.$$

The last exponents satisfy $(21/8 - 1)/(15/2 - 1) = 1/4$.

We keep the logarithm before applying the floor estimate. For the row law $p_x = P(x, \cdot)$,

$$\log(1/g(x)) \le -\log b + \mathbb{E}_{p_x} \log g.$$

For $v = 15/2$, Jensen applied under p_x to $g^v/(dp_x/d\mu)$ gives

$$v\, \mathbb{E}_{p_x} \log g \le \log \mathbb{E}_\mu g^v + D(p_x \| \mu).$$

The product row has entropy relative to μ equal to LI, where $I = (3/4)\log 3 - \log 2$. Thus $g(x)^{-2} \le b^{-2} M_b^2 \exp(4LI/15)$ at every vertex. This intermediate bound may grow exponentially with L.

Set $U = h/g = \pi/g^2$. Conjugation by $\sqrt{\pi}$ identifies K_{AA} with the compression of $S = M_{\sqrt{U}} P M_{\sqrt{U}}$ in the uniform inner product. This matrix is positive semidefinite because the Walsh eigenvalues of P are $\rho^{|T|}$. Translate the center to the all-plus vertex, take $t = 1/20$, and test the positive matrix on A against $v_x = \sqrt{U(x)}\, t^{2|x|/3}$. Its largest eigenvalue is at most $\max_{x \in A} (S_{AA} v)_x / v_x$ by diagonal similarity and the row-sum bound. Factor out $(3/4)^L$ from the noise row. The remaining ratio is bounded by

$$b^{-2} M_b^2 e^{4LI/15}\, (2^{L+1})^{1/3} \left[\sum_{y \in A} 3^{-3d(x,y)/2}\, t^{|y|-|x|} \right]^{2/3}.$$

Here Hölder uses $\sum_y \pi(y)^3 \le 2^{L+1}$ in counting measure. With $\kappa = 3^{-3/2} < 1/5$, extending the nonnegative sum over the cube factors it into

$$(1 + \kappa t)^{L-|x|} (1 + \kappa/t)^{|x|} \le (101/100)^L 5^{L/96}.$$

Combining the factors gives the claimed constant with $C_b = 2^{1/3} b^{-2} M_b^2$. For an explicit sign check, use $\log 3 < 1.099$, $\log 2 > 0.693$, $\log 5 < 1.610$ and $\log(101/100) < 0.01$. Substitution gives $\log \gamma < -0.003$. The first three rational bounds follow, for example, by truncating $2 \sum_{j \ge 0} z^{2j+1}/(2j+1)$ at $j = 12$ with $z = (x-1)/(x+1)$ and bounding the tail by $2z^{27}/(27(1-z^2))$. Markov contraction gives the cutoff at one. □

The two features of the proof matter at this radius. For comparison, the unweighted row test and $\|U\|_{27/11} \le H_b$ give

$$H_b \beta^L \max_{x \in A} \left(\sum_{y \in A} 3^{-27 d(x,y)/16} \right)^{16/27}, \qquad \beta = (3/4) 2^{11/27}.$$

Indeed the preceding estimates give $\|g^{-1}\|_{27} \le 2^{2/3}/b^3$ from $Pg = P_{1/4} h$, then Hölder gives the stated U moment. For the full ball of radius $\lfloor L/96 \rfloor$ and L divisible by 100, the center row includes the layer $L/100$. Since $\binom{L}{j} \ge (L/j)^j$, its sum is at least $(100/9)^{L/100}$. The resulting exponential base exceeds one: $\log \beta = \log 3 - (43/27) \log 2 > -0.008$ and $(16/2700) \log(100/9) > 0.012$. Thus that estimate with its norm-one cutoff gives no decay for these balls.

Corollary 2.2. *For any ball as in Theorem 2.1, $|\Phi_h(A)/\Phi_u(A) - 1| \le \epsilon_L/(1 - \epsilon_L)$ whenever $\epsilon_L = C_b \gamma^L < 1$, with $uPu = R\pi$ about its center.*

Proof. Averaging preserves mean one and the floor; Jensen preserves the cap. Positive reconstruction exists by Lemma 3.1 below. Both retentions lie in $[0, \epsilon_L]$. Subtracting their complementary conductances proves the bound. □

3 Failure of exact radial domination

Lemma 3.1. *For any positive w there is a unique positive solution of $uPu = w$. If $w = 1 + af$, $\|f\|_\infty \le 1$, $B = (\lceil L/2 \rceil + 2)/2$ and $4B^2|a| < 1$, then*

$$u = 1 + a(I + P)^{-1} f + e, \qquad \|e\|_\infty \le 4B^3 a^2.$$

Proof. Minimize $\frac{1}{2} \sum_{x,y} \mu(x) P(x,y) \exp(v_x + v_y) - \sum_x \mu(x) w(x) v_x$. Positive diagonals give strict convexity. For $m = \max v_x$ a diagonal term gives a lower bound of the form $ce^{2m} - m\mathbb{E}w$. On a sublevel set m is bounded above; positivity of all linear weights then also bounds each coordinate below. Stationarity gives $u = e^v$ as required.

Write $T = (I - P)/2$. Its absolute row sums are at most one, and its L^2 norm is at most one half. Therefore $\|T^j\|_{\infty \to \infty} \le \min(1, 2^{L/2 - j})$. Summation gives $\|(I + P)^{-1}\|_{\infty \to \infty} \le B$. With $H = (I + P)^{-1}$, solve $z = H(af - zPz)$ on the sup ball of radius $2B|a|$. The displayed condition makes the map a contraction into that ball. Its fixed point satisfies $\|z - aHf\|_\infty \le 4B^3 a^2$, and $1 + z$ is positive. This proves the expansion. □

Theorem 3.2. *Fix $\rho, b \in (0,1)$ and $C > 0$. Put $s = (1 + \rho)/(1 - \rho)$ and $\alpha = 1/(1 + 32s)$. For all sufficiently large $L = 4m$ there are potentials satisfying the standing hypotheses and exact matching through $\lfloor C\sqrt{L} \rfloor$ for which $r_h(A) > r_u(A)$. Here A is the ball about all-plus whose radius is the largest odd integer at most αL. The same family works with a fixed nonzero perturbation amplitude as m varies. At half noise $\alpha = 1/97$, so Corollary 2.2 simultaneously bounds the relative difference of conductances.*

Proof. Use four disjoint coordinate blocks $B_1, \dots, B_4$ of size m and write $\xi_j = \chi_{B_j}$. Set $v = (\xi_1 - \xi_2)(\xi_3 - \xi_4)/4$, $z = \rho^{2m}$ and $h_0 = 1 + \delta v$. Then $Pv = zv$, $Rv = 0$, and

$$v^2 = (1 - \xi_1\xi_2 - \xi_3\xi_4 + \xi_1\xi_2\xi_3\xi_4)/4, \qquad w = h_0 P h_0 = 1 + (1 + z)\delta v + z\delta^2 v^2.$$

The common energy is $Z = 1 + z\delta^2/4$. No nonconstant Walsh term of w has degree below $2m$. Choose $0 < \delta < 1$ with $(1 - \delta)^2/(1 + \delta^2/4) \ge b$ and $(1 + \delta)^{2s} \le 2$. Then $h = h_0/\sqrt{Z}$ satisfies the

claims about its density in all large dimensions. Reconstruct $u_0 P u_0 = Rw$ and set $u = u_0/\sqrt{Z}$. Its stationary mass agrees with that of h on every radial set.

Let $a = z\delta^2$, $f = Rv^2$, $D = (I+P)^{-1}f$. If φ_j is the average of Walsh characters of degree j, then on shell i

$$D_i = \tfrac{1}{4}\left[\tfrac{1}{2} - \frac{2\varphi_{L/2}(i)}{1+z} + \frac{(-1)^i}{1+z^2}\right].$$

On odd shells complementary half-size subsets cancel, so $D_i \le -1/16$ when $z^2 \le 1/3$. On even shells $D_i \le 7/8$. Write $q_i = P(x, A)$ on shell i and $W_i = \mu(|x| = i)q_i$. Rows of noise from adjacent vertices have likelihood ratio bounded by s pointwise. When i is below the radius r,

$$W_i/W_{i+1} \le s(i+1)/(L-i) \le sr/(L-r+1) \le 1/32.$$

Each even shell up to r pairs with the following odd shell. It follows that

$$\langle \mathbf{1}_A D, P\mathbf{1}_A\rangle \le -(9/264)\langle \mathbf{1}_A, P\mathbf{1}_A\rangle.$$

The unnormalized internal mass for h_0 is at least the reference $\langle \mathbf{1}_A, P\mathbf{1}_A\rangle$: the linear perturbation vanishes by spherical averaging and the quadratic term $\delta^2\langle \mathbf{1}_A v, P(\mathbf{1}_A v)\rangle$ is nonnegative. Lemma 3.1 gives uniformly $u_0(x)u_0(y) = 1 + a(D(x)+D(y)) + O(B^3a^2)$. Summing on the same internal edges gives at most $1-(9/264)a$ times the reference for large m, since $B^3 a \to 0$. The conditional retentions have equal unnormalized stationary denominators. This proves the strict comparison. The guaranteed fractional reduction of the internal mass in this proof can decay exponentially. □

4 Coordinate cuts and projected edges

Balls of positive radius fraction use all coordinates. A second regime consists of cuts identified after few coordinate queries. Here matching gives a direct comparison of edge probabilities, even for pairs of rare patterns.

Theorem 4.1. *At any $0 < \rho < 1$, assume exact matching through k and $\|\pi\|_p \le B_0$ for $p > 1+\rho$. No stationary floor is needed. Put $\eta^2 = \rho\max(1, 1/(p-1))$, $e = B_0\eta^{k+1}$, $u_m = (1+1/\rho)^{m/2}e$, and $\Delta_m = e^2 + 2u_m + u_m^2$. On any m coordinates the projected actual and ordinary stationary pair probabilities satisfy*

$$|J_I(i,j)/J_{0,I}(i,j) - 1| \le \Delta_m$$

for every pair of coordinate patterns. A cut D decided after at most $d \le k$ coordinate queries, of mass $\tau > 0$, consequently has $r_h(D) \le (1+\Delta_{\min(2d,L)})\tau^{(1-\rho)/(1+\rho)}$.

Proof. Put $Q = P_{\sqrt{\rho}}$, $c = \mathbb{E}h$ and $v = 1 - c^2 = \|Qh - c\|_2^2$. Let $p' = p/(p-1)$. For $p' > 2$, split $Q = P_{1/\sqrt{p'-1}} P_{\sqrt{\rho(p'-1)}}$. The first factor is contractive from L^2 to $L^{p'}$; on degrees $> k$ the second has L^2 norm at most η^{k+1}. For $p' \le 2$ use monotonicity and the noise multipliers. Thus the part of $g = Q(Qh)$ above degree k has p' norm at most $\eta^{k+1}\sqrt{v}$. Matching gives $\mathbb{E}\pi g \le c + e\sqrt{v}$. By Cauchy–Schwarz, $1 = (\mathbb{E}hg)^2 \le c\,\mathbb{E}(hg^2) = c\,\mathbb{E}\pi g$, hence $\sqrt{v} \le ce$.

Let Q_{out} apply Q only to the coordinates outside I. By the product kernel formula, $J_I(i,j)/J_{0,I}(i,j) = \langle F_i, F_j\rangle$ with $F_i = (Q_{\text{out}}h)(i,\cdot)$ in the complementary uniform space. Expand $Qh - c$ in Walsh characters on I with vector coefficients b_T. Their squared norms sum to v. Inverting Q on I multiplies the coefficients by $\rho^{-|T|/2}$; Cauchy–Schwarz now gives $\|F_i - c\|_2 \le (1+1/\rho)^{m/2}\sqrt{v} \le u_m$. Expanding the inner product and using $1 - c^2 = v \le e^2$ proves the estimate.

A decision tree has disjoint accepting leaves and their one-time probabilities match the uniform probabilities. A pair of leaves restricts at most $2d$ distinct coordinates. Sum the joint atom inequalities on these coordinates, then over all accepting pairs. Finally $\langle \mathbf{1}_D, P\mathbf{1}_D \rangle = \|Q\mathbf{1}_D\|_2^2 \leq \mu(D)^{2/(1+\rho)}$ by forward hypercontractivity. Division by $\tau = \mu(D)$ finishes the proof. □

5 Scope, antecedents and review context

Bonami's inequality and Boolean noise analysis provide the forward hypercontractive estimates [1, 2]. Polyanskiy and Samorodnitsky study improved hypercontractivity on small supports and concentration on the cube [3]. Their reference semigroup is the ordinary cube semigroup. Here the potential alters both stationary mass and transition probabilities; the row-entropy bound controls the extra reciprocal noise average. Boyd and collaborators analyze symmetry reduction for reversible chains with symmetries [4]. Reconstructing a new chain from spherical averages of stationary masses changes the transition law. Earlier work in this programme develops moment transfer and full spectrum estimates for specific Boolean Doob families [5]. Our estimates address killed operators, projected edges and a comparison between two different chains. Global priority, optimal ball radius and the full gap for unrestricted potentials remain open.

Model assessments and review context

The author used OpenAI Codex for analytic development and manuscript preparation. The mathematical checks represented here are same-context algebraic checks. No numerical oracle or proof-assistant verification is asserted. Any exact-PDF assessment, its score, model identification and involvement declaration belong to the version-specific publication record. No score from the earlier working paper has been carried over.

References

[1] A. Bonami, *Etude des coefficients de Fourier des fonctions de $L^p(G)$*, Ann. Inst. Fourier 20 (1970), 335–402. https://doi.org/10.5802/aif.357.

[2] R. O'Donnell, *Analysis of Boolean Functions*, corrected edition. https://arxiv.org/abs/2105.10386.

[3] Y. Polyanskiy and A. Samorodnitsky, *Improved log-Sobolev inequalities, hypercontractivity and uncertainty principle on the hypercube*, J. Funct. Anal. 277 (2019). https://arxiv.org/abs/1606.07491.

[4] S. Boyd, P. Diaconis, P. Parrilo and L. Xiao, *Symmetry Analysis of Reversible Markov Chains*, Internet Math. 2 (2005), 31–71. https://web.stanford.edu/~boyd/papers/symmetry.html.

[5] L. Eriksson, *Moment transfer and spectral bounds for Boolean Doob kernels*, AIRR Working paper, v3 (2026). https://airr.science/papers/ARR-2026-4T35VQ6MZB8F99F4/versions/v3/.