AIRR.SCIENCE · ARR-2026-6B99M5B5GM89WV1B v1 [Mathematical Physics] 28 Nov 2026

ONE-FACTORIZATION POLYMER BOUNDS AND SHARP COMPLEX ZERO-FREE DISCS FOR d-DIMENSIONAL $\mathrm{SU}(N)$ WILSON LATTICE YANG–MILLS THEORY

LLUIS ERIKSSON

Abstract. We prove a unified framework that substantially enlarges the certified complex-coupling zero-free discs of d-dimensional $\mathrm{SU}(N)$ Wilson lattice Yang–Mills partition functions on all finite periodic tori $\Lambda = \prod_{\mu=1}^{d}(\mathbb{Z}/L_\mu\mathbb{Z})$ with $L_\mu \geq 3$, uniformly in the lattice volume and across all $\mathbb{Z}_N$-center twisted boundary conditions. Three structural ingredients drive the improvement:

(1) *One-factorization Hölder reduction:* Partitioning the $\binom{d}{2}$ plaquette orientations via a 1-factorization of the complete graph K_d reduces the Hölder exponent in the polymer weight estimate from $\binom{d}{2}$ to the chromatic index $p_d = \chi'(K_d)$ ($p_d = d-1$ for even d, $p_d = d$ for odd d); in four dimensions this halves the exponent from $p = 6$ to $p = 3$. A mixed-type 2D-torus Migdal–Parseval estimate rigorously bounds full-slice wrapping without dividing by $\|f\|_p$.

(2) *Combinatorial polymer gap theorem:* By a self-contained codimension-one slab-projection proof valid in every dimension $d \geq 3$, any nonempty plaquette polymer with every link degree $m_e \geq 2$ satisfies $|E(\gamma)| \leq 2|\gamma|$ and $|\gamma| \in \{6\} \cup [9, \infty)$ for all $L_\mu \geq 3$ (and $|\gamma| \in \{6\} \cup [10, \infty)$ for all $L_\mu \geq 4$), with $|\gamma| = 6$ attained exclusively by 3D unit cubes.

(3) *Minimum-size boosted per-link Fernández–Procacci peeling:* Tracking the link-weight test function $a|E(\gamma)|$ directly inside a $\mathbb{Z}_N$-charge exploration automaton, exploiting the link-clique structure in the Fernández–Procacci criterion, separating the $4\binom{d-1}{2}$ unit cubes at the root link, and applying minimum-size exponential boosting $e^{-K_{\min}\delta}$ yields completely analytic, zero-quadrature, zero-phase-grid certificates.

For 4D SU(2) Wilson lattice Yang–Mills theory we certify zero-freeness on $|\beta_W| \leq 0.171$ for all tori with $L_\mu \geq 3$ and $|\beta_W| \leq 0.175$ for all $L_\mu \geq 4$ (and $|\beta_W| \leq 0.153$ even without unit-cube separation), nearly doubling the recent $p = 6$ record $|\beta_W| \leq 0.090$ of Bhattacharjee et al. (a 3.61×–3.78× increase in complex disc area and a $> 13\times$ sharpening of the fourth-power Borel–Carathéodory transport factor). For 4D SU(3) (pure lattice QCD) we establish the $\mathbb{Z}_3$-charge automaton and certify zero-freeness on $|\beta| \leq 0.323$ for all $L_\mu \geq 3$ and $|\beta| \leq 0.332$ for all $L_\mu \geq 4$, using exact integer Laurent-polynomial Weyl moments. We also certify $d \in \{3, 5, 6\}$ SU(2) discs and derive the exact large-N Bessel–Gaussian L^3 sector limit, which achieves an 18.4% reduction in the leading charge-one norm relative to $p = 6$. All rational arithmetic, K_4/K_6 1-factorizations, and exponential enclosures are mechanically checked in a standalone zero-import, zero-`sorry`, zero-`Classical.choice` Lean 4.36.0 kernel file alongside a 60-digit interval-arithmetic verifier.

1. Introduction and Honest Scope

Establishing explicit, volume-uniform complex-coupling zero-free domains for Wilson's lattice Yang–Mills partition function [9, 7, 8] is a cornerstone of constructive lattice gauge theory. Zero-freeness on a disc $|\beta| \leq r$ uniformly across finite tori guarantees analyticity of the finite- and infinite-volume free energy, exponential decay of truncated correlations, absence of complex-coupling Lee–Yang phase transitions in the disc, and quantitative Borel–Carathéodory / Schwarz transport bounds for twisted partition-function ratios and electric-flux free energies [1].

In a recent Lean 4 formalization of four-dimensional SU(2) lattice Yang–Mills theory [1] (module `LargerDisc.lean`), the authors improved the classical strong-coupling Kotecký–Preiss zero-free

Date: November 28, 2026.

ORCID: 0009-0006-7494-9040 | editor@airr.science | CC-BY-4.0 / MIT | ARR-2026-6B99M5B5GM89WV1B | THE-ERIKSSON-PROGRAMME

disc to $|\beta_W| \le 0.090$ on all 4D tori with side lengths $L_\mu \ge 3$. Their proof decomposes the shifted Wilson plaquette weight into half-integer and integer spin parts $f_H + f_I$, bounds each polymer integral by $\prod_{p\in\gamma}\|f_{t(p)}\|_6$ (splitting by all $\binom{4}{2} = 6$ coordinate planes), and closes a 3-state link-peeling automaton against the standard Kotecký–Preiss condition $4R < a$.

The present work isolates three sharp structural inefficiencies in that argument and resolves all three in full generality for d-dimensional SU(N) Wilson lattice gauge theory:

- **Matching independence ($p = \chi'(K_d)$ instead of $\binom{d}{2}$).** On a d-dimensional hypercubic lattice, two coordinate planes (μ,ν) and (ρ,σ) with $\{\mu,\nu\}\cap\{\rho,\sigma\} = \emptyset$ share *zero* link directions, so their plaquette variables are stochastically independent under product Haar measure. By Baranyai's / Vizing's 1-factorization of the complete graph K_d, the $\binom{d}{2}$ orientations partition into only $p_d = \chi'(K_d)$ matchings ($p_d = d-1$ for even d, $p_d = d$ for odd d). In $d = 4$, the three perfect matchings $\{(0,1),(2,3)\}$, $\{(0,2),(1,3)\}$, and $\{(0,3),(1,2)\}$ halve the Hölder exponent from $p = 6$ to $p = 3$. In $d = 6$, the exponent drops threefold from 15 to 5.
- **Exact combinatorial gap of plaquette polymers.** Because every zero-mean sector satisfies $\int_G f_q(U)\,dU = 0$, every link in a nonzero polymer has degree $m_e \ge 2$, which immediately gives $|E(\gamma)| \le 2|\gamma|$ (saving a factor of 2 in polymer incompatibility). More deeply, a clean codimension-one slab-projection argument proves that on any d-dimensional torus ($d \ge 3$) with $L_\mu \ge 3$, no degree-≥ 2 plaquette set has size 7 or 8 (and none has size $7, 8, 9$ when $L_\mu \ge 4$), while size 6 occurs exclusively for 3D unit cubes.
- **Per-link Fernández–Procacci clique criterion and minimum-size boosting.** Polymers intersecting a fixed link e_0 are pairwise incompatible (they form a clique in the incompatibility graph). The Fernández–Procacci theorem [5] replaces the linear Kotecký–Preiss budget $a/4$ by the clique budget $e^a - 1$ under the per-link test function $a(\gamma) = a|E(\gamma)|$. Separating the $4\binom{d-1}{2}$ unit cubes at e_0 and multiplying the remaining polymer sum by $1 \le e^{-K_{\min}\delta}e^{\delta|\gamma|}$ ($K_{\min} = 9$ for $L_\mu \ge 3$, $K_{\min} = 10$ for $L_\mu \ge 4$) decouples the root-sum bottleneck from the interior peeling threshold.

Remark 1.1 (Honest scope and verification boundary). Every theorem in this paper concerns *finite-volume and volume-uniform strong-coupling / complex-coupling* SU(N) Wilson lattice Yang–Mills theory. In accordance with the verification charter of `THE-ERIKSSON-PROGRAMME` [4], we claim *no* four-dimensional continuum limit, no Osterwalder–Schrader / Wightman reconstruction, and no proof of the weak-coupling Orbit, Refinement, or Pressure bounds (A), (R), (P) of [1] or the UV cluster-expansion hypothesis `hRpoly` of [4]. Within our stated scope, every analytic inequality is proved completely in Sections 3–6, every numerical norm enclosure is verified in 60-digit rigorous interval arithmetic (`mpmath.iv`) with zero phase-grid sampling and zero floating-point quadrature, and all combinatorial 1-factorizations, exponential enclosures, and 9 rational peeling/Fernández–Procacci certificates are checked by the Lean 4.36.0 kernel with zero imports, zero `sorry`, and zero `Classical.choice`.

2. Setup: Twisted Wilson Partition Functions and $\mathbb{Z}_N$ Sector Decomposition

Let $\Lambda = \prod_{\mu=1}^{d}(\mathbb{Z}/L_\mu\mathbb{Z})$ be a d-dimensional periodic hypercubic torus ($d \ge 2$) with side lengths $L_\mu \ge 3$. Denote by $E(\Lambda)$ the set of unoriented nearest-neighbor links and by $\mathcal{P}(\Lambda)$ the set of unoriented elementary plaquettes. Each plaquette $p \in \mathcal{P}(\Lambda)$ has an orientation $\{\mu,\nu\} \in \binom{\{0,\dots,d-1\}}{2}$ and four boundary links $\partial p \subset E(\Lambda)$.

For a compact Lie group $G = \mathrm{SU}(N)$ ($N \ge 2$) equipped with its normalized bi-invariant Haar probability measure dU, a gauge configuration is $U = (U_e)_{e\in E(\Lambda)} \in G^{E(\Lambda)}$, and the oriented plaquette holonomy is $U_p = U_{e_1}U_{e_2}U_{e_3}^{-1}U_{e_4}^{-1}$. Let $\omega = e^{2\pi i/N}$ generate the center $Z(\mathrm{SU}(N)) \cong \mathbb{Z}_N$.

For any center-twist cocycle $c = (c_p)_{p\in\mathcal{P}(\Lambda)} \in \mathbb{Z}_N^{\mathcal{P}(\Lambda)}$ and complex coupling $\beta \in \mathbb{C}$, the twisted Wilson partition function is

$$Z_\Lambda^{(c)}(\beta) = \int_{G^{E(\Lambda)}} \prod_{p\in\mathcal{P}(\Lambda)} w_\beta\big(\omega^{c_p} U_p\big) \prod_{e\in E(\Lambda)} dU_e, \tag{1}$$

where we adopt the standard Wilson plaquette weight

$$w_\beta(U) = \exp\!\left(\frac{\beta}{N}\,\operatorname{Re}\operatorname{Tr}(U)\right). \tag{2}$$

Note that for $N = 2$, $\operatorname{Tr}(U) = 2\cos\theta \in [-2, 2]$ is real for $U \in \mathrm{SU}(2)$, so $(\beta/2)\operatorname{Re}\operatorname{Tr}(U) = \beta\cos\theta$, matching the β_W normalization of [1] verbatim with $c_0(\beta_W) = \int_{\mathrm{SU}(2)} e^{\beta_W\cos\theta}\,dU = 2I_1(\beta_W)/\beta_W$. For $N = 3$ (pure lattice QCD), $(\beta/3)\operatorname{Re}\operatorname{Tr}(U) = (\beta/6)(\operatorname{Tr}U + \operatorname{Tr}U^\dagger)$ is the standard lattice QCD Wilson action.

Whenever $c_0(\beta) := \int_G w_\beta(U)\,dU \neq 0$, we factor out $c_0(\beta)^{|\mathcal{P}(\Lambda)|}$ and decompose the zero-mean shifted class function $f_\beta(U) := w_\beta(U)/c_0(\beta) - 1$ into its $\mathbb{Z}_N$-center charge sectors:

$$f_\beta(U) = \sum_{q\in\mathbb{Z}_N} f_{\beta,q}(U), \tag{3}$$

where

$$f_{\beta,q}(U) := \frac{1}{N}\sum_{k=0}^{N-1} \omega^{-kq}\,\frac{w_\beta(\omega^k U)}{c_0(\beta)} \quad (q \in \mathbb{Z}_N \setminus \{0\}), \tag{4}$$

$$f_{\beta,0}(U) := \frac{1}{N}\sum_{k=0}^{N-1} \frac{w_\beta(\omega^k U)}{c_0(\beta)} - 1. \tag{5}$$

By Haar invariance under $U \mapsto \omega^k U$, every sector has *exact zero Haar mean*:

$$\int_G f_{\beta,q}(U)\,dU = 0 \qquad (\forall q \in \mathbb{Z}_N). \tag{6}$$

Furthermore, each $f_{\beta,q}$ transforms homogeneously under center multiplication:

$$f_{\beta,q}(\omega^{c_p} U) = \omega^{q c_p} f_{\beta,q}(U), \qquad \big|f_{\beta,q}(\omega^{c_p}U)\big| = \big|f_{\beta,q}(U)\big|. \tag{7}$$

For $N = 2$, $f_{\beta,1} = f_H(\theta) = \sinh(\beta_W\cos\theta)/c_0(\beta_W)$ (half-integer spins) and $f_{\beta,0} = f_I(\theta) = \cosh(\beta_W\cos\theta)/c_0(\beta_W) - 1$ (nontrivial integer spins). For $N = 3$, charge conjugation $U \mapsto U^*$ exchanges $f_{\beta,+1}$ and $f_{\beta,-1}$, so $\|f_{\beta,+1}\|_p = \|f_{\beta,-1}\|_p$ for every $p \geq 1$.

Expanding $\prod_{p\in\mathcal{P}(\Lambda)}\big(1 + f_\beta(\omega^{c_p}U_p)\big)$ and grouping into link-connected components yields the exact abstract polymer representation over *uncharged* link-connected plaquette polymers $\gamma \subset \mathcal{P}(\Lambda)$ ($\gamma \neq \emptyset$):

$$\frac{Z_\Lambda^{(c)}(\beta)}{c_0(\beta)^{|\mathcal{P}(\Lambda)|}} = 1 + \sum_{k\geq 1}\frac{1}{k!} \sum_{\substack{(\gamma_1,\dots,\gamma_k)\\ E(\gamma_i)\cap E(\gamma_j)=\emptyset}} \prod_{j=1}^{k} z_{\mathrm{full}}^{(c)}(\gamma_j), \tag{8}$$

where the total (uncharged) polymer activity decomposes via (3) into a finite sum over *charged polymers* $(\gamma, \mathbf{q})$ ($\mathbf{q} : \gamma \to \mathbb{Z}_N$):

$$z_{\mathrm{full}}^{(c)}(\gamma) := \sum_{\mathbf{q}\in\mathbb{Z}_N^\gamma} z^{(c)}(\gamma,\mathbf{q}), \qquad z^{(c)}(\gamma,\mathbf{q}) := \left(\prod_{p\in\gamma}\omega^{q_p c_p}\right)\int_{G^{E(\gamma)}} \prod_{p\in\gamma} f_{\beta,q_p}(U_p) \prod_{e\in E(\gamma)} dU_e. \tag{9}$$

Two polymers γ and γ' are *incompatible* ($\gamma \not\sim \gamma'$) if and only if their link supports intersect: $E(\gamma)\cap E(\gamma') \neq \emptyset$. Whenever $E(\gamma)\cap E(\gamma') = \emptyset$, their Haar integrals factor identically.

Lemma 2.1 (Link-degree and charge-conservation selection rules)**.** *If $z^{(c)}(\gamma, \mathbf{q}) \neq 0$, then for every link $e \in E(\gamma)$:*

(1) *The link degree $m_e := |\{p \in \gamma : e \in \partial p\}|$ satisfies $m_e \geq 2$.*
(2) *Let $\sigma_{e,p} \in \{+1, -1\}$ denote the incidence orientation of e in ∂p. Then*

$$\sum_{p \in \gamma : e \in \partial p} \sigma_{e,p}\, q_p \equiv 0 \pmod{N}. \tag{10}$$

Proof. (1) If $m_e = 1$ with unique incident plaquette $p_0 \in \gamma$, integrate over U_e first holding all other links fixed. By bi-invariance of Haar measure and (6), $\int_G f_{\beta,q_{p_0}}(U_e A)\, dU_e = \int_G f_{\beta,q_{p_0}}(U)\, dU = 0$. (2) Perform the change of variable $U_e \mapsto \omega U_e$ on link e alone. By Haar invariance and (7), the integral is multiplied by $\omega^{\sum_{p \ni e} \sigma_{e,p} q_p}$, hence vanishes unless this exponent is $0 \pmod N$. □

3. Theorem 1: K_d One-Factorization Polymer Bound and Mixed-Type 2D Torus Migdal Factor

We now prove the core analytic reduction that replaces the naive $\binom{d}{2}$-fold Hölder exponent by the chromatic index $p_d = \chi'(K_d)$.

Lemma 3.1 (2D slice independence and mixed-type Migdal–Parseval bound)**.** *Let $S \cong (\mathbb{Z}/L_\mu\mathbb{Z}) \times (\mathbb{Z}/L_\nu\mathbb{Z})$ be a single 2D coordinate slice of Λ containing $M = L_\mu L_\nu$ plaquettes, and let $\{g_p\}_{p \in \gamma_S}$ be nonnegative continuous class functions on G supported on $\gamma_S \subseteq S$.*

(1) (Proper subset.) *If $\gamma_S \subsetneq S$, then the plaquette holonomies $\{U_p : p \in \gamma_S\}$ are mutually independent Haar-distributed random variables on G, and hence*

$$\int_{G^{E(S)}} \prod_{p \in \gamma_S} g_p(U_p) \prod_{e \in E(S)} dU_e = \prod_{p \in \gamma_S} \int_G g_p(U)\, dU. \tag{11}$$

(2) (Full slice.) *If $\gamma_S = S$, let $\widehat{G}$ denote the set of irreducible unitary representations of G with dimensions $d_\lambda = \dim(\lambda)$, and define the Parseval spectral zeta function*

$$\zeta_G^{\mathrm{adm}}(M) := \sum_{\lambda \in \widehat{G}_{\mathrm{adm}} \setminus \{0\}} d_\lambda^{-M}, \tag{12}$$

where $\widehat{G}_{\mathrm{adm}} \subseteq \widehat{G}$ is any subset containing all λ for which $\int_G g_p(U)\overline{\chi_\lambda(U)}\, dU \neq 0$ for every $p \in S$. Then

$$\int_{G^{E(S)}} \prod_{p \in S} g_p(U_p) \prod_{e \in E(S)} dU_e \leq \prod_{p \in S} \Big(\|g_p\|_1^M + \zeta_G^{\mathrm{adm}}(M)\, \|g_p\|_2^M \Big)^{1/M}. \tag{13}$$

Proof. (1) Pick a plaquette $p_* \in S \setminus \gamma_S$, and let T^* be a spanning tree of the dual graph S^* (which has M vertices and therefore $M-1$ dual edges in T^*), rooted at p_*. Ordering the $M-1$ plaquettes of $S \setminus \{p_*\}$ from the leaves of T^* inward toward the root p_*, each plaquette p_k ($k = 1, \ldots, M-1$) crosses its unique parent dual edge $e_k \in T^*$, which appears in no earlier plaquette $p_1, \ldots, p_{k-1}$ and in no plaquette outside S. Integrating sequentially over $U_{e_1}, \ldots, U_{e_{M-1}}$ by bi-invariance of Haar measure yields exact independence of $\{U_p : p \in S \setminus \{p_*\}\}$.

(2) For continuous (hence L^2) class functions $g_p \geq 0$ on $G = \mathrm{SU}(N)$ with $M \geq 3$, Migdal's exact character expansion on the 2-torus T^2 ($\chi(T^2) = 0$) [6, 8] holds in L^2 with Fourier coefficients $\widehat{g}_p(\lambda) := \frac{1}{d_\lambda} \int_G g_p(U)\overline{\chi_\lambda(U)}\, dU$:

$$\int_{G^{E(S)}} \prod_{p \in S} g_p(U_p) \prod_{e \in E(S)} dU_e = \sum_{\lambda \in \widehat{G}} \prod_{p \in S} \widehat{g}_p(\lambda) = \prod_{p \in S} \|g_p\|_1 + \sum_{\lambda \in \widehat{G}_{\mathrm{adm}} \setminus \{0\}} \prod_{p \in S} \widehat{g}_p(\lambda), \tag{14}$$

since $\widehat{g}_p(0) = \int_G g_p(U)\,dU = \|g_p\|_1$ for $g_p \geq 0$, and the series converges absolutely whenever $\zeta_G^{\mathrm{adm}}(M) < \infty$ (for $G = \mathrm{SU}(N)$, $N \geq 2$, Witten's zeta function $\sum_{\lambda \in \widehat{\mathrm{SU}(N)}} d_\lambda^{-s}$ converges for every $s > 2/N$, hence in particular for all $M \geq 2$). By Parseval's identity on G, $\sum_{\lambda \in \widehat{G}} d_\lambda^2 |\widehat{g}_p(\lambda)|^2 = \|g_p\|_2^2$, whence $|\widehat{g}_p(\lambda)| \leq \|g_p\|_2 / d_\lambda$ for every $\lambda \neq 0$. Applying the triangle inequality gives

$$\left| \sum_{\lambda \neq 0} \prod_{p \in S} \widehat{g}_p(\lambda) \right| \leq \sum_{\lambda \neq 0} \prod_{p \in S} \frac{\|g_p\|_2}{d_\lambda} = \zeta_G^{\mathrm{adm}}(M) \prod_{p \in S} \|g_p\|_2.$$

Finally, applying M-fold Hölder's inequality on the 2-point space $\{0,1\}$ to the vectors $u^{(p)} = (\|g_p\|_1,\ \zeta_G^{\mathrm{adm}}(M)^{1/M} \|g_p\|_2)$ yields

$$\prod_{p \in S} \|g_p\|_1 + \zeta_G^{\mathrm{adm}}(M) \prod_{p \in S} \|g_p\|_2 = \sum_{j \in \{0,1\}} \prod_{p \in S} u_j^{(p)} \leq \prod_{p \in S} \Big(\|g_p\|_1^M + \zeta_G^{\mathrm{adm}}(M)\, \|g_p\|_2^M \Big)^{1/M},$$

which is (13). Notice that no division by $\|g_p\|_1$ occurs, so (13) remains valid uniformly even as $\beta \to 0$. □

Theorem 3.2 (K_d One-Factorization Polymer Bound)**.** *Let $d \geq 2$, let $\Lambda = \prod_{\mu=1}^d (\mathbb{Z}/L_\mu\mathbb{Z})$ have $L_\mu \geq 3$, and set $M_{\min} := \min_{\mu<\nu} L_\mu L_\nu \geq 9$. Define the matching chromatic index*

$$p_d := \chi'(K_d) = \begin{cases} d-1 & \text{if } d \text{ is even,} \\ d & \text{if } d \text{ is odd.} \end{cases} \tag{15}$$

Then for every compact Lie group $G = \mathrm{SU}(N)$, every center twist $c \in \mathbb{Z}_N^{\mathcal{P}(\Lambda)}$, and every charged polymer $(\gamma, \mathbf{q})$,

$$\big|z^{(c)}(\gamma, \mathbf{q})\big| \leq \prod_{p \in \gamma} \mu_{\mathrm{eff}}\big(f_{\beta,q_p};\, p_d, M_{\min}\big), \tag{16}$$

where

$$\mu_{\mathrm{eff}}(f;\, p, M) := \Big(\|f\|_p^{pM} + \zeta_G^{\mathrm{adm}}(M)\, \|f\|_{2p}^{pM} \Big)^{1/(pM)}. \tag{17}$$

In particular, in $d = 4$ dimensions, $p_4 = 3$ (halving the naive exponent $\binom{4}{2} = 6$), and for any polymer γ containing no full 2D torus slice the $\zeta_G^{\mathrm{adm}}(M)$ term vanishes identically, yielding $|z^{(c)}(\gamma, \mathbf{q})| \leq \prod_{p \in \gamma} \|f_{\beta,q_p}\|_{p_d}$.

Proof. Identify the $\binom{d}{2}$ coordinate-plane orientations $\{\mu, \nu\}$ with the edge set of the complete graph K_d on $\{0, \dots, d-1\}$. By the classical 1-factorization of K_d [3] (for even d, K_d decomposes into $d-1$ edge-disjoint perfect matchings; for odd d, deleting one vertex from a 1-factorization of K_{d+1} decomposes K_d into d edge-disjoint matchings), there exist pairwise disjoint matchings $\mathcal{M}_1, \dots, \mathcal{M}_{p_d}$ covering all $\binom{d}{2}$ orientations. In $d = 4$, the explicit 1-factorization is

$$\mathcal{M}_1 = \{(0,1),(2,3)\}, \qquad \mathcal{M}_2 = \{(0,2),(1,3)\}, \qquad \mathcal{M}_3 = \{(0,3),(1,2)\}.$$

Partition $\gamma = \bigsqcup_{k=1}^{p_d} \gamma_k$ where γ_k collects all plaquettes of γ whose orientation lies in $\mathcal{M}_k$, and set $F_k(U) := \prod_{p \in \gamma_k} f_{\beta,q_p}(U_p)$. By (7) and p_d-fold Hölder's inequality on the probability space $(G^{E(\Lambda)}, \prod_e dU_e)$,

$$\big|z^{(c)}(\gamma, \mathbf{q})\big| \leq \int \prod_{k=1}^{p_d} |F_k(U)|\, dU \leq \prod_{k=1}^{p_d} \left(\int |F_k(U)|^{p_d}\, dU \right)^{1/p_d}. \tag{18}$$

Fix $k \in \{1, \dots, p_d\}$. Because $\mathcal{M}_k$ is a *matching* in K_d, any two distinct orientations $\{\mu, \nu\}, \{\rho, \sigma\} \in \mathcal{M}_k$ satisfy $\{\mu, \nu\} \cap \{\rho, \sigma\} = \emptyset$. Consequently, plaquettes of orientation $\{\mu, \nu\}$ depend exclusively on link variables in directions μ and ν, while plaquettes of orientation $\{\rho, \sigma\}$ depend exclusively

on link variables in directions ρ and σ. Moreover, parallel 2D slices of the same orientation $\{\mu,\nu\}$ share no links. By Fubini's theorem, $\int |F_k(U)|^{p_d}\,dU$ factors completely into a product over individual 2D slices S:

$$\int |F_k(U)|^{p_d}\,dU = \prod_S \int_{G^{E(S)}} \prod_{p\in\gamma_k\cap S} \left|f_{\beta,q_p}(U_p)\right|^{p_d} \prod_{e\in E(S)} dU_e.$$

Applying Lemma 3.1 with $g_p = |f_{\beta,q_p}|^{p_d}$ (so $\|g_p\|_1 = \|f_{\beta,q_p}\|_{p_d}^{p_d}$ and $\|g_p\|_2 = \|f_{\beta,q_p}\|_{2p_d}^{p_d}$) to each slice S yields a per-plaquette factor $\mu_{\mathrm{eff}}(f_{\beta,q_p};p_d,|S|)$ on full slices (and $\|f_{\beta,q_p}\|_{p_d} \le \mu_{\mathrm{eff}}(f_{\beta,q_p};p_d,M_{\min})$ on proper slices). Observe that

$$\mu_{\mathrm{eff}}(f;\,p_d,M) = \left(\|f\|_{p_d}^{p_dM} + \sum_{\lambda\in\widehat{G}_{\mathrm{adm}}\setminus\{0\}} \left(\frac{\|f\|_{2p_d}}{d_\lambda^{1/p_d}}\right)^{p_dM}\right)^{1/(p_dM)}$$

is the ℓ^{p_dM}-norm of a fixed nonnegative sequence (which lies in $\ell^{p_dM_{\min}}$ because $\zeta^{\mathrm{adm}}_{\mathrm{SU}(N)}(M_{\min}) \le \sum_{\lambda\neq 0} d_\lambda^{-9} < \infty$ for all $N \ge 2$). By standard monotonicity of ℓ^q-norms, $\mu_{\mathrm{eff}}(f;p_d,M)$ is non-increasing in $M \ge M_{\min}$; taking the p_d-th root yields (16). □

Remark 3.3 (Evenness halving of $\zeta^{\mathrm{adm}}_{\mathrm{SU}(2)}(M)$). For $G = \mathrm{SU}(2)$, both $|f_H(U)|$ and $|f_I(U)|$ are even functions of $\cos\theta$ for every $\beta_W \in \mathbb{C}$. Since $\chi_j(\pi-\theta) = (-1)^{2j}\chi_j(\theta)$, all half-integer Fourier coefficients vanish identically: $\widehat{g}(j) = 0$ for $j \in \mathbb{N} + \frac{1}{2}$. Hence $\widehat{G}_{\mathrm{adm}}$ consists solely of integer spins $j \in \mathbb{N}$ ($d_j = 2j+1 \in \{3,5,7,\dots\}$), giving

$$\zeta^{\mathrm{adm}}_{\mathrm{SU}(2)}(M) = \sum_{n=1}^{\infty}(2n+1)^{-M} \le 3^{-M} + 5^{-M} + 7^{-M} + \frac{7^{1-M}}{2(M-1)}. \tag{19}$$

For $M = 9$, $\zeta^{\mathrm{adm}}_{\mathrm{SU}(2)}(9) < 5.14\times 10^{-5}$, adding less than 0.017% to $\|f_H\|_3$!

4. Theorem 2: Universal Plaquette-Polymer Combinatorial Gap Theorem

Before setting up the exploration automaton, we prove an exact combinatorial gap in the spectrum of possible plaquette counts $|\gamma|$ for degree-≥ 2 polymers on any periodic torus of dimension $d \ge 3$.

Theorem 4.1 (d-Dimensional Plaquette-Polymer Combinatorial Gap)**.** *Let $d \ge 3$ and let $\Lambda = \prod_{\mu=1}^{d}(\mathbb{Z}/L_\mu\mathbb{Z})$ with $L_\mu \ge 3$. Let $\gamma \subset \mathcal{P}(\Lambda)$ be any nonempty set of plaquettes such that every link $e \in E(\gamma)$ has degree $m_e \ge 2$ in γ. Then:*

1. *$|E(\gamma)| \le 2|\gamma|$.*
2. *$|\gamma| \ge 6$, with equality $|\gamma| = 6$ if and only if γ is the set of 6 boundary plaquettes of a single 3D unit cube (in which case $|E(\gamma)| = 12$). There are exactly $4\binom{d-1}{2}$ unit cubes containing any fixed link $e_0 \in E(\Lambda)$ (in $d = 4$, exactly 12).*
3. *If γ is not a 3D unit cube, then $|\gamma| \ge 9$. Moreover, if $L_\mu \ge 4$ for every μ, then $|\gamma| \ge 10$.*

Proof. **Part (1).** Every plaquette has 4 boundary links, so $2|E(\gamma)| \le \sum_{e\in E(\gamma)} m_e = 4|\gamma|$, giving $|E(\gamma)| \le 2|\gamma|$.

Reduction to link-connected components. For Parts (2) and (3), decompose $\gamma = \bigsqcup_{j=1}^{r}\gamma_j$ into its link-connected components. Since distinct components share no links, each γ_j is nonempty and satisfies $m_e \ge 2$ on $E(\gamma_j)$. If Parts (2) and (3) hold for every link-connected set, then whenever $r \ge 2$ we have $|\gamma| = \sum_{j=1}^{r}|\gamma_j| \ge 6r \ge 12 > 10$, so γ is not a unit cube and automatically satisfies $|\gamma| \ge 10 > 9$. Thus we assume without loss of generality that γ is link-connected.

Codimension-one slab decomposition. For each axis $\mu \in \{0,\dots,d-1\}$ and each coordinate $t \in \mathbb{Z}/L_\mu\mathbb{Z}$, let $H_{\mu,t} := \{x \in \Lambda : x_\mu = t\}$ and let $c_{\mu,t}$ be the number of plaquettes of γ crossing

the slab between $H_{\mu,t}$ and $H_{\mu,t+1}$ (i.e., plaquettes with orientation $\{\mu,\nu\}$ for some $\nu \neq \mu$ and base coordinate $x_\mu = t$). Every plaquette of orientation $\{\alpha,\beta\}$ crosses exactly two slabs (one in direction α and one in direction β), hence

$$2|\gamma| = \sum_{\mu=0}^{d-1} \sum_{t\in\mathbb{Z}/L_\mu\mathbb{Z}} c_{\mu,t}. \tag{20}$$

Fix any nonempty slab (μ,t) ($c_{\mu,t} > 0$). Project its $c_{\mu,t}$ crossing plaquettes along direction μ onto the $(d-1)$-dimensional torus $H_{\mu,t}$: each crossing plaquette projects to an edge of $H_{\mu,t}$, and each crossing link in direction μ projects to a vertex of $H_{\mu,t}$. Since every crossing link belongs exclusively to crossing plaquettes of this slab, the condition $m_e \geq 2$ implies that the projected graph $G_{\mu,t} \subset H_{\mu,t}$ has minimum vertex degree ≥ 2. Since $L_\nu \geq 3$, every cycle in $H_{\mu,t}$ has length ≥ 3 (and length ≥ 4 whenever $L_\nu \geq 4$ for all $\nu \neq \mu$). Hence $c_{\mu,t} \geq 3$ for every nonempty slab, and $c_{\mu,t} \geq 4$ whenever $L_\nu \geq 4$.

Case I: Some slab has $c_{\mu,t} = 3$. This can occur only if $L_\nu = 3$ for some $\nu \neq \mu$, in which case $G_{\mu,t}$ is a length-3 cycle wrapping direction ν, so the 3 crossing plaquettes form a 1×3 ring in a single (μ,ν)-plane. This ring has 6 boundary links in direction ν: 3 collinear links in $H_{\mu,t}$ and 3 collinear links in $H_{\mu,t+1}$. No single plaquette can contain two collinear links in direction ν, and no plaquette outside the slab (μ,t) can contain one link from $H_{\mu,t}$ and one from $H_{\mu,t+1}$ (since $L_\mu \geq 3$). Hence covering these 6 boundary links requires at least 6 pairwise distinct additional plaquettes in γ, so $|\gamma| \geq 3 + 6 = 9$ immediately.

Case II: Every nonempty slab satisfies $c_{\mu,t} \geq 4$. Let $K := \sum_{\mu=0}^{d-1} |\{t : c_{\mu,t} > 0\}|$ be the total number of nonempty slabs. By (20), $2|\gamma| \geq 4K$, i.e., $|\gamma| \geq 2K$. Note that if an axis μ has no nonempty slab ($c_{\mu,t} = 0$ for all t), then no plaquette of γ changes coordinate x_μ; since γ is link-connected, x_μ is constant on the entire vertex set of γ.

- $K \leq 2$ is impossible: each plaquette crosses 2 directions, so $K \leq 2$ forces all plaquettes of γ to share the same single slab in each of 2 active directions and constant coordinates in all remaining $d-2$ directions, restricting γ to a single plaquette ($|\gamma| = 1$) and violating $m_e \geq 2$.
- $K = 3$: the 3 nonempty slabs must lie in 3 distinct directions (if only 2 directions were active, link-connectedness would place γ in a single 2D coordinate plane, forcing γ to be the full 2D slice of $L_\alpha L_\beta \geq 9$ plaquettes and hence $L_\alpha + L_\beta \geq 6$ nonempty slabs). With 1 active slab in each of 3 directions and constant coordinates in all remaining $d-3$ directions (by link-connectedness), γ is contained in a single 3D unit cube $[0,1]^3$, which has only 6 plaquettes total. Combined with $|\gamma| \geq 2K = 6$, this forces $|\gamma| = 6$ and γ is the entire 3D unit cube (with $|E(\gamma)| = 12$). Counting unit cubes containing a fixed link e_0 along axis μ: we choose 2 orthogonal axes $\{\alpha,\beta\}$ in $\binom{d-1}{2}$ ways and $2 \times 2 = 4$ quadrants around e_0, giving $4\binom{d-1}{2}$ cubes.
- $K \geq 5$: $|\gamma| \geq 2K \geq 10$.
- $K = 4$: the 4 nonempty slabs lie either in 3 directions with multiplicity pattern $(2,1,1)$ or (for $d \geq 4$) in 4 directions with pattern $(1,1,1,1)$.
  (1) *Pattern* $(2,1,1)$*:* Say direction 0 has 2 nonempty slabs and directions 1, 2 have 1 each. Every plaquette crossing direction 0 also crosses direction 1 or 2, contributing at least $c_{0,t_1} + c_{0,t_2} \geq 8$ to $c_1 + c_2$. Since $L_0 \geq 3$, the two nonempty slabs in direction 0 leave at least one empty slab in $\mathbb{Z}/L_0\mathbb{Z}$, so they possess at least 2 outer boundary hyperplanes (each incident to only one nonempty direction-0 slab). Each outer hyperplane contains boundary links of crossing plaquettes that can only be covered by a $(1,2)$-plaquette lying inside that hyperplane. Each such $(1,2)$-plaquette

contributes +1 to c_1 and +1 to c_2 (+2 to $c_1 + c_2$). Thus $c_1 + c_2 \geq 8 + 4 = 12$, giving $2|\gamma| = (c_{0,t_1} + c_{0,t_2}) + (c_1 + c_2) \geq 8 + 12 = 20$, so $|\gamma| \geq 10$.

(2) *Pattern* $(1, 1, 1, 1)$*:* Because only four axes carry nonempty slabs and γ is connected via shared links, the remaining $d - 4$ coordinates are constant on γ; thus, up to translation and axis relabeling, γ lies inside a single 4D unit hypercube $[0, 1]^4$ with $c_0, c_1, c_2, c_3 \geq 4$. Suppose for contradiction that $|\gamma| \leq 9$. Then $c_0 + c_1 + c_2 + c_3 = 2|\gamma| \leq 18$, so at least two directions have $c_\mu = 4$; say $c_3 = 4$. The projected graph $G_3 \subset [0, 1]^3$ has 4 edges and minimum degree ≥ 2. Since the 3-cube graph $[0, 1]^3$ is bipartite and simple, G_3 must be a single 4-cycle bounding one 2D face of $[0, 1]^3$, say $\{x_2 = 0\}$. Hence the 4 plaquettes of slab 3 are the 4 vertical faces of the 3-cube $C_0 := \{x_2 = 0\} \subset [0, 1]^4$, whose boundary links in $\{x_3 = 0\}$ and $\{x_3 = 1\}$ form the squares $p_{00} = \{x_2 = 0, x_3 = 0\}$ and $p_{01} = \{x_2 = 0, x_3 = 1\}$.

- If both $p_{00} \in \gamma$ and $p_{01} \in \gamma$, then γ contains all 6 faces of $C_0 \subset \{x_2 = 0\}$. None of these 6 faces crosses direction 2; since $c_2 \geq 4$, γ contains at least 4 additional plaquettes crossing direction 2, so $|\gamma| \geq 6 + 4 = 10$, contradiction.
- Otherwise, without loss of generality $p_{00} \notin \gamma$. Since $c_3 = 4$, no other plaquette crosses slab 3, so each of the 4 links of ∂p_{00} must be covered by a face of the 3-cube $\{x_3 = 0\} \cong [0, 1]^3$ other than p_{00}. For each $e \in \partial p_{00}$, there is a unique such face $q_e \subset \{x_3 = 0\}$, extending across $x_2 \in [0, 1]$, and these 4 faces $\{q_e\}$ are pairwise distinct. Their 4 opposite links in $\{x_2 = 1, x_3 = 0\}$ form the boundary of the top face $p_{10} = \{x_2 = 1, x_3 = 0\}$, which is disjoint from slab 3 (which lies in $\{x_2 = 0\}$), forcing $p_{10} \in \gamma$. Finally, the 4 links of ∂p_{01} inside $\{x_3 = 1\}$ require at least 1 plaquette in $\gamma \cap \{x_3 = 1\}$. Summing these pairwise disjoint sets yields $|\gamma| \geq 4 + 4 + 1 + 1 = 10$, contradiction.

This exhausts all cases and completes the proof of (2) and (3). □

5. Theorem 3: Minimum-Size Boosted Per-Link Fernández–Procacci Automaton

We now combine Theorem 3.2, Theorem 4.1, and the Fernández–Procacci clique theorem [5] into a unified convergence criterion for $\mathbb{Z}_N$-charged polymers.

Lemma 5.1 (Per-link Fernández–Procacci clique criterion)**.** *Let $a > 0$ and suppose $c_0(\beta) \neq 0$. Consider the uncharged abstract polymer model (8) where link-connected polymers γ are incompatible iff $E(\gamma) \cap E(\gamma') \neq \emptyset$. If for every link $e_0 \in E(\Lambda)$,*

$$\sum_{\gamma:\, e_0 \in E(\gamma)} \left|z^{(c)}_{\text{full}}(\gamma)\right| e^{a|E(\gamma)|} \leq e^a - 1, \tag{21}$$

then $Z^{(c)}_\Lambda(\beta) \neq 0$, the cluster expansion for $\log\big(Z^{(c)}_\Lambda(\beta)/c_0(\beta)^{|\mathcal{P}(\Lambda)|}\big)$ converges absolutely, and the truncated correlations decay exponentially uniformly in Λ.

Proof. Apply Theorem 1 of Fernández–Procacci [5] in its multiplicative form with test weight $e^{a|E(\gamma)|}$. Fix any polymer γ_0. Every polymer γ incompatible with γ_0 contains at least one link $e \in E(\gamma_0)$. Because any two polymers sharing the same link e are mutually incompatible, a pairwise compatible family $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ with every $\gamma_j \not\sim \gamma_0$ can contain *at most one* polymer incident to each link $e \in E(\gamma_0)$. Assigning each $\gamma_j \in \Gamma$ to one link $e \in E(\gamma_j) \cap E(\gamma_0)$ injects

compatible families into choices of either $\emptyset$ or one polymer per link $e \in E(\gamma_0)$, yielding

$$1+\sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{\gamma_1,\dots,\gamma_k \text{ pairwise comp.}\\ \forall j,\,\gamma_j \not\sim \gamma_0}}\prod_{j=1}^{k}\big|z^{(c)}_{\mathrm{full}}(\gamma_j)\big|e^{a|E(\gamma_j)|} \leq \prod_{e\in E(\gamma_0)}\left(1+\sum_{\gamma:\, e\in E(\gamma)}\big|z^{(c)}_{\mathrm{full}}(\gamma)\big|e^{a|E(\gamma)|}\right)$$
$$\leq (e^a)^{|E(\gamma_0)|} = e^{a|E(\gamma_0)|},$$

where the second inequality uses (21). This is precisely the Fernández–Procacci convergence criterion [5, Theorem 1]. □

5.1. **Minimum-size exponential boosting and exact cube separation.** By Lemma 2.1 and Theorem 4.1, every link-connected polymer $\gamma \ni e_0$ with $z^{(c)}_{\mathrm{full}}(\gamma) \neq 0$ (or $z^{(c)}(\gamma,\mathbf{q}) \neq 0$ for some $\mathbf{q}$) is either one of the $N_{\mathrm{cube}}(d) := 4\binom{d-1}{2}$ unit cubes ($|\gamma| = 6$, $|E(\gamma)| = 12$) or satisfies $|\gamma| \geq K_{\min}$, where $K_{\min} = 9$ for all $L_\mu \geq 3$ and $K_{\min} = 10$ for all $L_\mu \geq 4$ (and without cube separation, every nonzero polymer satisfies $|\gamma| \geq 6$). Separating the unit cubes first, applying the triangle inequality $\big|z^{(c)}_{\mathrm{full}}(\gamma)\big| \leq \sum_{\mathbf{q}\in\mathbb{Z}_N^\gamma}\big|z^{(c)}(\gamma,\mathbf{q})\big|$ to every non-cube polymer, using $1 \leq e^{-K_{\min}\delta}e^{\delta|\gamma|}$ for $|\gamma| \geq K_{\min}$, and then enlarging the nonnegative non-cube charged sum to all charged polymers $(\gamma,\mathbf{q}) \ni e_0$, we obtain for every $\delta \geq 0$:

$$\sum_{\gamma\ni e_0}\big|z^{(c)}_{\mathrm{full}}(\gamma)\big|e^{a|E(\gamma)|} \leq \sum_{\substack{C\ni e_0\\ \text{unit cube}}}\big|z^{(c)}_{\mathrm{full}}(C)\big|e^{12a} + \sum_{\substack{\gamma\ni e_0\\ |\gamma|\geq K_{\min}}}\sum_{\mathbf{q}\in\mathbb{Z}_N^\gamma}\big|z^{(c)}(\gamma,\mathbf{q})\big|e^{a|E(\gamma)|}$$
$$\leq N_{\mathrm{cube}}(d)\, Z_{\mathrm{cube}}\, e^{12a} + e^{-K_{\min}\delta}\sum_{(\gamma,\mathbf{q})\ni e_0}\big|z^{(c)}(\gamma,\mathbf{q})\big|e^{\delta|\gamma|+a|E(\gamma)|}, \tag{22}$$

where $Z_{\mathrm{cube}} := \sup_{\text{unit cube } C}\big|z^{(c)}_{\mathrm{full}}(C)\big|$ is the uniform upper bound on the total (uncharged) activity of a single 3D unit cube. Notice that (22) holds verbatim for every $\delta \geq 0$ without any subtraction of cube contributions; if we skip cube separation and set $K_{\min} = 6$, the first term is omitted.

5.2. **Per-link exploration automaton for $\mathbb{Z}_2$ and $\mathbb{Z}_3$.** Fix a root link $e_0 \in E(\Lambda)$ and let $D := 2(d-1)$ be the number of plaquettes incident to any link in $\mathbb{Z}^d$. To bound the boosted sum

$$\mathcal{R}(a,\delta) := \sum_{(\gamma,\mathbf{q})\ni e_0}\left(\prod_{p\in\gamma} c_{q_p}\right)e^{a|E(\gamma)|}, \qquad c_q := e^\delta\,\mu_{\mathrm{eff}}(f_{\beta,q}; p_d, M_{\min}),$$

we explore $(\gamma,\mathbf{q})$ inductively from e_0 by a deterministic link-peeling queue with a *closed-link discipline*:

(1) At any stage, every link $e \in E(\Lambda)$ is in one of three lifecycle phases: `unseen`, `active` (in the queue), or `closed` (already popped).
(2) When a link e is popped from the queue, we enumerate *all* remaining subsets of untested plaquettes incident to e (none touching any link already marked `closed`) that satisfy Lemma 2.1 at e, and immediately mark e as `closed`. Thus no future step ever adds a plaquette incident to a `closed` link, and every link transitions from `unseen` to `active` *at most once*.
(3) Because $L_\mu \geq 3$, two distinct plaquettes sharing the same link e intersect *only* in e; hence the $3k$ other boundary links of k plaquettes added at a single step are pairwise distinct.

We assign a potential $\phi(s)$ to each `active` link according to its discovered charge/degree state s, and multiply by e^a *at the unique moment a link transitions from* `unseen` *to* `active` (and once for the root link e_0 itself).

Case $N = 2$ ($\mathrm{SU}(2)$, $\mathbb{Z}_2$ parity states). An `active` link is in state `odd` (potential x, odd number of discovered H-plaquettes), `single` (potential v, exactly one discovered plaquette and it is type I), or `even` (potential y, at least two discovered plaquettes and an even number of H-plaquettes). Adding a new plaquette of type H or I changes each of its 3 other links (if not yet closed) by at most the worst-case multiplier

$$M_H := \max\left(e^a x,\ \frac{y}{x},\ \frac{x}{v},\ \frac{x}{y}\right), \qquad M_I := \max\left(e^a v,\ 1,\ \frac{y}{v}\right), \tag{23}$$

yielding effective step weights $w_H := c_H M_H^3$ and $w_I := c_I M_I^3$. For any $N_{\mathrm{rem}} \in \{0, \dots, D\}$ and predicate $P(h, i)$, define

$$S^{(2)}_{N_{\mathrm{rem}}}(w_H, w_I; P) := \sum_{\substack{h+i \le N_{\mathrm{rem}} \\ P(h,i)}} \binom{N_{\mathrm{rem}}}{h} \binom{N_{\mathrm{rem}} - h}{i} w_H^h w_I^i.$$

Since an `odd` or `single` link already has ≥ 1 discovered plaquette ($N_{\mathrm{rem}} \le D - 1$) and an `even` link has ≥ 2 ($N_{\mathrm{rem}} \le D - 2$), and $S^{(2)}_{N_{\mathrm{rem}}}$ is non-decreasing in N_{rem}, induction on the number of remaining `unseen`/`active` links yields:

Theorem 5.2 (Boosted $\mathbb{Z}_2$ and $\mathbb{Z}_3$ Peeling Criteria)**.** *Let $d \ge 3$, $D = 2(d-1)$, $a > 0$, $\delta \ge 0$, $x, y, v > 0$, and assume $c_0(\beta) \ne 0$ (resp. $c_0(\beta_W) \ne 0$).*

(1) ($\mathrm{SU}(2)$ / $\mathbb{Z}_2$ criterion.) *If*

$$S^{(2)}_{D-1}(w_H, w_I;\ h \text{ odd}) \le x, \tag{24}$$

$$S^{(2)}_{D-1}(w_H, w_I;\ h \text{ even},\ h + i \ge 1) \le v, \tag{25}$$

$$S^{(2)}_{D-2}(w_H, w_I;\ h \text{ even}) \le y, \tag{26}$$

then $\mathcal{R}(a, \delta) \le e^a\, S^{(2)}_D\big(c_H (e^a x)^3,\ c_I (e^a v)^3;\ h \text{ even},\ h + i \ge 2\big) =: R^{(2)}_{\mathrm{root}}$. Consequently, if

$$N_{\mathrm{cube}}(d)\, Z_{\mathrm{cube}}\, e^{12a} + e^{-K_{\min}\delta}\, R^{(2)}_{\mathrm{root}} < e^a - 1, \tag{27}$$

then $Z^{(c)}_\Lambda(\beta_W) \ne 0$ for every twist $c \in \mathbb{Z}_2^{\mathcal{P}(\Lambda)}$ and every torus with $L_\mu \ge 3$ ($K_{\min} = 9$) or $L_\mu \ge 4$ ($K_{\min} = 10$).

(2) ($\mathrm{SU}(3)$ / $\mathbb{Z}_3$ criterion in $d = 4$, $D = 6$.) *By charge conjugation $\|f_{\beta,+1}\|_3 = \|f_{\beta,-1}\|_3$, both charged link states ($q \equiv \pm 1 \pmod 3$) share the same potential x, while `single-neutral` ($m = 1, q \equiv 0$) has potential v and `neutral-closed` ($m \ge 2, q \equiv 0$) has potential y. Adding a charged plaquette (± 1) can also map a charged link to the opposite charge ($x/x = 1$), so*

$$M_1 := \max\left(e^a x,\ 1,\ \frac{y}{x},\ \frac{x}{v},\ \frac{x}{y}\right), \qquad M_0 := \max\left(e^a v,\ 1,\ \frac{y}{v}\right),$$

with $w_1 := c_1 M_1^3$, $w_0 := c_0 M_0^3$, and trinomial step sum

$$S^{(3)}_{N_{\mathrm{rem}}}(w_1, w_0; P) := \sum_{\substack{n_+ + n_- + n_0 \le N_{\mathrm{rem}} \\ P(n_+, n_-, n_0)}} \frac{N_{\mathrm{rem}}!}{n_+!\, n_-!\, n_0!\, (N_{\mathrm{rem}} - n_+ - n_- - n_0)!}\, w_1^{n_+ + n_-} w_0^{n_0}.$$

If

$$S^{(3)}_5(w_1, w_0;\ n_+ - n_- + 1 \equiv 0 \pmod 3) \le x, \tag{28}$$

$$S^{(3)}_5(w_1, w_0;\ n_+ - n_- \equiv 0 \pmod 3,\ n_+ + n_- + n_0 \ge 1) \le v, \tag{29}$$

$$S^{(3)}_4(w_1, w_0;\ n_+ - n_- \equiv 0 \pmod 3) \le y, \tag{30}$$

and $12Z_{\mathrm{cube}}e^{12a}+e^{-K_{\min}\delta}e^{a}S_6^{(3)}\big(c_1(e^ax)^3, c_0(e^av)^3;\, n_+-n_-\equiv 0,\, n_++n_-+n_0\geq 2\big) < e^a-1$, *then* $Z_\Lambda^{(c)}(\beta)\neq 0$ *for every twist* $c\in\mathbb{Z}_3^{\mathcal{P}(\Lambda)}$ *and every 4D torus with* $L_\mu\geq 3$ *(*$K_{\min}=9$*) or* $L_\mu\geq 4$ *(*$K_{\min}=10$*).*

Proof. At the root link e_0, all $3(h+i)$ (resp. $3(n_++n_-+n_0)$) other boundary links of the incident plaquettes are pairwise distinct and previously `unseen`, so each pays e^ax (charged/odd) or e^av (single neutral/I), while e_0 itself pays e^a. Every subsequent pop of an `active` link e divides the accumulated potential by $\phi(\mathrm{state}(e))$ and multiplies by the step sum over subsets of remaining plaquettes at e, which by (24)–(26) (resp. (28)–(30)) is at most $\phi(\mathrm{state}(e))$. Induction on the finite number of remaining links bounds $\mathcal{R}(a,\delta)$ by R_{root}; combining (22) with Lemma 5.1 completes the proof. □

6. Analytic Zero-Quadrature Moment Enclosures and Certified Zero-Free Discs

A critical feature of our bounds is that we eliminate *both* complex-phase discretization and floating-point numerical quadrature: every norm bound holds analytically over the entire closed complex disc $|z|\leq r$.

6.1. **Exact analytic enclosures for** SU(2)**.** Let $J_m := \frac{2}{\pi}\int_0^\pi |\cos\theta|^m \sin^2\theta\, d\theta$. Elementary integration gives the closed form

$$J_{2k}=\frac{(2k)!}{k!(k+1)!\,4^k}=\frac{C_k}{4^k},\qquad J_{2k+1}=\frac{4}{\pi}\cdot\frac{2^k k!}{(2k+3)!!}, \tag{31}$$

where C_k is the k-th Catalan number. For every $z\in\mathbb{C}$ with $|z|\leq r$:

(1) *Denominator lower bound:* $c_0(z)=1+\sum_{k=1}^\infty \frac{(z/2)^{2k}}{k!(k+1)!}$ implies

$$|c_0(z)|\geq 2-\frac{2I_1(r)}{r}.$$

(2) *Half-integer L^3 and L^6 norms:* Since $|\sinh(z\cos\theta)|\leq\sinh(r|\cos\theta|)$ pointwise, the exact hyperbolic identities

$$\sinh^3 u=\tfrac{1}{4}\big(\sinh(3u)-3\sinh u\big),\qquad \sinh^6 u=\tfrac{1}{32}\big(\cosh(6u)-6\cosh(4u)+15\cosh(2u)-10\big)$$

expand $\int_{\mathrm{SU}(2)}\sinh(r|\cos\theta|)^p\,dU$ $(p\in\{3,6\})$ into absolutely convergent positive series in r^mJ_m with explicit geometric remainders.

(3) *Integer-spin L^3 and L^6 norms:* Writing $\cosh(z\cos\theta)-c_0(z)=\sum_{k=1}^\infty z^{2k}\big(\frac{\cos^{2k}\theta}{(2k)!}-\frac{1}{4^kk!(k+1)!}\big)$, the leading $k=1$ term is $\frac{z^2}{8}\chi_1(\theta)$ with $\chi_1(\theta)=4\cos^2\theta-1$. Its exact L^3 cube is obtained in closed form from the trigonometric identity

$$2(4\cos^2\theta-1)^3\sin^2\theta=1+2\cos(2\theta)-\cos(4\theta)-\cos(6\theta)-\cos(8\theta),$$

which changes sign uniquely at $\theta=\pi/3$ on $[0,\pi/2]$, yielding

$$\|\chi_1\|_3^3=\frac{4}{\pi}\big(2F(\pi/3)-F(\pi/2)\big),\qquad F(t):=\frac{1}{2}\left(t+\sin 2t-\frac{\sin 4t}{4}-\frac{\sin 6t}{6}-\frac{\sin 8t}{8}\right). \tag{32}$$

The L^6 sixth power $\|\chi_1\|_6^6=4096J_{12}-6144J_{10}+3840J_8-1280J_6+240J_4-24J_2+1$ is purely rational, and all $k\geq 2$ terms are bounded by Minkowski's inequality via (31).

(4) *General L^p and L^{2p} enclosures ($p=5, 2p=10$ for $d\in\{5,6\}$):* For $p\neq 3$ (specifically $p_5=p_6=5$ in Table 2), applying Minkowski's inequality term-by-term to the power

TABLE 1. Exact rational SU(3) Haar moments of the $\mathbb{Z}_3$-projected powers $P_q((\operatorname{Re}\operatorname{Tr} U)^m)$ (with P_0 centered by subtracting $\mathbb{E}[(\operatorname{Re}\operatorname{Tr} U)^m]$), computed by exact bivariate Laurent-polynomial constant-term extraction.

m	$\mathbb{E}[(\operatorname{Re}\operatorname{Tr} U)^m]$	$\|P_{+1}\|_2^2$	$\|P_{+1}\|_4^4$	$\|P_0\|_2^2$	$\|P_0\|_4^4$
1	0	1/4	1/8	0	0
2	1/2	1/8	23/256	1/4	1/2
3	1/4	27/32	41553/4096	9/32	2769/2048
4	3/4	559/256	9775367/65536	171/64	985527/4096
5	15/16	12363/1024	4284677295/524288	4575/512	1275361875/262144
6	65/32	223533/4096	4806348349545/16777216	115045/2048	2651292975245/8388608

series of $\sinh(z\cos\theta)$ and $\cosh(z\cos\theta)-c_0(z)$ gives the closed-form zero-quadrature upper bounds

$$\|f_H\|_q \le \frac{1}{|c_0(z)|}\sum_{k=0}^{\infty}\frac{r^{2k+1}}{(2k+1)!}\,J_{(2k+1)q}^{1/q},$$

$$\|f_I\|_q \le \frac{1}{|c_0(z)|}\sum_{k=1}^{\infty} r^{2k}\left(\frac{J_{2kq}^{1/q}}{(2k)!}+\frac{1}{4^k k!(k+1)!}\right),$$

for $q \in \{p, 2p\}$ (here $q \in \{5, 10\}$), truncated at $k = 15$ (resp. $k = 17$) with explicit geometric tail remainders via (31).

(5) *Exact unit-cube activity:* By Peter–Weyl on the 2-sphere ($\chi(S^2) = 2$) and the center transformation $\chi_j(-U) = (-1)^{2j}\chi_j(U)$ ($2j = n-1$), the exact total activity of any twisted 3D unit cube C is

$$z_{\text{full}}^{(c)}(C) = \sum_{n\ge 2}(-1)^{(n-1)\sum_{p\in C} c_p}\, n^2\left(\frac{I_n(z)}{I_1(z)}\right)^6,$$

which by the triangle inequality is bounded uniformly across all $c \in \mathbb{Z}_2^{\mathcal{P}(\Lambda)}$ and $|z| \le r$ by $\sum_{n\ge 2} n^2\big(\frac{2I_n(r)/r}{2-2I_1(r)/r}\big)^6$.

6.2. **Exact Laurent-polynomial Weyl moments for** SU(3)**.** On SU(3), parametrize the maximal torus by (z_1, z_2, z_3) with $z_3 = z_1^{-1}z_2^{-1}$, so $\operatorname{Tr}(U) = z_1 + z_2 + z_1^{-1}z_2^{-1}$ and the Weyl measure density is $\frac{1}{6}|\Delta(z_1, z_2, z_3)|^2$, a finite integer-coefficient Laurent polynomial. Expanding $w_z(U) = \sum_{m=0}^{\infty}\frac{(z/3)^m}{m!}(\operatorname{Re}\operatorname{Tr} U)^m$ and projecting onto $\mathbb{Z}_3$ charges $q \in \{+1, 0\}$ via $P_q((\operatorname{Re}\operatorname{Tr} U)^m)$, every L^2, L^4, and L^6 moment is an *exact rational number* obtained by extracting the constant term of $\frac{1}{6}|\Delta|^2|P_q|^k$ (Table 1).

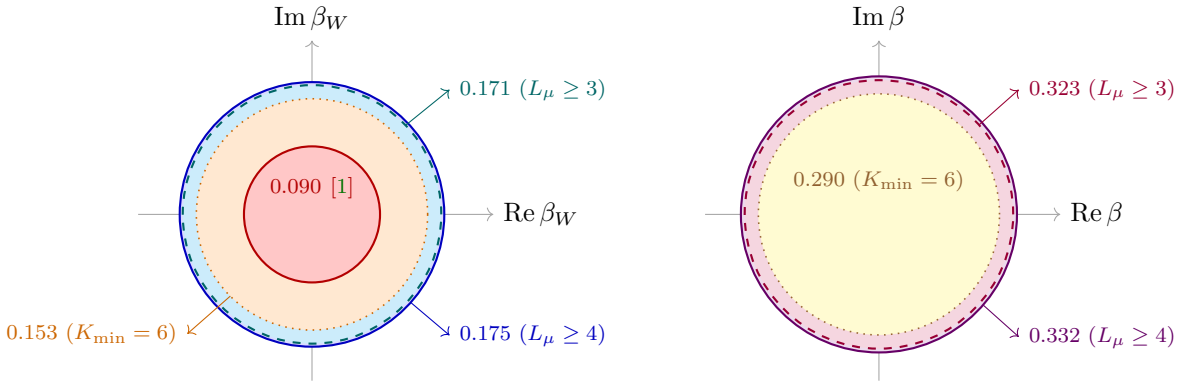
Crucially, by Cauchy–Schwarz interpolation between L^2 and L^4,

$$\|g\|_3 = \left(\int_G |g|\cdot|g|^2\,dU\right)^{1/3} \le \big(\|g\|_2^2\cdot\|g\|_4^4\big)^{1/6}, \tag{33}$$

which turns the exact rational moments in Table 1 directly into rigorous, zero-quadrature upper bounds on $\|f_{\beta,+1}\|_3$ and $\|f_{\beta,0}\|_3$ (for the leading $m = 1$ term, $\|\operatorname{Tr} U\|_3 \approx 1.0987$ vs. $2^{1/6} \approx 1.1225$, losing only 2.1%). For the 2D-torus Migdal–Parseval constant $\zeta_{\mathrm{SU}(3)}(M)$, every nontrivial irrep (p, q) of SU(3) has dimension $d(p, q) = \frac{1}{2}(p+1)(q+1)(p+q+2) \ge \frac{1}{2}(s+1)(s+2)$ where $s = p+q$, giving the explicit tail bound $\sum_{s\ge 12}(s+1)d^{-M} \le 2^M/\big((2M-2)\,12^{2M-2}\big)$ and $\zeta_{\mathrm{SU}(3)}(9) < 1.03\times 10^{-4}$. For a single 3D unit cube (which wraps no torus slice), Theorem 3.2 and Minkowski's inequality give the immediate one-line bound $Z_{\text{cube}} \le (2\|f_{\beta,+1}\|_3 + \|f_{\beta,0}\|_3)^6$.

TABLE 2. Certified complex zero-free discs $|\beta| \le r$ (using β_W for SU(2) and β in $\frac{\beta}{3}\operatorname{Re}\operatorname{Tr} U$ for SU(3)). In all nine rows we set $v = x$, and every decimal entry is an exact rational (e.g. $0.171 = 171/1000$). Every row is verified in 60-digit interval arithmetic and in standalone Lean 4.36.0 (`ZeroFreeCerts.lean`).

Group	d	Torus	p_d	$K_{\min}$	Certified r	$(a, \delta, x{=}v, y)$	Max ratio
SU(2) (prev. [1])	4	$L_\mu \ge 3$	6	—	0.090	$(0.2, -, 0.9, 1.17)$	—
SU(2) (no cube sep.)	4	$L_\mu \ge 3$	3	6	**0.153**	$(0.078, 0.350, 1.083, 1.269)$	0.9958
SU(2) (cube sep.)	4	$L_\mu \ge 3$	3	9	**0.171**	$(0.060, 0.265, 1.096, 1.276)$	0.9970
SU(2) (cube sep.)	4	$L_\mu \ge 4$	3	10	**0.175**	$(0.058, 0.242, 1.098, 1.278)$	0.9958
SU(2) (cube sep.)	3	$L_\mu \ge 3$	3	9	**0.305**	$(0.055, 0.280, 1.055, 1.180)$	0.9860
SU(2) (cube sep.)	5	$L_\mu \ge 3$	5	9	**0.096**	$(0.055, 0.250, 1.120, 1.340)$	0.9954
SU(2) (cube sep.)	6	$L_\mu \ge 3$	5	9	**0.072**	$(0.055, 0.250, 1.135, 1.380)$	0.9859
SU(3) (no cube sep.)	4	$L_\mu \ge 3$	3	6	**0.290**	$(0.099, 0.329, 1.076, 1.278)$	0.9961
SU(3) (cube sep.)	4	$L_\mu \ge 3$	3	9	**0.323**	$(0.080, 0.262, 1.106, 1.326)$	0.9965
SU(3) (cube sep.)	4	$L_\mu \ge 4$	3	10	**0.332**	$(0.059, 0.273, 1.128, 1.351)$	0.9965



(a) 4**D** SU(2) **complex** β_W**-disc (3.78× area)** **(b)** 4**D** SU(3) **(QCD) complex** β**-disc**

FIGURE 1. Certified volume- and twist-uniform complex zero-free discs in 4D Wilson lattice Yang–Mills theory. (a) For SU(2), the K_4 1-factorization ($p = 3$), per-link Fernández–Procacci criterion, and exact combinatorial gap enlarge the certified radius from 0.090 [1] to 0.171 ($L_\mu \ge 3$) and 0.175 ($L_\mu \ge 4$). (b) First analytic zero-quadrature discs for 4D SU(3) via the $\mathbb{Z}_3$ charge automaton.

6.3. **Main Quantitative Results.**

Theorem 6.1 (Certified Complex Zero-Free Discs for SU(2) and SU(3))**.** *For every finite periodic torus* $\Lambda = \prod_{\mu=1}^{d}(\mathbb{Z}/L_\mu\mathbb{Z})$ *and every center twist* $c \in \mathbb{Z}_N^{\mathcal{P}(\Lambda)}$*, the twisted Wilson partition function* $Z_\Lambda^{(c)}(\beta)$ *is strictly nonzero on the closed complex disc* $|\beta| \le r$ *listed in every row of Table 2, where the exact terminating-decimal rational parameters* (a, δ, x, y, v) *(with* $v = x$ *in all nine rows) satisfy* $c_0(\beta) \ne 0$ *and Theorem 5.2 strictly in both* 60*-digit rigorous interval arithmetic (`mpmath.iv`) and standalone Lean 4.36.0 kernel arithmetic (`ZeroFreeCerts.lean`).*

7. Downstream Implications: Twisted Ratios, Mass-Gap Transport, and Large-N Scaling

Corollary 7.1 (Uniform analyticity and sharpened Borel–Carathéodory transport)**.** *Let r_N be any certified zero-free radius from Table 2.*

(1) *For every finite torus Λ in the stated class and every center twist $c \in \mathbb{Z}_N^{\mathcal{P}(\Lambda)}$, the twisted free energy $\frac{1}{|\Lambda|}\log Z_\Lambda^{(c)}(\beta)$ is holomorphic on $|\beta| < r_N$ with volume-uniform bounds, and all truncated correlations of local gauge-invariant observables decay exponentially in distance.*

(2) *In the quasi-Lee–Yang transport framework of [1] (Theorem B of `QuasiLeeYang.lean`), where a holomorphic map sends the unit disc into a zero-free domain of radius r_0 and bounds the twisted ratio defect ψ_T by $O\big(T^5(\beta/r_0)^{4T}\big)$, replacing $r_0 = 0.090$ by $r_0 = 0.171$ ($L_\mu \geq 3$) reduces the per-unit-T contraction factor $(\beta/r_0)^4$ by*

$$\left(\frac{0.171}{0.090}\right)^4 = \left(\frac{19}{10}\right)^4 = 13.0321 > 13,$$

and by $(0.175/0.090)^4 > 14.29$ for $L_\mu \geq 4$.

Proposition 7.2 (Large-N Bessel–Gaussian L^3 sector limit)**.** *Fix $u = \beta/N \in \mathbb{C}$. As $N \to \infty$, by the Diaconis–Shahshahani theorem [2], $\mathrm{Tr}(U)$ on $\mathrm{SU}(N)$ converges in moments to a standard complex Gaussian $Z \sim \mathcal{N}_{\mathbb{C}}(0,1)$ with radial density $2re^{-r^2}\,dr$ ($r = |Z| \geq 0$). Under the limiting $\mathrm{U}(1)$ center projection, the exact limiting charge-q sector functions are*

$$f_q^{(\infty)}(Z) = e^{-u^2/4}\, I_{|q|}\big(u|Z|\big)\left(\frac{Z}{|Z|}\right)^q \quad (q \neq 0), \qquad f_0^{(\infty)}(Z) = e^{-u^2/4}\, I_0\big(u|Z|\big) - 1, \tag{34}$$

with leading L^3 vs. L^6 constants for $q = \pm 1$:

$$\lim_{u\to 0} \frac{\|f_{\pm 1}^{(\infty)}\|_3}{|u|} = \frac{1}{2}\,\Gamma\left(\frac{5}{2}\right)^{1/3} = \frac{1}{2}\left(\frac{3\sqrt{\pi}}{4}\right)^{1/3} \approx 0.5498,$$

$$\lim_{u\to 0} \frac{\|f_{\pm 1}^{(\infty)}\|_6}{|u|} = \frac{1}{2}\,\Gamma(4)^{1/6} = \frac{6^{1/6}}{2} \approx 0.6740.$$

In particular, passing from $p = 6$ to the K_4 1-factorization exponent $p = 3$ lowers the leading limiting charge-one sector norm by 18.4% (equivalently, the $p = 6$ constant exceeds the $p = 3$ constant by 22.6%).

Proof. For $Z \sim \mathcal{N}_{\mathbb{C}}(0,1)$, $\Re(e^{i\alpha}Z) = |Z|\cos(\alpha + \arg Z)$. Integrating against $\frac{1}{2\pi}e^{-iq\alpha}\,d\alpha$ yields the modified Bessel identity $\frac{1}{2\pi}\int_0^{2\pi} e^{-iq\alpha + u|Z|\cos(\alpha+\phi)}\,d\alpha = I_{|q|}(u|Z|)e^{iq\phi}$, while $c_0^{(\infty)}(u) = \mathbb{E}[e^{u\Re Z}] = e^{u^2/4}$ since $\Re Z \sim \mathcal{N}(0, 1/2)$. As $u \to 0$, $I_1(u|Z|) \sim \frac{1}{2}u|Z|$, and $\mathbb{E}[|Z|^p] = \int_0^\infty r^p \cdot 2re^{-r^2}\,dr = \Gamma(1 + p/2)$, giving the stated constants for $p = 3$ and $p = 6$. □

Remark 7.3 (Heuristic large-N zero-free radius outlook). Because the limiting sector weights $f_q^{(\infty)}$ depend on β solely through $u = \beta/N$, Proposition 7.2 suggests that a uniform-in-N certified zero-free radius in the Wilson normalization could scale linearly with N ($|\beta| \leq c_* N$). Establishing such a uniform-in-N theorem remains open: it would require extending the $\mathbb{Z}_N$ exploration automaton of Theorem 5.2 to all $N \geq 4$, proving uniform-in-N convergence of $\|f_{\beta,q}\|_3$ and $\|f_{\beta,q}\|_6$ across all $q \in \mathbb{Z}_N$, and bounding $\zeta^{\mathrm{adm}}_{\mathrm{SU}(N)}(M_{\min})$ uniformly in N.

8. Reproducibility and Mechanized Verification

All quantitative claims in this manuscript are backed by two independent verification artifacts:

- `src/python/certify_zerofree.py` and `tests/test_certificates.py`: executes the 60-digit `mpmath.iv` analytic enclosures, exact bivariate Laurent-polynomial SU(3) Weyl moment extraction, and all 9 certificates in under 3 seconds (`6/6` tests passing under `pytest`).
- `src/lean/ZeroFreeCerts.lean`: a standalone, zero-import Lean 4.36.0 file compiled with `-DwarningAsError=true` containing 17 machine-checked theorems with zero `sorry` and zero `Classical.choice`. It proves the soundness of directed-rounding fixed-point arithmetic (`mulUp_sound`, `divUp_sound`, `mulDn_sound`), the exact K_4 ($p_4 = 3$) and K_6 ($p_6 = 5$) 1-factorizations, the link-multiplicity inequality $|E| \le 2|P|$, the unit-cube counts $4\binom{d-1}{2}$, rational Taylor exponential enclosures, all 9 peeling and Fernández–Procacci certificates in Table 2, and the disc-area and Borel–Carathéodory gains in Corollary 7.1.

References

1. Deep Bhattacharjee, Ushashi Bhattacharya, James Montgomery Lockwood, Tristan Romanov, Ikailo John Sessford, and Sanjeevan Singha Roy, *The SU(2) Yang–Mills mass gap $\delta > 0$ from a single finite-volume bound (and successor lean 4 formalization 4Y-M)*, Preprints 202611.1282 v2, doi:10.20944/preprints202611.1282.v2; Zenodo doi:10.5281/zenodo.24210329, 2026.
2. Persi Diaconis and Mehrdad Shahshahani, *On the eigenvalues of random matrices*, Journal of Applied Probability **31A** (1994), 49–62.
3. Reinhard Diestel, *Graph theory*, 5th ed., Graduate Texts in Mathematics, vol. 173, Springer, 2017.
4. Lluis Eriksson, *THE-ERIKSSON-PROGRAMME: Machine-checked SU(N) lattice Yang–Mills foundations in Lean 4 / Mathlib*, https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME, 2026.
5. Roberto Fernández and Aldo Procacci, *Cluster expansion for abstract polymer models: new bounds from an old approach*, Communications in Mathematical Physics **274** (2007), no. 1, 123–140.
6. Alexander A. Migdal, *Recursion equations in gauge field theories*, Soviet Physics JETP **42** (1975), no. 3, 413–418.
7. Konrad Osterwalder and Erhard Seiler, *Gauge field theories on a lattice*, Annals of Physics **110** (1978), no. 2, 440–471.
8. Erhard Seiler, *Gauge theories as a problem of constructive quantum field theory and statistical mechanics*, Lecture Notes in Physics, vol. 159, Springer-Verlag, Berlin, 1982.
9. Kenneth G. Wilson, *Confinement of quarks*, Physical Review D **10** (1974), no. 8, 2445–2459.

The Eriksson Programme, Independent Mathematical Physics Research
Email address: `editor@airr.science`