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UNIVERSAL NON-COMMUTATIVE REVERSE-KL LAWS VIA BKM–CHEBYSHEV MONOTONICITY AND COMPLETE MODIFIED LOGARITHMIC SOBOLEV CONSTANTS OF GENERALIZED QUANTUM DEPOLARIZING AND CONDITIONAL-EXPECTATION SEMIGROUPS

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Abstract. The complete modified logarithmic Sobolev inequality (cMLSI) governs the uniform exponential decay of quantum relative entropy in the presence of arbitrary entangled reference systems. Whereas the single-system constant $\alpha_1(\mathcal{L}_\sigma)$ of a generalized quantum depolarizing semigroup $\mathcal{L}_\sigma(\rho) = \sigma - \rho$ was computed exactly by Müller-Hermes, Stilck França, and Wolf (2016) as $\frac{1}{2}(1 + \kappa_2(\lambda_{\min}(\sigma)))$, determining the exact complete constant $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma)$ has remained open even for the qubit tracial channel. Recently, Gao and Zhao (arXiv:2608.19685, 2026) introduced BKM/Hockey-Stick reverse-ratio lower bounds, index-achieving pure-state upper bounds, tensor-stable product estimates, and tight first-order asymptotics for generalized quantum depolarization and dephasing.

Building upon and complementing this framework with fully self-contained proofs, we establish four sharp structural and asymptotic results. First, using the exact Bogoliubov–Kubo–Mori (BKM) Fisher integral representations of forward and reverse Umegaki relative entropy together with operator anti-monotonicity and Chebyshev's order inequality, we prove a universal two-sided non-commutative reverse-KL law: for any distinct faithful states ρ, τ $(\rho \neq \tau)$ satisfying $m\tau \leq \rho \leq M\tau$ $(0 < m < 1 < M)$,

$$\gamma(M) \;\leq\; \frac{D(\tau\|\rho)}{D(\rho\|\tau)} \;\leq\; \frac{1}{\gamma(1/m)} \;=\; \frac{m - 1 - \log m}{1 - m + m\log m}, \qquad \gamma(M) := \frac{M - 1 - \log M}{M\log M - M + 1},$$

with both endpoints optimal on their respective operator-dominance cones. Second, for every idempotent CPTP projection $\mathcal{E}_*$ with finite complete Pimsner–Popa index $C_{\mathrm{cb}}(\mathcal{E}_*) \in (1, \infty)$, we recover the strict bound $\alpha_{\mathrm{cb}}(\mathcal{E}_* - \mathrm{id}) \geq \frac{1}{2}(1 + \gamma(C_{\mathrm{cb}}(\mathcal{E}_*))) > \frac{1}{2}$, establish $C_{\mathrm{cb}}(\mathcal{E}_\sigma) = \mathrm{Tr}(\sigma^{-1})$ explicitly (raising the qubit depolarizing baseline from 0.5 to $0.81701240\ldots$), and give a short self-contained chain-rule proof of complete tensorization for N-site non-tracial spin chains. Third, constructing an explicit commuting σ-inverted Choi isotropic family and proving via an elementary Maclaurin/harmonic-number argument that the binary reverse-KL constant $\kappa_2(p)$ is strictly increasing on $(0, 1/2]$, we prove universal strict entanglement degradation

$$\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) \;\leq\; \frac{1}{2}\Big(1 + \kappa_2\big(\mathrm{Tr}(\sigma^{-1})^{-1}\big)\Big) \;<\; \frac{1}{2}\big(1 + \kappa_2(\lambda_{\min}(\sigma))\big) \;=\; \alpha_1(\mathcal{L}_\sigma)$$

for every full-rank state σ in every dimension $d \geq 2$; in particular $\alpha_{\mathrm{cb}}(\mathcal{L}_{I_2/2}) \leq 0.95470639\cdots < 1$. Fourth, we derive exact two-term asymptotic expansions of $\gamma(M)$, $\kappa_2(1/M)$, and the optimal binary witness $q_*(1/M) = \frac{2}{\log M}(1 + O(\frac{\log\log M}{\log M}))$ as $M \to \infty$. Together with $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) - \frac{1}{2} = \frac{1+o(1)}{2\log \mathrm{Tr}(\sigma^{-1})}$ as $\mathrm{Tr}(\sigma^{-1}) \to \infty$, this reveals a sharp entanglement collapse dichotomy—tracial depolarizing semigroups suffer an exact factor-of-2 asymptotic collapse (approached strictly from below at finite d), whereas dephasing and strongly hierarchical Gibbs depolarizing semigroups exhibit zero asymptotic collapse—and motivates an exact harmonic-mean cMLSI conjecture supported to 10^{-13} by unconstrained bipartite optimization.

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ORCID: 0009-0006-7494-9040. Code, 100-digit tables, figures, and verification suite archived at `https://github.com/lluiseriksson/quantum-cmlsi-reverse-kl` (release tag `v1.0.1`).

1. Introduction

Modified logarithmic Sobolev inequalities (MLSIs) provide the sharpest general route to rapid mixing and thermalization of classical and quantum Markov semigroups [3, 8]. For a quantum Markov semigroup $(e^{t\mathcal{L}})_{t\geq 0}$ in the Schrödinger picture converging to a stationary conditional expectation $\mathcal{E}_*$, the MLSI constant $\alpha_1(\mathcal{L})$ is the largest constant $\alpha \geq 0$ such that

$$D\big(e^{t\mathcal{L}}(\rho)\,\|\,\mathcal{E}_*(\rho)\big) \;\leq\; e^{-2\alpha t}\, D\big(\rho\,\|\,\mathcal{E}_*(\rho)\big) \qquad \text{for all } t\geq 0 \text{ and all states } \rho\in\mathcal{D}(\mathcal{H}_A), \tag{1}$$

where $D(\rho\|\tau) := \mathrm{Tr}[\rho(\log\rho - \log\tau)]$ is the Umegaki quantum relative entropy (taking the value $+\infty$ if $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\tau)$). Equivalently, $\alpha_1(\mathcal{L})$ is the optimal constant in the differential entropy-production inequality

$$\mathrm{EP}_{\mathcal{L}}(\rho) \;:=\; -\,\frac{d}{dt}\Big|_{t=0} D\big(e^{t\mathcal{L}}(\rho)\,\|\,\mathcal{E}_*(\rho)\big) \;\geq\; 2\alpha\, D\big(\rho\,\|\,\mathcal{E}_*(\rho)\big). \tag{2}$$

In contrast to classical Markov chains, quantum MLSIs do not tensorize automatically because of entanglement across subsystems [2, 5]. To restore tensorization, one introduces the *complete* modified logarithmic Sobolev constant (cMLSI) [5, 4, 6]:

$$\alpha_{\mathrm{cb}}(\mathcal{L}) \;:=\; \inf_{d_B\geq 1} \alpha_1\big(\mathcal{L}\otimes \mathrm{id}_{M_{d_B}}\big). \tag{3}$$

Because $\alpha_{\mathrm{cb}}(\mathcal{L}_1\otimes\mathrm{id} + \mathrm{id}\otimes\mathcal{L}_2) \geq \min\{\alpha_{\mathrm{cb}}(\mathcal{L}_1), \alpha_{\mathrm{cb}}(\mathcal{L}_2)\}$, sharp bounds on α_{cb} for elementary local generators directly govern mixing times of many-body open quantum systems and noisy quantum circuits [6, 4, 1].

The canonical benchmark is the generalized quantum depolarizing semigroup on $\mathcal{H}_A = \mathbb{C}^d$ with full-rank invariant state $\sigma\in\mathcal{D}_+(\mathbb{C}^d)$, generated in the Schrödinger picture by

$$\mathcal{L}_\sigma(\rho) \;:=\; \mathcal{E}_\sigma(\rho) - \rho, \qquad \mathcal{E}_\sigma(\rho) \;:=\; \mathrm{Tr}(\rho)\,\sigma, \qquad e^{t\mathcal{L}_\sigma}(\rho) = e^{-t}\rho + (1-e^{-t})\,\mathrm{Tr}(\rho)\,\sigma.$$

For a single system ($d_B = 1$), Müller-Hermes, Stilck França, and Wolf [9] proved that $\alpha_1(\mathcal{L}_\sigma)$ coincides with the classical MLSI constant of a complete-graph Markov chain with stationary law equal to the spectrum of σ. Explicitly, introducing the binary reverse-KL constant

$$\kappa_2(p) \;:=\; \inf_{q\in(0,1)\setminus\{p\}} \frac{d_2(p\|q)}{d_2(q\|p)}, \qquad d_2(a\|b) := a\log\frac{a}{b} + (1-a)\log\frac{1-a}{1-b}, \tag{4}$$

for $p\in(0,1/2]$ (with $\kappa_2(1/2)=1$), the exact single-system constant is

$$\alpha_1(\mathcal{L}_\sigma) \;=\; \frac{1}{2}\Big(1+\kappa_2\big(\lambda_{\min}(\sigma)\big)\Big), \tag{5}$$

where $\lambda_{\min}(\sigma)$ denotes the smallest eigenvalue of σ; in particular, $\alpha_1(\mathcal{L}_{I_2/2}) = 1$ for the tracial qubit depolarizing semigroup.

For the *complete* constant $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma)$, however, the picture has remained dramatically incomplete. For any idempotent completely positive trace-preserving (CPTP) projection $\mathcal{E}_*$ ($\mathcal{E}_*^2 = \mathcal{E}_*$) and generator $\mathcal{L} = \mathcal{E}_* - \mathrm{id}$, the entropy production satisfies the exact symmetric identity (Lemma 3.2)

$$\mathrm{EP}_{\mathcal{L}\otimes\mathrm{id}_B}(\rho_{AB}) \;=\; D(\rho_{AB}\|\tau_{AB}) + D(\tau_{AB}\|\rho_{AB}), \qquad \tau_{AB} := (\mathcal{E}_*\otimes\mathrm{id}_B)(\rho_{AB}). \tag{6}$$

Dropping the non-negative reverse relative entropy $D(\tau_{AB}\|\rho_{AB}) \geq 0$ immediately yields the classical universal baseline $\alpha_{\mathrm{cb}}(\mathcal{E}_* - \mathrm{id}) \geq \frac{1}{2}$ (decay rate $2\alpha_{\mathrm{cb}} \geq 1$) [2, 5, 4]. Very recently, Gao and Zhao [7] made a major advance beyond this baseline by establishing tight upper and lower bounds for relative entropy contraction of generalized quantum depolarizing and conditional-expectation dynamics, using sharp reverse ratios derived from both Hockey-Stick f-divergences and the Bogoliubov–Kubo–Mori (BKM) Fisher metric together with index-achieving pure states, tensor-stable estimates, and tight first-order asymptotics. Even so, the exact finite-dimensional value of $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma)$ remains open (even for the tracial qubit $\sigma = I_2/2$), as do several fundamental structural questions: the sharp two-sided reverse-KL envelope under lower dominance $m\tau \leq \rho$, a

universal proof that entanglement strictly degrades $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) < \alpha_1(\mathcal{L}_\sigma)$ for *every* full-rank state σ in every dimension $d \geq 2$, the exact second-order asymptotic separation between $\gamma(M)$ and $\kappa_2(1/M)$ and its effect on finite-dimensional collapse ratios, and the sharp spectral dichotomy governing when entanglement halves the MLSI excess versus leaving it asymptotically unchanged.

Main Contributions and Relation to Prior Work. Developing a self-contained BKM–Chebyshev and binary-entropic analysis that complements and sharpens [7], we establish four interlocking results:

(1) **Universal Two-Sided Non-Commutative Reverse-KL Law (Theorem 2.4).** For any finite-dimensional Hilbert space $\mathcal{H}$ and any faithful states $\rho, \tau \in \mathcal{D}_+(\mathcal{H})$ $(\rho \neq \tau)$ satisfying $m\tau \leq \rho \leq M\tau$ $(0 < m < 1 < M)$, we prove

$$\gamma(M) \;\leq\; \frac{D(\tau\|\rho)}{D(\rho\|\tau)} \;\leq\; \frac{1}{\gamma(1/m)} \;=\; \frac{m-1-\log m}{1-m+m\log m}, \qquad \gamma(M) := \frac{M-1-\log M}{M\log M - M + 1},$$

and prove that both constants $\gamma(M)$ and $\gamma(1/m)^{-1}$ are sharp on their respective operator-dominance cones. While the one-sided BKM reverse-ratio lower bound aligns with the independent BKM analysis of [7], our short unified proof via $(1+(c-1)t)F(t)$ and Chebyshev's inequality yields both the lower and upper envelopes simultaneously and proves cone optimality on both sides.

(2) **Complete Pimsner–Popa Indices, Explicit cMLSI Lower Bounds, and Short Chain-Rule Tensorization (Theorems 3.3–3.4 and Corollary 3.5).** For any idempotent CPTP projection $\mathcal{E}_*$ with finite complete Pimsner–Popa index $C_{\mathrm{cb}}(\mathcal{E}_*) \in (1,\infty)$, Theorem 2.4 recovers the strict bound $\alpha_{\mathrm{cb}}(\mathcal{E}_* - \mathrm{id}) \geq \frac{1}{2}(1+\gamma(C_{\mathrm{cb}}(\mathcal{E}_*))) > \frac{1}{2}$ (cf. [7]). We give short self-contained proofs that $C_{\mathrm{cb}}(\mathcal{E}_\sigma) = \mathrm{Tr}(\sigma^{-1})$ for generalized depolarizing projections and $C_{\mathrm{cb}}(\mathcal{P}) = r$ for r-block orthogonal pinching (giving $\alpha_{\mathrm{cb}}(\mathcal{L}_{I_2/2}) \geq \frac{3\log 4}{2(4\log 4-3)} = 0.817012403669\ldots$ and $\alpha_{\mathrm{cb}}(\Delta_2 - \mathrm{id}) \geq 0.897174862390\ldots$), together with an elementary six-line chain-rule proof of complete tensorization for N-site non-tracial depolarizing spin chains.

(3) **Commuting σ-Inverted Choi Isotropic Family, Elementary Strict Monotonicity of κ_2, and Universal Entanglement Degradation (Theorem 4.1, Proposition 4.2, and Corollary 4.3).** Introducing the σ-inverted Choi vector $|\psi_\sigma\rangle = p_{\mathrm{eff}}^{1/2} \sum_{i=1}^d \lambda_i^{-1/2} |i\rangle_A |i\rangle_B$ with effective harmonic probability $p_{\mathrm{eff}} := \mathrm{Tr}(\sigma^{-1})^{-1}$, we construct an explicit commuting bipartite family $\rho_{AB}(q)$ with invariant marginal $\rho_B = p_{\mathrm{eff}}\sigma^{-1}$ and prove $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) \leq \frac{1}{2}(1+\kappa_2(\mathrm{Tr}(\sigma^{-1})^{-1}))$. Complementing [7], we give a completely elementary, self-contained Maclaurin-series and harmonic-number proof that $\kappa_2(p)$ is strictly increasing on $(0,1/2]$ and that its global minimizer lies strictly in $(p,1/2)$. Because $\mathrm{Tr}(\sigma^{-1})^{-1} < \lambda_{\min}(\sigma) \leq \frac{1}{2}$ for every $d \geq 2$ and every full-rank σ, this proves universal strict entanglement degradation $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) < \alpha_1(\mathcal{L}_\sigma)$ across all dimensions and all faithful fixed points (in particular $\alpha_{\mathrm{cb}}(\mathcal{L}_{I_2/2}) \leq 0.954706390023\cdots < 1$).

(4) **Exact Two-Term Asymptotics and the Factor-of-2 vs. Zero Entanglement Collapse Dichotomy (Theorems 5.1–5.2).** Refining the first-order asymptotics of [7], we prove exact two-term expansions as $M \to \infty$:

$$\gamma(M) = \frac{1}{\log M}\left(1 + \frac{1}{\log M} + O\left(\frac{1}{(\log M)^2}\right)\right),$$
$$\kappa_2\left(\frac{1}{M}\right) = \frac{1}{\log M}\left(1 + \frac{\log\log M + 2 - \log 2}{\log M} + O\left(\frac{(\log\log M)^2}{(\log M)^2}\right)\right),$$

together with the exact optimal witness scaling $q_*(1/M) = \frac{2}{\log M}(1 + O(\frac{\log\log M}{\log M}))$. Comparing $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) - \frac{1}{2} = \frac{1+o(1)}{2\log \mathrm{Tr}(\sigma^{-1})}$ with $\alpha_1(\mathcal{L}_\sigma) - \frac{1}{2} = \frac{1+o(1)}{2\log(\lambda_{\min}(\sigma)^{-1})}$ establishes a sharp entanglement collapse dichotomy: tracial and bounded-condition-number depolarizing semigroups suffer an exact factor-of-2 asymptotic entanglement collapse (approached strictly from below at finite d, as predicted by our second-order term; Remark 5.3), whereas d-dimensional dephasing and strongly hierarchical Gibbs depolarizing semigroups exhibit zero asymptotic collapse.

Finally, unconstrained non-commuting BFGS optimizations across $d_A, d_B \in \{2, 3\}$ match $\frac{1}{2}(1 + \kappa_2(\mathrm{Tr}(\sigma^{-1})^{-1}))$ to 10^{-13} for every tested tracial and non-tracial σ (and $\frac{1}{2}(1 + \kappa_2(1/d))$ for dephasing), motivating Conjecture 6.1.

2. Universal Non-Commutative Reverse-KL Law via BKM–Chebyshev Monotonicity

Throughout, $\mathcal{H}$ is a finite-dimensional complex Hilbert space, $\mathcal{B}(\mathcal{H})$ is the algebra of linear operators on $\mathcal{H}$, $\mathcal{D}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) : \rho \geq 0, \mathrm{Tr}\,\rho = 1\}$ is the set of quantum states, and $\mathcal{D}_+(\mathcal{H}) = \{\rho \in \mathcal{D}(\mathcal{H}) : \rho > 0\}$ is the subset of faithful (strictly positive definite) states. All logarithms are natural. For $A > 0$ and Hermitian $\Delta = \Delta^* \in \mathcal{B}(\mathcal{H})$, the Bogoliubov–Kubo–Mori (BKM) Fisher information (or Kubo–Mori metric quadratic form) is

$$\mathcal{I}_{\mathrm{BKM}}(A; \Delta) \;:=\; \int_0^\infty \mathrm{Tr}\Big(\Delta(A + sI)^{-1}\Delta(A + sI)^{-1}\Big)\, ds. \tag{7}$$

In an eigenbasis $\{|k\rangle\}$ of A with eigenvalues $a_k > 0$, the elementary integral $\int_0^\infty (a_j + s)^{-1}(a_k + s)^{-1}\, ds = \frac{\log a_j - \log a_k}{a_j - a_k}$ (understood as a_j^{-1} when $a_j = a_k$) yields

$$\mathcal{I}_{\mathrm{BKM}}(A; \Delta) \;=\; \sum_{j,k} \frac{\log a_j - \log a_k}{a_j - a_k}\, |\langle j|\Delta|k\rangle|^2 \;>\; 0 \qquad (\Delta \neq 0). \tag{8}$$

Lemma 2.1 (Exact BKM integral representations)**.** *Let $\rho, \tau \in \mathcal{D}_+(\mathcal{H})$, set $\Delta := \rho - \tau$ and $\rho_t := (1 - t)\tau + t\rho \in \mathcal{D}_+(\mathcal{H})$ for $t \in [0, 1]$, and define $F(t) := \mathcal{I}_{\mathrm{BKM}}(\rho_t; \Delta)$. Then*

$$D(\rho\|\tau) \;=\; \int_0^1 (1 - t)\, F(t)\, dt, \qquad D(\tau\|\rho) \;=\; \int_0^1 t\, F(t)\, dt. \tag{9}$$

Proof. Using the integral representation $\log X = \int_0^\infty \big((1 + s)^{-1} I - (X + sI)^{-1}\big)\, ds$ for $X > 0$, differentiating along the smooth positive path $t \mapsto \rho_t$ gives the standard Fréchet derivative

$$\frac{d}{dt} \log \rho_t \;=\; \int_0^\infty (\rho_t + sI)^{-1}\Delta(\rho_t + sI)^{-1}\, ds.$$

Consequently, by cyclicity of the trace and $\int_0^\infty \rho_t(\rho_t + sI)^{-2}\, ds = I$,

$$\mathrm{Tr}\left(\rho_t \frac{d}{dt} \log \rho_t\right) = \int_0^\infty \mathrm{Tr}\Big(\Delta(\rho_t + sI)^{-1}\rho_t(\rho_t + sI)^{-1}\Big)\, ds = \mathrm{Tr}(\Delta) = \mathrm{Tr}(\rho) - \mathrm{Tr}(\tau) = 0.$$

Consider the smooth function $h(t) := D(\rho_t\|\tau) = \mathrm{Tr}[\rho_t(\log \rho_t - \log \tau)]$ on $[0, 1]$. Differentiating once and using the identity above gives

$$h'(t) = \mathrm{Tr}\big(\Delta(\log \rho_t - \log \tau)\big) + \mathrm{Tr}\left(\rho_t \frac{d}{dt} \log \rho_t\right) = \mathrm{Tr}\big(\Delta(\log \rho_t - \log \tau)\big).$$

Differentiating a second time yields

$$h''(t) = \mathrm{Tr}\left(\Delta \frac{d}{dt} \log \rho_t\right) = \mathcal{I}_{\mathrm{BKM}}(\rho_t; \Delta) = F(t).$$

Since $h(0) = 0$ and $h'(0) = 0$, Taylor's theorem with integral remainder gives

$$D(\rho\|\tau) = h(1) = h(0) + h'(0) + \int_0^1 (1-t)h''(t)\,dt = \int_0^1 (1-t)\,F(t)\,dt.$$

Applying the identical calculation to $\tilde{h}(u) := D(\rho_{1-u}\|\rho)$, which satisfies $\tilde{h}(0) = \tilde{h}'(0) = 0$ and $\tilde{h}''(u) = \mathcal{I}_{\mathrm{BKM}}(\rho_{1-u}; -\Delta) = F(1-u)$, yields

$$D(\tau\|\rho) = \tilde{h}(1) = \int_0^1 (1-u)\,F(1-u)\,du = \int_0^1 t\,F(t)\,dt. \qquad \square$$

Lemma 2.2 (Operator anti-monotonicity and (-1)-homogeneity of $\mathcal{I}_{\mathrm{BKM}}$)**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$ satisfy $0 < A \le B$, let $c > 0$, and let $\Delta = \Delta^* \in \mathcal{B}(\mathcal{H})$. Then:*

(a) $\mathcal{I}_{\mathrm{BKM}}(cA; \Delta) = c^{-1}\mathcal{I}_{\mathrm{BKM}}(A; \Delta)$*;*
(b) $\mathcal{I}_{\mathrm{BKM}}(B; \Delta) \le \mathcal{I}_{\mathrm{BKM}}(A; \Delta)$*.*

Proof. Part (a) follows immediately from the substitution $s = cu$ in (7). For (b), fix any $s \ge 0$ and set $X := (A + sI)^{-1}$ and $Y := (B + sI)^{-1}$. Since $0 < A + sI \le B + sI$, Löwner's theorem (operator anti-monotonicity of inversion) gives $0 < Y \le X$. Define $C := X^{-1/2}YX^{-1/2}$ and $W := X^{1/2}\Delta X^{1/2}$. Both C and W are Hermitian, and $0 < C \le I$, hence $\|C\|_{\mathrm{op}} \le 1$ and $C^2 \le I$. Using cyclicity of the trace and the Cauchy–Schwarz inequality for the Hilbert–Schmidt inner product $\langle U, V\rangle_{\mathrm{HS}} = \mathrm{Tr}(U^*V)$, we obtain

$$\begin{aligned}\mathrm{Tr}(\Delta Y \Delta Y) &= \mathrm{Tr}(WCWC) = \langle CW,\, WC\rangle_{\mathrm{HS}} \le \|CW\|_{\mathrm{HS}}\,\|WC\|_{\mathrm{HS}} \\ &= \mathrm{Tr}(WC^2W) \le \mathrm{Tr}(W^2) = \mathrm{Tr}(\Delta X \Delta X),\end{aligned}$$

where we used $\|CW\|_{\mathrm{HS}}^2 = \mathrm{Tr}(WC^2W) = \|WC\|_{\mathrm{HS}}^2$ and $WC^2W \le W^2$. Integrating over $s \in [0, \infty)$ yields $\mathcal{I}_{\mathrm{BKM}}(B; \Delta) \le \mathcal{I}_{\mathrm{BKM}}(A; \Delta)$. $\square$

Lemma 2.3 (Weighted Chebyshev ratio inequality)**.** *Let $w : [0,1] \to (0, \infty)$ be continuous and let $G : [0,1] \to (0, \infty)$ be continuous and monotone non-decreasing (resp. monotone non-increasing). Then*

$$\frac{\int_0^1 t\,w(t)\,G(t)\,dt}{\int_0^1 (1-t)\,w(t)\,G(t)\,dt} \;\ge\; (\textit{resp.}\;\le)\; \frac{\int_0^1 t\,w(t)\,dt}{\int_0^1 (1-t)\,w(t)\,dt}.$$

Proof. Cross-multiplying the positive denominators, the difference of products equals

$$\begin{aligned}&\int_0^1\int_0^1 \big[t(1-s) - (1-t)s\big]\,w(t)\,w(s)\,G(t)\,dt\,ds = \int_0^1\int_0^1 (t-s)\,w(t)\,w(s)\,G(t)\,dt\,ds \\ &\quad = \frac{1}{2}\int_0^1\int_0^1 (t-s)\big(G(t) - G(s)\big)\,w(t)\,w(s)\,dt\,ds,\end{aligned}$$

which is non-negative (resp. non-positive) whenever G is non-decreasing (resp. non-increasing). $\square$

We now combine Lemmas 2.1–2.3 to obtain our first main theorem.

Theorem 2.4 (Universal Two-Sided Non-Commutative Reverse-KL Law)**.** *Let $\rho, \tau \in \mathcal{D}_+(\mathcal{H})$ be distinct faithful quantum states. Define*

$$\gamma(M) \;:=\; \frac{M - 1 - \log M}{M\log M - M + 1} \qquad (M > 1). \tag{10}$$

(a) Upper operator dominance. *If $\rho \le M\tau$ for some $M > 1$, then*

$$\frac{D(\tau\|\rho)}{D(\rho\|\tau)} \;\ge\; \gamma(M), \qquad \frac{\mathrm{Tr}\big((\rho-\tau)(\log\rho - \log\tau)\big)}{D(\rho\|\tau)} \;\ge\; 1 + \gamma(M) \;=\; \frac{(M-1)\log M}{M\log M - M + 1}. \tag{11}$$

(b) Lower operator dominance. *If $m\tau \le \rho$ for some $m \in (0,1)$, then*

$$\frac{D(\tau\|\rho)}{D(\rho\|\tau)} \;\le\; \frac{1}{\gamma(1/m)} \;=\; \frac{m-1-\log m}{1-m+m\log m}. \tag{12}$$

(c) Sharpness. *For every $M>1$ and every $m \in (0,1)$ and every $\dim \mathcal{H} \ge 2$, the constants $\gamma(M)$ in (11) and $\gamma(1/m)^{-1}$ in (12) are optimal on the respective operator-dominance cones $\{\rho \le M\tau\}$ and $\{m\tau \le \rho\}$.*

Proof. For any real constant $c > 0$ and any $0 \le s \le t \le 1$, direct expansion gives the exact operator identity

$$\big(1+(c-1)s\big)\rho_t - \big(1+(c-1)t\big)\rho_s \;=\; (t-s)(\rho - c\tau). \tag{13}$$

Indeed, the coefficient of ρ is $(1+(c-1)s)t - (1+(c-1)t)s = t-s$, and the coefficient of τ is $(1+(c-1)s)(1-t) - (1+(c-1)t)(1-s) = c(s-t) = -(t-s)c$.

Proof of (a). Taking $c = M > 1$ in (13), the assumption $\rho \le M\tau$ gives

$$\rho_t \;\le\; \frac{1+(M-1)t}{1+(M-1)s}\,\rho_s \qquad (0 \le s \le t \le 1).$$

By Lemma 2.2(a)–(b), it follows that

$$F(t) = \mathcal{I}_{\mathrm{BKM}}(\rho_t;\Delta) \;\ge\; \frac{1+(M-1)s}{1+(M-1)t}\,\mathcal{I}_{\mathrm{BKM}}(\rho_s;\Delta) \;=\; \frac{1+(M-1)s}{1+(M-1)t}\,F(s).$$

Thus $G_M(t) := (1+(M-1)t)F(t) > 0$ is continuous and monotone non-decreasing on $[0,1]$. Writing $F(t) = w_M(t)G_M(t)$ with $w_M(t) := (1+(M-1)t)^{-1} > 0$, Lemmas 2.1 and 2.3 yield

$$\frac{D(\tau\|\rho)}{D(\rho\|\tau)} = \frac{\int_0^1 t\, w_M(t)\, G_M(t)\, dt}{\int_0^1 (1-t)\, w_M(t)\, G_M(t)\, dt} \;\ge\; \frac{\int_0^1 t\, w_M(t)\, dt}{\int_0^1 (1-t)\, w_M(t)\, dt}.$$

Evaluating the two elementary integrals via $u = 1+(M-1)t \in [1,M]$ gives

$$\int_0^1 \frac{t\,dt}{1+(M-1)t} = \frac{M-1-\log M}{(M-1)^2}, \qquad \int_0^1 \frac{(1-t)\,dt}{1+(M-1)t} = \frac{M\log M - M + 1}{(M-1)^2}.$$

Taking their ratio yields $D(\tau\|\rho)/D(\rho\|\tau) \ge \gamma(M)$. Adding 1 to both sides and noting $\mathrm{Tr}((\rho-\tau)(\log\rho - \log\tau)) = D(\rho\|\tau) + D(\tau\|\rho)$ gives the second inequality in (11).

Proof of (b). Taking $c = m \in (0,1)$ in (13), the assumption $m\tau \le \rho$ gives $(1-(1-m)s)\rho_t \ge (1-(1-m)t)\rho_s$ for $0 \le s \le t \le 1$, where $1-(1-m)t \ge m > 0$. By Lemma 2.2, $G_m(t) := (1-(1-m)t)F(t)$ is monotone non-increasing on $[0,1]$. Evaluating the elementary integrals via $u = 1-(1-m)t \in [m,1]$ gives $\int_0^1 t(1-(1-m)t)^{-1}\,dt = (m-1-\log m)/(1-m)^2$ and $\int_0^1 (1-t)(1-(1-m)t)^{-1}\,dt = (1-m+m\log m)/(1-m)^2$. Applying Lemma 2.3 with $w_m(t) := (1-(1-m)t)^{-1}$ therefore yields

$$\frac{D(\tau\|\rho)}{D(\rho\|\tau)} \;\le\; \frac{\int_0^1 t(1-(1-m)t)^{-1}\,dt}{\int_0^1 (1-t)(1-(1-m)t)^{-1}\,dt} = \frac{m-1-\log m}{1-m+m\log m} = \frac{1}{\gamma(1/m)},$$

which also follows equivalently by exchanging (ρ,τ) in (a) since $\tau \le m^{-1}\rho$ (and noting $\gamma(1/m) = (1-m+m\log m)/(m-1-\log m) \in (0,1)$ for $m \in (0,1)$).

Proof of (c). Fix $M > 1$ and let $d := \dim\mathcal{H} \ge 2$. For $t \in (0, M^{-1})$ and any $\varepsilon \in (0,(d-2)^{-1})$ (taking $\varepsilon = 0$ if $d = 2$), set $c_\varepsilon := 1-(d-2)\varepsilon \in (0,1]$ and define the faithful states

$$\tau_{t,\varepsilon} := \mathrm{diag}\big(c_\varepsilon t,\, c_\varepsilon(1-t),\, \varepsilon, \ldots, \varepsilon\big), \qquad \rho_{t,\varepsilon} := \mathrm{diag}\big(c_\varepsilon M t,\, c_\varepsilon(1-Mt),\, \varepsilon, \ldots, \varepsilon\big) \in \mathcal{D}_+(\mathcal{H}).$$

Clearly $\mathrm{Tr}(\tau_{t,\varepsilon}) = \mathrm{Tr}(\rho_{t,\varepsilon}) = 1$ and $\rho_{t,\varepsilon} \le M\tau_{t,\varepsilon}$ because $Mt \le M \cdot t$, $1 - Mt \le 1 - t \le M(1-t)$, and $\varepsilon \le M\varepsilon$. Because the final $d-2$ diagonal entries are identical, they contribute zero to both $D(\tau_{t,\varepsilon}\|\rho_{t,\varepsilon})$ and $D(\rho_{t,\varepsilon}\|\tau_{t,\varepsilon})$, while the common factor $c_\varepsilon > 0$ cancels identically in the ratio:

$$D(\tau_{t,\varepsilon}\|\rho_{t,\varepsilon}) = c_\varepsilon \left(t\log(1/M) + (1-t)\log\frac{1-t}{1-Mt} \right) = c_\varepsilon t(M - 1 - \log M) + O(t^2),$$
$$D(\rho_{t,\varepsilon}\|\tau_{t,\varepsilon}) = c_\varepsilon \left(Mt\log M + (1-Mt)\log\frac{1-Mt}{1-t} \right) = c_\varepsilon t(M\log M - M + 1) + O(t^2),$$

so $\lim_{t\to 0^+} D(\tau_{t,\varepsilon}\|\rho_{t,\varepsilon})/D(\rho_{t,\varepsilon}\|\tau_{t,\varepsilon}) = \gamma(M)$ independently of ε. Exchanging $\rho_{t,\varepsilon}$ and $\tau_{t,\varepsilon}$ with $M = 1/m$ proves sharpness of $\gamma(1/m)^{-1}$. □

Remark 2.5 (Monotonicity and boundary values of γ)**.** Since $\gamma(M)$ is the ratio of the integrals of $t(1+(M-1)t)^{-1}$ and $(1-t)(1+(M-1)t)^{-1}$, Lemma 2.3 applied to the ratio of weights $(1+(M_2-1)t)^{-1}/(1+(M_1-1)t)^{-1}$ (which is strictly decreasing in t for $1 < M_1 < M_2$) immediately implies that $\gamma : (1,\infty) \to (0,1)$ is strictly decreasing, with $\lim_{M\to 1^+}\gamma(M) = 1$ and $\lim_{M\to\infty}\gamma(M) = 0$.

3. Complete Pimsner–Popa Index and Universal Strict cMLSI Lower Bounds

We now apply Theorem 2.4 to quantum Markov semigroups driven by idempotent CPTP projections $\mathcal{E}_* : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ ($\mathcal{E}_*^2 = \mathcal{E}_*$). This class encompasses all Schrödinger-picture quantum conditional expectations onto finite-dimensional von Neumann subalgebras (including generalized non-tracial depolarizing maps, full and block dephasing/pinching maps, and partial-trace reset channels).

Definition 3.1 (Complete Pimsner–Popa index)**.** Let $\mathcal{E}_* : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ be a CPTP idempotent map. Its *complete Pimsner–Popa index* is

$$C_{\mathrm{cb}}(\mathcal{E}_*) := \inf\Big\{ C \ge 1 \ :\ \rho_{AB} \le C\,(\mathcal{E}_* \otimes \mathrm{id}_B)(\rho_{AB}) \quad \forall\, d_B \ge 1,\ \forall\, \rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathbb{C}^{d_B}) \Big\}, \tag{14}$$

with $\inf\emptyset := +\infty$. This agrees with the inverse of the completely bounded Pimsner–Popa constant studied in [5].

Lemma 3.2 (Exact entropy production for idempotent CPTP projections)**.** *Let $\mathcal{E}_* : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ be a CPTP map with $\mathcal{E}_*^2 = \mathcal{E}_*$, and let $\mathcal{L} := \mathcal{E}_* - \mathrm{id}$. Then for every $d_B \ge 1$ and every $\rho_{AB} \in \mathcal{D}_+(\mathcal{H}_A \otimes \mathbb{C}^{d_B})$ with $\tau_{AB} := (\mathcal{E}_* \otimes \mathrm{id}_B)(\rho_{AB}) \in \mathcal{D}_+(\mathcal{H}_A \otimes \mathbb{C}^{d_B})$, the semigroup $\rho_t := (e^{t\mathcal{L}} \otimes \mathrm{id}_B)(\rho_{AB}) = e^{-t}\rho_{AB} + (1-e^{-t})\tau_{AB}$ satisfies $(\mathcal{E}_* \otimes \mathrm{id}_B)(\rho_t) = \tau_{AB}$ for all $t \ge 0$, and*

$$\mathrm{EP}_{\mathcal{L}\otimes\mathrm{id}_B}(\rho_{AB}) \ =\ -\,\frac{d}{dt}\Big|_{t=0} D(\rho_t\|\tau_{AB}) \ =\ D(\rho_{AB}\|\tau_{AB}) + D(\tau_{AB}\|\rho_{AB}). \tag{15}$$

Proof. Because $\mathcal{E}_*^2 = \mathcal{E}_*$, linearity gives $(\mathcal{E}_* \otimes \mathrm{id}_B)(\rho_t) = e^{-t}\tau_{AB} + (1-e^{-t})\tau_{AB} = \tau_{AB}$ identically for all $t \ge 0$. Moreover, $\dot\rho_0 = \tau_{AB} - \rho_{AB}$ has trace zero, so $\mathrm{Tr}(\rho_{AB}\,\frac{d}{dt}\big|_{t=0}\log\rho_t) = \mathrm{Tr}(\dot\rho_0) = 0$ as in Lemma 2.1. Differentiating $D(\rho_t\|\tau_{AB})$ at $t=0$ therefore gives

$$-\,\frac{d}{dt}\Big|_{t=0} D(\rho_t\|\tau_{AB}) = -\,\mathrm{Tr}\big((\tau_{AB}-\rho_{AB})(\log\rho_{AB} - \log\tau_{AB})\big) = D(\rho_{AB}\|\tau_{AB}) + D(\tau_{AB}\|\rho_{AB}).$$ □

Theorem 3.3 (Universal strict cMLSI lower bound for idempotent projections)**.** *Let $\mathcal{E}_* : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ be a CPTP idempotent map ($\mathcal{E}_*^2 = \mathcal{E}_*$) with finite complete Pimsner–Popa index $C_{\mathrm{cb}}(\mathcal{E}_*) \in (1,\infty)$, and let $\mathcal{L} := \mathcal{E}_* - \mathrm{id}$. Then*

$$\alpha_{\mathrm{cb}}(\mathcal{L}) \ \ge\ \frac{1}{2}\Big(1 + \gamma\big(C_{\mathrm{cb}}(\mathcal{E}_*)\big)\Big) \ >\ \frac{1}{2}. \tag{16}$$

Equivalently, for every auxiliary dimension $d_B \geq 1$, every state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathbb{C}^{d_B})$, and every $t \geq 0$,

$$D\big((e^{t\mathcal{L}} \otimes \mathrm{id}_B)(\rho_{AB}) \,\|\, (\mathcal{E}_* \otimes \mathrm{id}_B)(\rho_{AB})\big) \;\leq\; \exp\Big(-\big(1+\gamma(C_{\mathrm{cb}}(\mathcal{E}_*))\big)t\Big)\, D\big(\rho_{AB} \,\|\, (\mathcal{E}_* \otimes \mathrm{id}_B)(\rho_{AB})\big).$$

Proof. Fix $d_B \geq 1$, set $M := C_{\mathrm{cb}}(\mathcal{E}_*) > 1$, and let $\alpha_0 := \frac{1}{2}(1+\gamma(M))$. First assume $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathbb{C}^{d_B})$ is faithful on the support of $\tau_{AB} := (\mathcal{E}_* \otimes \mathrm{id}_B)(\rho_{AB})$, i.e. $\mathrm{supp}(\rho_{AB}) = \mathrm{supp}(\tau_{AB})$ (note that $\rho_{AB} \leq M\tau_{AB}$ always implies $\mathrm{supp}(\rho_{AB}) \subseteq \mathrm{supp}(\tau_{AB})$). Restricting both operators to $\mathcal{K} := \mathrm{supp}(\tau_{AB})$, both $\rho_{AB}|_{\mathcal{K}}$ and $\tau_{AB}|_{\mathcal{K}}$ belong to $\mathcal{D}_+(\mathcal{K})$ and satisfy $\rho_{AB}|_{\mathcal{K}} \leq M\tau_{AB}|_{\mathcal{K}}$; moreover, the trajectory $\rho_t = e^{-t}\rho_{AB} + (1-e^{-t})\tau_{AB}$ remains supported on $\mathcal{K}$ for all $t \geq 0$, so Lemma 3.2 applies verbatim on $\mathcal{K}$. If $\rho_{AB} = \tau_{AB}$ there is nothing to prove; otherwise Lemma 3.2 and Theorem 2.4(a) give

$$\mathrm{EP}_{\mathcal{L}\otimes\mathrm{id}_B}(\rho_{AB}) = D(\rho_{AB}\|\tau_{AB}) + D(\tau_{AB}\|\rho_{AB}) \;\geq\; \big(1+\gamma(M)\big)\, D(\rho_{AB}\|\tau_{AB}) = 2\alpha_0\, D(\rho_{AB}\|\tau_{AB}).$$

Now let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathbb{C}^{d_B})$ be arbitrary (possibly rank-deficient inside $\mathrm{supp}(\tau_{AB})$). For any $\varepsilon \in (0,1)$, set $\rho^{(\varepsilon)} := (1-\varepsilon)\rho_{AB} + \varepsilon\tau_{AB}$. Since $\mathcal{E}_*$ is idempotent, $(\mathcal{E}_* \otimes \mathrm{id}_B)(\rho^{(\varepsilon)}) = \tau_{AB}$, and $\mathrm{supp}(\rho^{(\varepsilon)}) = \mathrm{supp}(\tau_{AB})$. Moreover, the entire trajectory $\rho_t^{(\varepsilon)} := (e^{t\mathcal{L}} \otimes \mathrm{id}_B)(\rho^{(\varepsilon)})$ satisfies $\mathrm{supp}(\rho_t^{(\varepsilon)}) = \mathrm{supp}(\tau_{AB})$ for all $t \geq 0$. Applying the differential inequality above along $t \mapsto \rho_t^{(\varepsilon)}$ and integrating via Grönwall's lemma yields

$$D\big(\rho_t^{(\varepsilon)} \,\|\, \tau_{AB}\big) \;\leq\; e^{-2\alpha_0 t}\, D\big(\rho^{(\varepsilon)} \,\|\, \tau_{AB}\big) \qquad \text{for all } t \geq 0.$$

Sending $\varepsilon \to 0^+$ and using continuity of Umegaki relative entropy on the finite-dimensional support $\mathrm{supp}(\tau_{AB})$ yields $D(\rho_t\|\tau_{AB}) \leq e^{-2\alpha_0 t} D(\rho_{AB}\|\tau_{AB})$. Taking the infimum over $d_B \geq 1$ proves (16). □

We next compute the exact complete Pimsner–Popa index for the two fundamental families of quantum conditional expectations.

Theorem 3.4 (Exact complete Pimsner–Popa indices)**.** *Let $\mathcal{H}_A = \mathbb{C}^d$ ($d \geq 2$).*

(a) Generalized depolarizing projection. *For every faithful state $\sigma \in \mathcal{D}_+(\mathbb{C}^d)$, the projection $\mathcal{E}_\sigma(\rho) = \mathrm{Tr}(\rho)\,\sigma$ has exact complete Pimsner–Popa index*

$$C_{\mathrm{cb}}(\mathcal{E}_\sigma) \;=\; \mathrm{Tr}(\sigma^{-1}) \;\geq\; d^2, \tag{17}$$

with equality $C_{\mathrm{cb}}(\mathcal{E}_\sigma) = d^2$ if and only if $\sigma = I_d/d$. Consequently,

$$\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) \;\geq\; \frac{1}{2}\Big(1+\gamma\big(\mathrm{Tr}(\sigma^{-1})\big)\Big). \tag{18}$$

(b) Orthogonal r-block pinching. *Let $\{P_k\}_{k=1}^r$ ($r \geq 2$) be non-zero mutually orthogonal projections on $\mathbb{C}^d$ summing to I_d, and let $\mathcal{P}(\rho) := \sum_{k=1}^r P_k\rho P_k$. Then $C_{\mathrm{cb}}(\mathcal{P}) = r$, and hence*

$$\alpha_{\mathrm{cb}}(\mathcal{P} - \mathrm{id}) \;\geq\; \frac{1}{2}\big(1+\gamma(r)\big). \tag{19}$$

In particular, for d-dimensional dephasing $\Delta_d(\rho) = \sum_{i=1}^d |i\rangle\langle i|\rho|i\rangle\langle i|$, we have $C_{\mathrm{cb}}(\Delta_d) = d$ and $\alpha_{\mathrm{cb}}(\Delta_d - \mathrm{id}) \geq \frac{1}{2}(1+\gamma(d))$.

Proof. Proof of (a). Let $H(\sigma) := \mathrm{Tr}(\sigma^{-1})$. By convexity of the operator inequality $\rho_{AB} \leq H(\sigma)(\sigma \otimes \rho_B)$ in ρ_{AB}, it suffices to verify it for pure bipartite states $\rho_{AB} = |\psi\rangle\langle\psi|$ on $\mathbb{C}^d \otimes \mathbb{C}^{d_B}$. Write the Schmidt decomposition $|\psi\rangle = \sum_{k=1}^r \sqrt{\mu_k}\,|u_k\rangle_A \otimes |v_k\rangle_B$ with $\mu_k > 0$, orthonormal $\{|u_k\rangle_A\}_{k=1}^r \subset \mathbb{C}^d$, and orthonormal $\{|v_k\rangle_B\}_{k=1}^r \subset \mathbb{C}^{d_B}$ ($r \leq d$). Then $\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \sum_{k=1}^r \mu_k |v_k\rangle\langle v_k|$. Letting $\rho_B^{-1/2}$ denote the Moore–Penrose pseudoinverse on $\mathcal{K}_B := \mathrm{supp}(\rho_B) =$

$\mathrm{span}\{|v_k\rangle_B\}$, define $|\phi\rangle := (\sigma^{-1/2}\otimes\rho_B^{-1/2})|\psi\rangle = \sum_{k=1}^r(\sigma^{-1/2}|u_k\rangle_A)\otimes|v_k\rangle_B$. Because $\{|v_k\rangle_B\}$ is orthonormal and $\{|u_k\rangle_A\}$ extends to an orthonormal basis of $\mathbb{C}^d$,

$$\|\phi\|^2 \;=\; \sum_{k=1}^r\langle u_k|\sigma^{-1}|u_k\rangle \;\le\; \mathrm{Tr}(\sigma^{-1}) \;=\; H(\sigma).$$

Since $|\phi\rangle\langle\phi| \le \|\phi\|^2\, I_A\otimes P_{\mathcal{K}_B}$ and $|\psi\rangle = (\sigma^{1/2}\otimes\rho_B^{1/2})|\phi\rangle$, conjugating by $\sigma^{1/2}\otimes\rho_B^{1/2}$ gives

$$|\psi\rangle\langle\psi| \;\le\; \|\phi\|^2\,(\sigma\otimes\rho_B) \;\le\; H(\sigma)\,(\sigma\otimes\rho_B).$$

Hence $C_{\mathrm{cb}}(\mathcal{E}_\sigma)\le H(\sigma)$. Conversely, writing $\sigma=\sum_{i=1}^d\lambda_i|i\rangle\langle i|$ and taking $d_B=d$ with the σ-inverted Choi unit vector $|\psi_\sigma\rangle := H(\sigma)^{-1/2}\sum_{i=1}^d\lambda_i^{-1/2}|i\rangle_A|i\rangle_B$, we have $\rho_B = H(\sigma)^{-1}\sum_{i=1}^d\lambda_i^{-1}|i\rangle\langle i|$ and

$$(\sigma\otimes\rho_B)|\psi_\sigma\rangle = H(\sigma)^{-3/2}\sum_{i=1}^d\lambda_i\cdot\lambda_i^{-1}\cdot\lambda_i^{-1/2}|i\rangle_A|i\rangle_B = H(\sigma)^{-1}|\psi_\sigma\rangle. \tag{20}$$

Thus $\langle\psi_\sigma|(\sigma\otimes\rho_B)|\psi_\sigma\rangle = H(\sigma)^{-1}$, forcing $C\ge H(\sigma)$ whenever $|\psi_\sigma\rangle\langle\psi_\sigma|\le C(\sigma\otimes\rho_B)$. Therefore $C_{\mathrm{cb}}(\mathcal{E}_\sigma)=H(\sigma)$. By Cauchy–Schwarz, $d^2=(\sum_i\lambda_i^{1/2}\lambda_i^{-1/2})^2\le(\sum_i\lambda_i)(\sum_i\lambda_i^{-1})=H(\sigma)$ with equality iff all $\lambda_i=1/d$. Applying Theorem 3.3 gives (18).

Proof of (b). For any $\rho_{AB}\ge0$, letting $\widetilde{P}_k := P_k\otimes I_B$ and $\omega := e^{2\pi i/r}$, the discrete Fourier identity $\frac{1}{r}\sum_{m=0}^{r-1}U^m\rho_{AB}U^{-m} = (\mathcal{P}\otimes\mathrm{id}_B)(\rho_{AB})$ with unitary $U := \sum_{k=1}^r\omega^k\widetilde{P}_k$ implies $r(\mathcal{P}\otimes\mathrm{id}_B)(\rho_{AB})-\rho_{AB} = \sum_{m=1}^{r-1}U^m\rho_{AB}U^{-m}\ge0$, so $C_{\mathrm{cb}}(\mathcal{P})\le r$. Conversely, picking unit vectors $|e_k\rangle\in\mathrm{ran}(P_k)$ and setting $|\phi_+\rangle := r^{-1/2}\sum_{k=1}^r|e_k\rangle$ gives $\mathcal{P}(|\phi_+\rangle\langle\phi_+|)=r^{-1}\sum_{k=1}^r|e_k\rangle\langle e_k|$ and $\langle\phi_+|\mathcal{P}(|\phi_+\rangle\langle\phi_+|)|\phi_+\rangle = r^{-1}$, so $C_{\mathrm{cb}}(\mathcal{P})=r$. Theorem 3.3 yields (19). □

As an immediate consequence, for the tracial qubit depolarizing semigroup ($d=2$, $C_{\mathrm{cb}}=4$) we obtain

$$\alpha_{\mathrm{cb}}(\mathcal{L}_{I_2/2}) \;\ge\; \frac{1}{2}\big(1+\gamma(4)\big) = \frac{3\log4}{2(4\log4-3)} = 0.81701240366958\ldots,$$

improving the universal lower bound $1/2$ by $+63.4\%$; for qubit dephasing ($C_{\mathrm{cb}}=2$) we obtain $\alpha_{\mathrm{cb}}(\Delta_2-\mathrm{id})\ge\frac{\log2}{2(2\log2-1)} = 0.89717486239052\ldots$. Moreover, complete tensorization immediately lifts Theorem 3.4(a) to arbitrary N-site non-tracial depolarizing spin chains with no dimension or system-size degradation.

Corollary 3.5 (Complete tensorization for N-site non-tracial depolarizing spin chains)**.** *Let $\sigma_1,\ldots,\sigma_N$ be faithful states on $\mathbb{C}^{d_1},\ldots,\mathbb{C}^{d_N}$, and consider the N-site product depolarizing generator $\mathcal{L}^{(N)} := \sum_{j=1}^N\mathrm{id}^{\otimes(j-1)}\otimes\mathcal{L}_{\sigma_j}\otimes\mathrm{id}^{\otimes(N-j)}$ with stationary state $\omega := \bigotimes_{j=1}^N\sigma_j$. Then*

$$\alpha_{\mathrm{cb}}\big(\mathcal{L}^{(N)}\big) \;\ge\; \min_{1\le j\le N}\alpha_{\mathrm{cb}}(\mathcal{L}_{\sigma_j}) \;\ge\; \frac{1}{2}\Big(1+\gamma\Big(\max_{1\le j\le N}\mathrm{Tr}(\sigma_j^{-1})\Big)\Big). \tag{21}$$

In particular, for N-qubit tracial depolarizing noise, every state ρ and $t\ge0$ satisfy

$$D\big(e^{t\mathcal{L}^{(N)}}(\rho)\,\|\,(I_2/2)^{\otimes N}\big) \;\le\; e^{-1.6340248\,t}\,D\big(\rho\,\|\,(I_2/2)^{\otimes N}\big).$$

Proof. Fix any ancilla $\mathbb{C}^{d_B}$ and any faithful state $\rho\in\mathcal{D}_+(\mathbb{C}^{d_1}\otimes\cdots\otimes\mathbb{C}^{d_N}\otimes\mathbb{C}^{d_B})$ with marginal $\rho_B := \mathrm{Tr}_{A_1\ldots A_N}(\rho)$ and local reset states $\tau^{(j)} := (\mathcal{E}_{\sigma_j}\otimes\mathrm{id}_{\setminus j})(\rho) = \sigma_j\otimes\rho_{\setminus j}$. Let $\tau_{\mathrm{tot}} := \omega\otimes\rho_B$. First, for each $j\in\{1,\ldots,N\}$, $\log\tau_{\mathrm{tot}}-\log\tau^{(j)} = I_{A_j}\otimes\big(\log(\bigotimes_{k\ne j}\sigma_k\otimes\rho_B)-\log\rho_{\setminus j}\big)$ acts trivially on A_j. Since $\mathrm{Tr}_{A_j}(\rho-\tau^{(j)}) = \rho_{\setminus j}-\rho_{\setminus j}=0$, we have $\mathrm{Tr}((\rho-\tau^{(j)})(\log\tau_{\mathrm{tot}}-\log\tau^{(j)}))=0$, and

hence

$$\begin{aligned}\mathrm{EP}_{\mathcal{L}^{(N)}\otimes\mathrm{id}_B}(\rho) &= \sum_{j=1}^{N}\mathrm{Tr}\big((\rho-\tau^{(j)})(\log\rho-\log\tau_{\mathrm{tot}})\big)\\ &= \sum_{j=1}^{N}\mathrm{Tr}\big((\rho-\tau^{(j)})(\log\rho-\log\tau^{(j)})\big) = \sum_{j=1}^{N}\mathrm{EP}_{\mathcal{L}_{\sigma_j}\otimes\mathrm{id}_{\setminus j}}(\rho).\end{aligned}$$

Second, denoting $\rho_{j\dots N,B} := \mathrm{Tr}_{A_1\dots A_{j-1}}(\rho)$ (with $\rho_{N+1\dots N,B} := \rho_B$), the von Neumann entropies telescope identically:

$$\begin{aligned}\sum_{j=1}^{N} D\big(\rho_{j\dots N,B}\,\|\,\sigma_j\otimes\rho_{j+1\dots N,B}\big) &= \sum_{j=1}^{N}\Big(-S(\rho_{j\dots N,B})+S(\rho_{j+1\dots N,B})-\mathrm{Tr}(\rho_{A_j}\log\sigma_j)\Big)\\ &= -S(\rho)+S(\rho_B)-\sum_{j=1}^{N}\mathrm{Tr}(\rho_{A_j}\log\sigma_j) = D(\rho\,\|\,\omega\otimes\rho_B).\end{aligned}$$

By the data-processing inequality for quantum relative entropy under $\mathrm{Tr}_{A_1\dots A_{j-1}}$, each summand satisfies $D(\rho_{j\dots N,B}\|\sigma_j\otimes\rho_{j+1\dots N,B}) \le D(\rho\|\sigma_j\otimes\rho_{\setminus j}) = D(\rho\|\tau^{(j)})$. Combining the two displays with $\mathrm{EP}_{\mathcal{L}_{\sigma_j}\otimes\mathrm{id}_{\setminus j}}(\rho) \ge 2\alpha_{\mathrm{cb}}(\mathcal{L}_{\sigma_j})D(\rho\|\tau^{(j)})$ yields

$$\mathrm{EP}_{\mathcal{L}^{(N)}\otimes\mathrm{id}_B}(\rho) \;\ge\; 2\Big(\min_{1\le j\le N}\alpha_{\mathrm{cb}}(\mathcal{L}_{\sigma_j})\Big)\sum_{j=1}^{N}D(\rho\|\tau^{(j)}) \;\ge\; 2\Big(\min_{1\le j\le N}\alpha_{\mathrm{cb}}(\mathcal{L}_{\sigma_j})\Big)\,D(\rho\,\|\,\omega\otimes\rho_B).$$

For general (possibly rank-deficient) states $\rho\in\mathcal{D}(\mathbb{C}^{d_1}\otimes\cdots\otimes\mathbb{C}^{d_N}\otimes\mathbb{C}^{d_B})$, let $\mathcal{K}_B := \mathrm{supp}(\rho_B)\subseteq\mathbb{C}^{d_B}$. Because $\mathrm{supp}(\rho)\subseteq\mathbb{C}^{d_1}\otimes\cdots\otimes\mathbb{C}^{d_N}\otimes\mathcal{K}_B$ and $\mathcal{L}^{(N)}\otimes\mathrm{id}_B$ leaves ρ_B invariant, both ρ and its entire trajectory remain supported on $\mathcal{H}_{\mathrm{red}} := \mathbb{C}^{d_1}\otimes\cdots\otimes\mathbb{C}^{d_N}\otimes\mathcal{K}_B$, on which $\omega\otimes\rho_B|_{\mathcal{K}_B}\in\mathcal{D}_+(\mathcal{H}_{\mathrm{red}})$ is strictly positive. Applying Grönwall's lemma on $\mathcal{H}_{\mathrm{red}}$ to the faithful trajectory $t\mapsto(e^{t\mathcal{L}^{(N)}}\otimes\mathrm{id}_{\mathcal{K}_B})((1-\varepsilon)\rho+\varepsilon\omega\otimes\rho_B)$ and sending $\varepsilon\to0^+$ extends the exponential decay to all states. Applying Theorem 3.4(a) and the monotonicity of γ concludes the proof. □

4. The σ-Inverted Choi Isotropic Family and Universal Entanglement Degradation

We now construct an explicit bipartite entangled family that tightly bounds $\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma)$ from above and proves that entanglement strictly degrades the MLSI constant for every full-rank state σ in every dimension $d\ge2$.

Theorem 4.1 (σ-inverted Choi isotropic upper bound)**.** *Let $\sigma\in\mathcal{D}_+(\mathbb{C}^d)$ ($d\ge2$) be any faithful state, and set $p_{\mathrm{eff}} := \mathrm{Tr}(\sigma^{-1})^{-1}\in(0,1/d^2]$. Then*

$$\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) \;\le\; \frac{1}{2}\Big(1+\kappa_2(p_{\mathrm{eff}})\Big) \;=\; \frac{1}{2}\Big(1+\kappa_2\big(\mathrm{Tr}(\sigma^{-1})^{-1}\big)\Big). \tag{22}$$

Proof. Write the spectral decomposition $\sigma=\sum_{i=1}^{d}\lambda_i|i\rangle\langle i|$ with $\lambda_i>0$, and take an auxiliary space $\mathcal{H}_B=\mathbb{C}^d$. Consider the σ-inverted Choi unit vector $|\psi_\sigma\rangle := p_{\mathrm{eff}}^{1/2}\sum_{i=1}^{d}\lambda_i^{-1/2}|i\rangle_A|i\rangle_B$, its rank-one projection $P_\sigma := |\psi_\sigma\rangle\langle\psi_\sigma|$, and its marginal state $\rho_B := \mathrm{Tr}_A(P_\sigma) = p_{\mathrm{eff}}\sum_{i=1}^{d}\lambda_i^{-1}|i\rangle\langle i|\in\mathcal{D}_+(\mathbb{C}^d)$. Set $\tau_{AB} := \sigma\otimes\rho_B\in\mathcal{D}_+(\mathbb{C}^d\otimes\mathbb{C}^d)$.

By (20), $|\psi_\sigma\rangle$ is an exact eigenvector of τ_{AB} with eigenvalue p_{eff}:

$$\tau_{AB}\,|\psi_\sigma\rangle \;=\; p_{\mathrm{eff}}\,|\psi_\sigma\rangle.$$

Therefore τ_{AB} commutes with P_σ and decomposes orthogonally as $\tau_{AB} = p_{\text{eff}} P_\sigma + (1 - p_{\text{eff}})\tau_\perp$, where

$$\tau_\perp \;:=\; \frac{\tau_{AB} - p_{\text{eff}} P_\sigma}{1 - p_{\text{eff}}} \;\geq\; 0$$

is a state supported strictly on the orthogonal complement $(I - P_\sigma)(\mathbb{C}^d \otimes \mathbb{C}^d)$ (since $\tau_{AB} > 0$, every eigenvalue of τ_{AB} on $\operatorname{ran}(I - P_\sigma)$ is strictly positive, so $\operatorname{supp}(\tau_\perp) = \operatorname{ran}(I - P_\sigma)$). Moreover,

$$\operatorname{Tr}_A(\tau_\perp) \;=\; \frac{\operatorname{Tr}_A(\sigma \otimes \rho_B) - p_{\text{eff}} \operatorname{Tr}_A(P_\sigma)}{1 - p_{\text{eff}}} \;=\; \frac{\rho_B - p_{\text{eff}} \rho_B}{1 - p_{\text{eff}}} \;=\; \rho_B.$$

Now, for any $q \in (0, 1)$, define the σ-isotropic bipartite state

$$\rho_{AB}(q) \;:=\; q\, P_\sigma + (1 - q)\, \tau_\perp \;\in\; \mathcal{D}_+(\mathbb{C}^d \otimes \mathbb{C}^d). \tag{23}$$

By construction, $\operatorname{Tr}_A(\rho_{AB}(q)) = q\rho_B + (1 - q)\rho_B = \rho_B$ identically for all $q \in (0, 1)$, so $(\mathcal{E}_\sigma \otimes \mathrm{id}_B)(\rho_{AB}(q)) = \sigma \otimes \rho_B = \tau_{AB}$. Furthermore, because P_σ and $\tau_\perp$ have mutually orthogonal supports,

$$\begin{aligned} D\big(\rho_{AB}(q) \,\|\, \tau_{AB}\big) &= q \log \frac{q}{p_{\text{eff}}} + (1 - q) \log \frac{1 - q}{1 - p_{\text{eff}}} \;=\; d_2(q \| p_{\text{eff}}), \\ D\big(\tau_{AB} \,\|\, \rho_{AB}(q)\big) &= p_{\text{eff}} \log \frac{p_{\text{eff}}}{q} + (1 - p_{\text{eff}}) \log \frac{1 - p_{\text{eff}}}{1 - q} \;=\; d_2(p_{\text{eff}} \| q). \end{aligned}$$

By Lemma 3.2, for every $q \in (0, 1) \setminus \{p_{\text{eff}}\}$,

$$\alpha_{\text{cb}}(\mathcal{L}_\sigma) \;\leq\; \frac{\mathrm{EP}_{\mathcal{L}_\sigma \otimes \mathrm{id}_B}(\rho_{AB}(q))}{2\, D(\rho_{AB}(q) \| \tau_{AB})} \;=\; \frac{1}{2}\left(1 + \frac{d_2(p_{\text{eff}} \| q)}{d_2(q \| p_{\text{eff}})}\right).$$

Taking the infimum over $q \in (0, 1) \setminus \{p_{\text{eff}}\}$ gives (22). □

To conclude strict entanglement degradation $\alpha_{\text{cb}}(\mathcal{L}_\sigma) < \alpha_1(\mathcal{L}_\sigma)$ rigorously for *every* full-rank state σ without appeal to numerics, we prove that $\kappa_2(p)$ is strictly increasing on $(0, 1/2]$.

Proposition 4.2 (Strict monotonicity of κ_2 and localization of $q_*(p)$)**.** *For every $p \in (0, 1/2)$, let $R(p, q) := d_2(p\|q)/d_2(q\|p)$ for $q \in (0, 1) \setminus \{p\}$. Then:*

(a) *$R(p, q) < 1$ for all $q \in (p, 1 - p)$, whereas $R(p, q) \geq 1$ for all $q \in (0, p) \cup [1 - p, 1)$. In particular, $\kappa_2(1/2) = 1$ and $\kappa_2(p) < 1$ for all $p \in (0, 1/2)$.*

(b) *For every $p \in (0, 1/2)$, $\partial_q R(p, q) > 0$ for all $q \in [1/2, 1 - p]$. Consequently, every global minimizer $q_*(p)$ of $q \mapsto R(p, q)$ lies strictly in $(p, 1/2)$.*

(c) *The function $p \mapsto \kappa_2(p)$ is strictly increasing on $(0, 1/2]$.*

Proof. Applying Lemma 2.1 to the classical binary states $(q, 1 - q)$ and $(p, 1 - p)$ with $x(t) := (1 - t)p + tq$ and $w(t) := \big(x(t)(1 - x(t))\big)^{-1}$ gives the exact representation

$$d_2(p\|q) = (q - p)^2 \int_0^1 t\, w(t)\, dt, \qquad d_2(q\|p) = (q - p)^2 \int_0^1 (1 - t)\, w(t)\, dt, \tag{24}$$

so $R(p, q) = \big(\int_0^1 t w(t)\, dt\big) / \big(\int_0^1 (1 - t) w(t)\, dt\big)$.

Proof of (a). Notice that $d_2(p\|q) - d_2(q\|p) = (q - p)^2 \int_{1/2}^1 (2t - 1)\big(w(t) - w(1 - t)\big)\, dt$. Moreover, $x(t) + x(1 - t) = p + q$ and $x(t)(1 - x(t)) - x(1 - t)(1 - x(1 - t)) = (x(t) - x(1 - t))(1 - p - q) = (2t - 1)(q - p)(1 - p - q)$. For $t \in (1/2, 1)$ and $p \in (0, 1/2)$:

- if $q \in (p, 1 - p)$, then $(q - p)(1 - p - q) > 0$, so $w(t) < w(1 - t)$ and $R(p, q) < 1$;
- if $q \in (0, p)$, then $q - p < 0$ and $1 - p - q > 0$, so $w(t) > w(1 - t)$ and $R(p, q) > 1$;
- if $q \in [1 - p, 1)$, then $q - p > 0$ and $1 - p - q \leq 0$, so $w(t) \geq w(1 - t)$ and $R(p, q) \geq 1$.

At $p = 1/2$, $1-p-q = 1/2-q$ has the opposite sign to $q-p$, giving $R(1/2,q) > 1$ for all $q \neq 1/2$, while $\lim_{q\to 1/2} R(1/2,q) = 1$, so $\kappa_2(1/2) = 1$.

Proof of (b). Fix $p \in (0,1/2)$ and define $u(q) := \partial_q d_2(p\|q) = \frac{q-p}{q(1-q)}$ and $v(q) := \partial_q d_2(q\|p) = \log\frac{q(1-p)}{p(1-q)}$. Then $\partial_q R(p,q) = B(p,q)/d_2(q\|p)^2$, where

$$B(p,q) \;:=\; u(q)\, d_2(q\|p) - v(q)\, d_2(p\|q).$$

Differentiating $B(p,q)$ in q, the cross terms $u(q)v(q) - v(q)u(q)$ cancel identically:

$$\partial_q B(p,q) \;=\; u'(q)\, d_2(q\|p) - v'(q)\, d_2(p\|q).$$

Direct differentiation gives $v'(q) = \frac{1}{q(1-q)}$ and $u'(q) = \frac{1-p}{(1-q)^2} + \frac{p}{q^2}$, whose difference simplifies algebraically to

$$u'(q) - v'(q) \;=\; \frac{q^2(1-p) + p(1-q)^2 - q(1-q)}{q^2(1-q)^2} \;=\; \frac{(q-p)(2q-1)}{q^2(1-q)^2} \;\geq\; 0 \qquad \forall\, q \in \Big[\frac{1}{2}, 1\Big).$$

On $q \in [1/2, 1-p]$, Part (a) gives $d_2(q\|p) \geq d_2(p\|q) > 0$, so $\partial_q B(p,q) \geq v'(q)(d_2(q\|p) - d_2(p\|q)) + (u'(q) - v'(q))d_2(q\|p) \geq 0$ (and strictly positive for $q > 1/2$).

It remains to verify $B(p,1/2) > 0$ for every $p \in (0,1/2)$. Writing $p = \frac{1-x}{2}$ with $x \in (0,1)$, the standard Maclaurin series valid for $|x| < 1$ are

$$d_2\Big(\tfrac{1}{2}\,\Big\|\,\tfrac{1-x}{2}\Big) = -\frac{1}{2}\log(1-x^2) = \sum_{k=1}^{\infty}\frac{x^{2k}}{2k}, \qquad d_2\Big(\tfrac{1-x}{2}\,\Big\|\,\tfrac{1}{2}\Big) = \sum_{k=1}^{\infty}\frac{x^{2k}}{2k(2k-1)},$$

while $u(1/2) = 2x$ and $v(1/2) = \log\frac{1+x}{1-x} = \sum_{m=0}^{\infty}\frac{2x^{2m+1}}{2m+1}$. Multiplying these absolutely convergent series yields $B(\frac{1-x}{2}, \frac{1}{2}) = \sum_{n=1}^{\infty} c_n x^{2n+1}$ with

$$c_n \;=\; \frac{1}{n} - \sum_{k=1}^{n}\frac{1}{k(2k-1)(2n-2k+1)}.$$

Using the partial-fraction identities $\frac{1}{(2k-1)(2n-2k+1)} = \frac{1}{2n}\big(\frac{1}{2k-1} + \frac{1}{2n-2k+1}\big)$, $\frac{1}{k(2k-1)} = \frac{2}{2k-1} - \frac{1}{k}$, and $\frac{1}{k(2n+1-2k)} = \frac{1}{2n+1}\big(\frac{1}{k} + \frac{2}{2n+1-2k}\big)$, together with the re-indexing $\sum_{k=1}^{n}\frac{1}{2n+1-2k} = \sum_{k=1}^{n}\frac{1}{2k-1} =: O_n$ and $H_n := \sum_{k=1}^{n}\frac{1}{k}$, we obtain the exact closed form

$$c_n \;=\; \frac{D_n}{n(2n+1)}, \qquad D_n \;:=\; (2n+1) + nH_n - 2(n+1)O_n.$$

Direct evaluation gives $D_1 = D_2 = 0$ and $D_3 = \frac{7}{30} > 0$. For every $n \geq 2$, the first difference is

$$D_{n+1} - D_n \;=\; \frac{4n-1}{2n+1} - (2O_n - H_n).$$

However, $2O_n - H_n = \sum_{k=1}^{n}\frac{1}{k(2k-1)} < \sum_{k=1}^{\infty} 2\big(\frac{1}{2k-1} - \frac{1}{2k}\big) = 2\log 2 < \frac{7}{5}$ (the last inequality follows from $e^{7/10} > 1 + \frac{7}{10} + \frac{49}{200} + \frac{343}{6000} > 2.002 > 2$), whereas $\frac{4n-1}{2n+1} \geq \frac{7}{5}$ for all $n \geq 2$. Hence $D_{n+1} > D_n$ for all $n \geq 2$, so $D_n > 0$ and $c_n > 0$ for all $n \geq 3$. Therefore $B(p,1/2) > 0$ for all $p \in (0,1/2)$, and combining with $\partial_q B(p,q) \geq 0$ yields $B(p,q) > 0$ on $[1/2, 1-p]$. Since $R(p,q) \geq 1 > R(p,1/2)$ outside $(p, 1-p)$ and $\lim_{q\to p^+} R(p,q) = 1$, the continuous function $q \mapsto R(p,q)$ attains its global minimum at some $q_*(p) \in (p, 1/2)$.

Proof of (c). Let $0 < p_1 < p_2 \leq 1/2$. If $p_2 = 1/2$, Part (a) gives $\kappa_2(p_1) < 1 = \kappa_2(1/2)$. Now suppose $p_2 < 1/2$, and let $q := q_*(p_2) \in (p_2, 1/2)$ by Part (b). Set $x_i(t) := (1-t)p_i + tq$ and $w_i(t) := (x_i(t)(1-x_i(t)))^{-1}$ for $i = 1, 2$. Then for every $t \in [0,1)$, we have $0 < x_1(t) < x_2(t) < q < \frac{1}{2}$, which

implies both $(1-q)x_1(t) < q(1-x_1(t))$ (since $x_1(t) < q$) and $x_2(t) < 1-x_2(t)$ (since $x_2(t) < 1/2$). Multiplying these two positive inequalities gives $(1-q)x_1(t)x_2(t) < q(1-x_1(t))(1-x_2(t))$, whence

$$\frac{d}{dt}\log\frac{w_1(t)}{w_2(t)} = \frac{d}{dt}\log\frac{x_2(t)(1-x_2(t))}{x_1(t)(1-x_1(t))} = (p_2-p_1)\left(\frac{1-q}{(1-x_1(t))(1-x_2(t))} - \frac{q}{x_1(t)x_2(t)}\right) \;<\; 0.$$

Hence $G(t) := w_1(t)/w_2(t)$ is strictly decreasing on $[0,1]$. By Lemma 2.3 (whose double integral is strictly negative when G is strictly decreasing) and (24),

$$\kappa_2(p_1) \;\le\; R(p_1,q) = \frac{\int_0^1 t\, w_2(t)\, G(t)\, dt}{\int_0^1 (1-t)\, w_2(t)\, G(t)\, dt} \;<\; \frac{\int_0^1 t\, w_2(t)\, dt}{\int_0^1 (1-t)\, w_2(t)\, dt} = R(p_2,q) = \kappa_2(p_2). \qquad \square$$

Corollary 4.3 (Universal strict entanglement degradation for all depolarizing semigroups)**.** *For every dimension $d \ge 2$ and every faithful state $\sigma \in \mathcal{D}_+(\mathbb{C}^d)$,*

$$\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) \;\le\; \frac{1}{2}\Big(1+\kappa_2\big(\mathrm{Tr}(\sigma^{-1})^{-1}\big)\Big) \;<\; \frac{1}{2}\Big(1+\kappa_2\big(\lambda_{\min}(\sigma)\big)\Big) \;=\; \alpha_1(\mathcal{L}_\sigma). \tag{25}$$

In particular, for the tracial qubit depolarizing semigroup,

$$\begin{aligned} 0.8170124036\cdots = \frac{1+\gamma(4)}{2} \;&\le\; \alpha_{\mathrm{cb}}(\mathcal{L}_{I_2/2}) \\ &\le\; \frac{1+\kappa_2(1/4)}{2} = 0.9547063900\ldots \;<\; 1 \;=\; \alpha_1(\mathcal{L}_{I_2/2}). \end{aligned}$$

Proof. Because $d \ge 2$ and every eigenvalue λ_i of σ is strictly positive, $\mathrm{Tr}(\sigma^{-1}) = \lambda_{\min}(\sigma)^{-1} + \sum_{i\neq i_{\min}} \lambda_i^{-1} > \lambda_{\min}(\sigma)^{-1}$, while $\lambda_{\min}(\sigma) \le 1/d \le 1/2$. Therefore $0 < \mathrm{Tr}(\sigma^{-1})^{-1} < \lambda_{\min}(\sigma) \le 1/2$. Combining Theorem 4.1, Proposition 4.2(c), and the exact single-system identity (5) of [9] proves (25). $\square$

By contrast, for orthogonal r-block pinching (including d-dimensional dephasing), a single-system coherent isotropic witness already matches the Pimsner–Popa index r without ancilla entanglement:

Proposition 4.4 (Pinching and dephasing upper bound)**.** *Let $\mathcal{P}(\rho) = \sum_{k=1}^r P_k \rho P_k$ be an orthogonal r-block pinching ($r \ge 2$). Then*

$$\frac{1}{2}\big(1+\gamma(r)\big) \;\le\; \alpha_{\mathrm{cb}}(\mathcal{P}-\mathrm{id}) \;\le\; \alpha_1(\mathcal{P}-\mathrm{id}) \;\le\; \frac{1}{2}\Big(1+\kappa_2(1/r)\Big). \tag{26}$$

Proof. Pick unit vectors $|e_k\rangle \in \mathrm{ran}(P_k)$, set $\mathcal{S} := \mathrm{span}\{|e_1\rangle,\ldots,|e_r\rangle\}$, $|\phi_+\rangle := r^{-1/2}\sum_{k=1}^r |e_k\rangle$, $Q := |\phi_+\rangle\langle\phi_+|$, and $\tau := r^{-1}\sum_{k=1}^r |e_k\rangle\langle e_k|$. Then $\tau|_{\mathcal{S}} \in \mathcal{D}_+(\mathcal{S})$ satisfies $\tau|\phi_+\rangle = r^{-1}|\phi_+\rangle$, and for every $q \in (0,1)$ the state $\rho(q) := qQ + \frac{1-q}{1-r^{-1}}(\tau - r^{-1}Q)$ is faithful on $\mathcal{S}$, commutes with τ, and satisfies $\mathcal{P}(\rho(q)) = \tau$ (so the trajectory $e^{-t}\rho(q)+(1-e^{-t})\tau$ remains supported on $\mathcal{S}$ and Lemma 3.2 applies verbatim on $\mathcal{S}$), giving $D(\tau\|\rho(q))/D(\rho(q)\|\tau) = d_2(r^{-1}\|q)/d_2(q\|r^{-1})$. Taking the infimum over q yields $\alpha_1(\mathcal{P}-\mathrm{id}) \le \frac{1}{2}(1+\kappa_2(1/r))$, while the lower bound is Theorem 3.4(b). $\square$

5. Exact Two-Term Asymptotics and the Factor-of-2 Collapse Dichotomy

Although $\gamma(M)$ and $\kappa_2(1/M)$ differ at finite M (e.g. 0.6340 vs. 0.9094 at $M=4$), we now prove that they share the *exact same* leading asymptotic $1/\log M$ as $M \to \infty$, which unconditionally pins down the exact asymptotic cMLSI excess.

Theorem 5.1 (Exact two-term asymptotics of $\gamma(M)$ and $\kappa_2(1/M)$)**.** *As $M \to \infty$ with $L := \log M$,*

$$\gamma(M) = \frac{1}{L}\left(1 + \frac{1}{L} + O\left(\frac{1}{L^2}\right)\right), \tag{27}$$

$$\kappa_2\left(\frac{1}{M}\right) = \frac{1}{L}\left(1 + \frac{\log L + 2 - \log 2}{L} + O\left(\frac{(\log L)^2}{L^2}\right)\right). \tag{28}$$

Moreover, every minimizer $q_*(1/M)$ of $q \mapsto R(1/M, q)$ satisfies $q_*(1/M) = \frac{2}{L}\big(1 + O\big(\frac{\log L}{L}\big)\big)$.

Proof. Proof of (27). Dividing numerator and denominator of (10) by ML gives

$$\gamma(M) = \frac{1 - (1+L)/M}{L(1 - 1/L + 1/(ML))} = \frac{1}{L} \cdot \frac{1 + O(Le^{-L})}{1 - L^{-1} + O(L^{-1}e^{-L})} = \frac{1}{L}\left(1 + \frac{1}{L} + O\left(\frac{1}{L^2}\right)\right).$$

Upper bound in (28). Set $p := 1/M = e^{-L}$ and evaluate $R(p,q) = d_2(p\|q)/d_2(q\|p)$ at the explicit trial point $q_0 := 2/L$ (valid for $L > 4$). Since $-\log(1-q_0) = q_0 + \frac{1}{2}q_0^2 + O(q_0^3)$ and $(1-q_0)\log(1-q_0) = -q_0 + \frac{1}{2}q_0^2 + O(q_0^3)$, while $p\log(q_0/p) = O(Le^{-L})$, we obtain

$$d_2(p\|q_0) = -\log(1-q_0) + O(Le^{-L}) = \frac{2}{L}\left(1 + \frac{1}{L} + O\left(\frac{1}{L^2}\right)\right),$$

$$d_2(q_0\|p) = q_0\big(L + \log q_0 - 1\big) + \frac{q_0^2}{2} + O(q_0^3) = 2\left(1 + \frac{\log 2 - \log L - 1}{L} + O\left(\frac{1}{L^2}\right)\right).$$

Dividing gives

$$\kappa_2(p) \;\le\; R(p, q_0) \;=\; \frac{1}{L}\left(1 + \frac{\log L + 2 - \log 2}{L} + O\left(\frac{(\log L)^2}{L^2}\right)\right).$$

Lower bound in (28). By Proposition 4.2(b), it suffices to bound $R(p,q)$ from below for $q \in (p, 1/2]$. Split $(p, 1/2]$ into four regimes for sufficiently large L:

(1) *Regime I:* $q \in (p, M^{-1/2}]$. Here $q/p \le M^{1/2}$ and $(1-q)/(1-p) \le 1$, so the binary vector $(q, 1-q)$ satisfies $(q, 1-q) \le M^{1/2}(p, 1-p)$. By Theorem 2.4(a) and (27),

$$R(p,q) \;\ge\; \gamma(M^{1/2}) \;=\; \frac{2}{L} + O\left(\frac{1}{L^2}\right) \;>\; \frac{1}{L}\left(1 + \frac{\log L + 2 - \log 2}{L}\right).$$

(2) *Regime II:* $q \in [M^{-1/2}, L^{-2}]$. Here $pL/q \le LM^{-1/2} = O(L^{-3})$ and $\log q \le -2\log L$. Using $-\log(1-q) \ge q$ and $(1-q)\log(1-q) \le -q + q^2$, we obtain $d_2(p\|q)/q \ge 1 - O(L^{-3})$ and $d_2(q\|p)/q \le L - 2\log L - 1 + O(L^{-2})$, whence

$$R(p,q) \;\ge\; \frac{1 - O(L^{-3})}{L - 2\log L} \;=\; \frac{1}{L}\left(1 + \frac{2\log L}{L} + O\left(\frac{(\log L)^2}{L^2}\right)\right),$$

strictly exceeding the target bound for large L.

(3) *Regime III:* $q \in [\frac{4\log L}{L}, \frac{1}{2}]$. For all $q \in (0, 1/2]$, Taylor's series gives $-\log(1-q) \ge q + \frac{q^2}{2}$ and $(1-q)\log(1-q) \le -q + (2\log 2 - 1)q$, while $\log q \le -\log 2$ and $q^{-1} \le \frac{L}{4\log L}$. Hence

$$d_2(p\|q) \ge q\Big(1 + \frac{q}{2}\Big) - O(Le^{-L}) \ge q\left(1 + \frac{2\log L}{L} - O\left(\frac{L^2}{\log L}e^{-L}\right)\right),$$

and $d_2(q\|p) \le q\big(L - \log 2 - 1 + (2\log 2 - 1) + O(\frac{L}{\log L}e^{-L})\big) = q(L + \log 2 - 2 + O(\frac{L}{\log L}e^{-L})) < qL$ for large L. Thus $R(p,q) \ge \frac{1}{L}\big(1 + \frac{2\log L}{L} - O(\frac{L^2}{\log L}e^{-L})\big)$, again strictly exceeding the target bound.

(4) *Regime IV:* $q \in [L^{-2}, \frac{4\log L}{L}]$. Write $q = u/L$ with $u \in [L^{-1}, 4\log L]$. Here $pL/q \le L^3 e^{-L} = O(L^{-3})$, $q^2 \le 16(\log L)^2/L^2$, and $|\log u| \le \log L$, so that $\delta_L(u) := (\log u - \log L - 1)/L$ satisfies $|\delta_L(u)| \le (2\log L+1)/L = O\big(\frac{\log L}{L}\big)$. Using $-\log(1-q) \ge q+\frac{q^2}{2}$ and $(1-q)\log(1-q) \le -q+q^2$, we have

$$\frac{d_2(p\|q)}{q} \ge 1 + \frac{u}{2L} - O\left(\frac{1}{L^2}\right), \qquad \frac{d_2(q\|p)}{q} \le L\left(1 + \delta_L(u) + O\left(\frac{\log L}{L^2}\right)\right).$$

Since $|\delta_L(u)| = O\big(\frac{\log L}{L}\big)$, expanding $(1+\delta_L(u))^{-1} = 1 - \delta_L(u) + O((\log L)^2/L^2)$ yields

$$R(p,q) \;\ge\; \frac{1}{L}\left(1 + \frac{\log L + 1 + \frac{u}{2} - \log u}{L} - O\left(\frac{(\log L)^2}{L^2}\right)\right).$$

The convex function $f(u) := \frac{u}{2} - \log u$ achieves its unique global minimum on $(0,\infty)$ at $u_* = 2$, where $f(2) = 1 - \log 2$, proving the matching lower bound in (28).

Finally, any minimizer $q_* = u_*/L$ must lie in Regime IV for large L, and comparing the upper and lower bounds gives $f(u_*) - f(2) = O\big(\frac{(\log L)^2}{L}\big)$. Since f is strictly convex with $f''(2) = \frac{1}{4}$, this implies $u_* = 2 + o(1)$ (in particular $u_* \in [1,3]$ for all sufficiently large L). Substituting $q_* = u_*/L$ with $u_* \in [1,3]$ into the exact first-order optimality condition $B(p,q_*) = 0$ from Proposition 4.2(b) gives

$$\begin{aligned}
\frac{q_* - p}{q_*(1-q_*)} d_2(q_*\|p) &= \left(1 + \frac{u_*}{L} + O(L^{-2})\right)\left(u_* + \frac{u_*(\log u_* - \log L - 1)}{L} + O(L^{-2})\right) \\
&= u_* + \frac{u_*(\log u_* - \log L - 1 + u_*)}{L} + O\left(\frac{\log L}{L^2}\right), \\
\log\left(\frac{q_*(1-p)}{p(1-q_*)}\right) d_2(p\|q_*) &= \left(L + \log u_* - \log L + O(L^{-1})\right)\left(\frac{u_*}{L} + \frac{u_*^2}{2L^2} + O(L^{-3})\right) \\
&= u_* + \frac{u_*(\log u_* - \log L) + u_*^2/2}{L} + O\left(\frac{\log L}{L^2}\right).
\end{aligned}$$

Subtracting the two expansions cancels the $u_*(\log u_* - \log L)/L$ terms identically and yields

$$0 \;=\; B(p,q_*) \;=\; \frac{u_*(u_*-2)}{2L} + O\left(\frac{\log L}{L^2}\right),$$

whence $u_* = 2 + O\big(\frac{\log L}{L}\big)$ and $q_*(1/M) = \frac{2}{L}\big(1 + O\big(\frac{\log L}{L}\big)\big)$. □

Theorem 5.2 (Exact asymptotic cMLSI excess and entanglement collapse dichotomy)**.** (a) *For any sequence of full-rank states $\sigma_n \in \mathcal{D}_+(\mathbb{C}^{d_n})$ ($d_n \ge 2$) with $\mathrm{Tr}(\sigma_n^{-1}) \to \infty$,*

$$\alpha_{\mathrm{cb}}(\mathcal{L}_{\sigma_n}) - \frac{1}{2} \;=\; \frac{1+o(1)}{2\log \mathrm{Tr}(\sigma_n^{-1})}. \tag{29}$$

(b) *If moreover $\lambda_{\min}(\sigma_n) \to 0$, the asymptotic entanglement collapse ratio is governed exactly by the logarithmic ratio of the single-system and complete Pimsner–Popa indices:*

$$\frac{\alpha_{\mathrm{cb}}(\mathcal{L}_{\sigma_n}) - \frac{1}{2}}{\alpha_1(\mathcal{L}_{\sigma_n}) - \frac{1}{2}} \;=\; \big(1+o(1)\big)\,\frac{\log\big(\lambda_{\min}(\sigma_n)^{-1}\big)}{\log \mathrm{Tr}(\sigma_n^{-1})} \;\in\; \Big[\frac{1}{2} - o(1),\, 1\Big]. \tag{30}$$

In particular:

- (Exact factor-of-2 collapse for tracial and bounded-condition-number depolarizing.) *Whenever $\lambda_{\max}(\sigma_n)/\lambda_{\min}(\sigma_n) = O(1)$ and $d_n \to \infty$ (in particular for tracial $\sigma_d =$*

I_d/d), $\mathrm{Tr}(\sigma_n^{-1}) = \Theta(d_n^2)$ *while* $\lambda_{\min}(\sigma_n)^{-1} = \Theta(d_n)$, *so*

$$\lim_{n\to\infty} \frac{\alpha_{\mathrm{cb}}(\mathcal{L}_{\sigma_n}) - \frac{1}{2}}{\alpha_1(\mathcal{L}_{\sigma_n}) - \frac{1}{2}} \;=\; \frac{1}{2}.$$

- (Zero asymptotic collapse for strongly hierarchical Gibbs spectra.) *Whenever* $\log d_n = o\big(\log(\lambda_{\min}(\sigma_n)^{-1})\big)$ *(e.g. geometric Gibbs states* $\lambda_j \propto e^{-\beta j}$ *with fixed* $\beta > 0$*),* $\mathrm{Tr}(\sigma_n^{-1}) \le d_n\, \lambda_{\min}(\sigma_n)^{-1}$ *gives* $\log \mathrm{Tr}(\sigma_n^{-1}) = (1+o(1)) \log(\lambda_{\min}(\sigma_n)^{-1})$*, so the ratio in* (30) *converges to* 1.

(c) (Zero asymptotic collapse for dephasing / r-block pinching.) *For any sequence of orthogonal* r*-block pinchings* $\mathcal{P}_r$ *with* $r \to \infty$ *(in particular* d*-dimensional dephasing* Δ_d*),*

$$\alpha_{\mathrm{cb}}(\mathcal{P}_r - \mathrm{id}) - \frac{1}{2} \;=\; \frac{1+o(1)}{2\log r} \;=\; \big(1+o(1)\big)\Big(\alpha_1(\mathcal{P}_r - \mathrm{id}) - \frac{1}{2}\Big).$$

Proof. Combining Theorems 3.4(a) and 4.1 with Theorem 5.1 gives

$$\frac{1+o(1)}{2\log \mathrm{Tr}(\sigma_n^{-1})} = \frac{\gamma(\mathrm{Tr}(\sigma_n^{-1}))}{2} \le \alpha_{\mathrm{cb}}(\mathcal{L}_{\sigma_n}) - \frac{1}{2} \le \frac{\kappa_2(\mathrm{Tr}(\sigma_n^{-1})^{-1})}{2} = \frac{1+o(1)}{2\log \mathrm{Tr}(\sigma_n^{-1})},$$

proving (a). For (b), (5) and (28) give $\alpha_1(\mathcal{L}_{\sigma_n}) - \frac{1}{2} = \frac{1+o(1)}{2\log(\lambda_{\min}(\sigma_n)^{-1})}$. Taking the quotient yields (30). The universal lower bound $\frac{1}{2}$ on the logarithmic ratio follows from the elementary inequality $\mathrm{Tr}(\sigma^{-1}) = \sum_{i=1}^d \lambda_i^{-1} \le d\lambda_{\min}(\sigma)^{-1} \le \lambda_{\min}(\sigma)^{-2}$ (since $\lambda_{\min}(\sigma) \le 1/d$), which gives $\log \mathrm{Tr}(\sigma^{-1}) \le 2\log(\lambda_{\min}(\sigma)^{-1})$. Both limiting regimes follow immediately. Part (c) follows identically from Proposition 4.4 and Theorem 5.1. □

Remark 5.3 (Finite-dimension non-monotonicity of the collapse ratio)**.** Interestingly, for tracial depolarizing $\mathcal{L}_{I_d/d}$, the upper collapse ratio $\kappa_2(1/d^2)/\kappa_2(1/d)$ dips below $1/2$ at intermediate dimensions (0.4969 at $d = 16$, 0.4121 at $d = 1024$) before rising back to $1/2$ as $d \to \infty$. This is explained quantitatively by (28): with $L = \log d$,

$$\begin{aligned}\frac{\kappa_2(1/d^2)}{\kappa_2(1/d)} &= \frac{1}{2} \cdot \frac{1 + (\log(2L) + 2 - \log 2)/(2L) + O((\log L)^2/L^2)}{1 + (\log L + 2 - \log 2)/L + O((\log L)^2/L^2)} \\ &= \frac{1}{2}\left(1 - \frac{\log L + 2 - 2\log 2}{2L} + O\left(\frac{(\log L)^2}{L^2}\right)\right),\end{aligned}$$

so the ratio approaches $1/2$ strictly from below at rate $O((\log\log d)/\log d)$.

6. High-Precision Numerical Verification and Open Conjecture

All theorems, identities, and asymptotic constants have been verified by a reproducible computational suite (`scripts/verify_cmlsi.py`, `tests/test_cmlsi.py`) using 100-digit `mpmath` arithmetic, 64-point Gauss–Legendre BKM quadrature, and multi-start unconstrained BFGS optimization over the full bipartite state space $\mathcal{D}_+(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$.

Across 80 random non-commuting 4×4 state pairs, 64-point Gauss–Legendre quadrature verifies the BKM identities (9) to maximum error 8.79×10^{-14} (forward) and 8.75×10^{-13} (reverse), with strict empirical monotonicity $\min \Delta G_M(t) = +1.96\times10^{-3} > 0$ and $\max \Delta G_m(t) = -3.81\times10^{-4} < 0$. Figure 1 illustrates both the universal Reverse-KL envelope $\gamma(M)$ versus $\kappa_2(1/M)$ and the two-sided BKM–Chebyshev monotonicity mechanism.

Table 1 lists our rigorous lower and upper bounds for tracial depolarizing $\mathcal{L}_{I_d/d}$ and d-dimensional dephasing $\Delta_d - \mathrm{id}$ across $d = 2, \ldots, 1024$. Table 2 compares our analytic bounds against unconstrained global BFGS minimization over full-rank bipartite states for tracial and non-tracial qubit and qutrit depolarizing semigroups (as well as dephasing), while Table 3 confirms the two-term

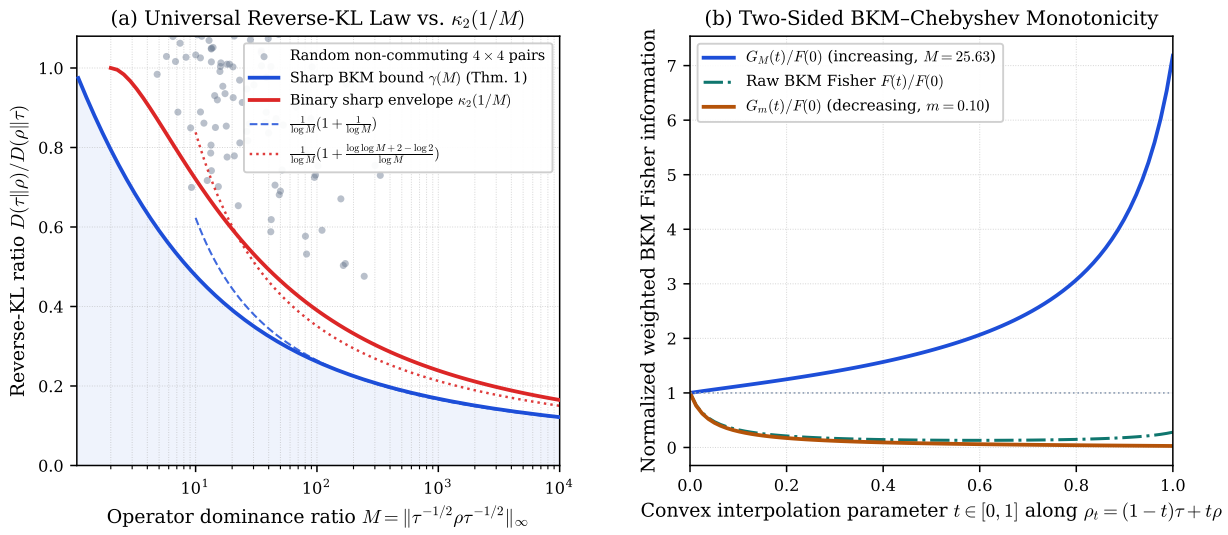


FIGURE 1. (a) Universal non-commutative reverse-KL lower bound $\gamma(M)$ (Theorem 2.4), sharp binary constant $\kappa_2(1/M)$, their two-term asymptotic laws (Theorem 5.1), and 140 random non-commuting 4×4 state pairs lying strictly above $\gamma(M)$. (b) Two-sided BKM–Chebyshev monotonicity along $\rho_t=(1-t)\tau+t\rho$ for a random non-commuting 4×4 pair ($m=0.10$, $M=25.63$): $G_M(t)/F(0)$ is strictly increasing while $G_m(t)/F(0)$ is strictly decreasing.

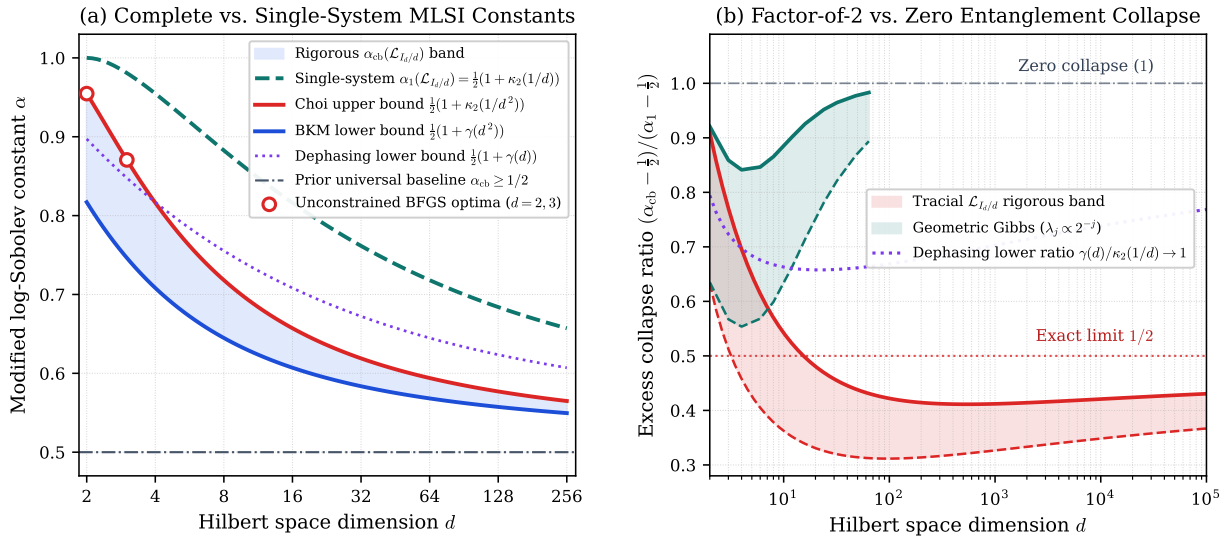


FIGURE 2. (a) Rigorous cMLSI band $[\frac{1}{2}(1+\gamma(d^2)),\ \frac{1}{2}(1+\kappa_2(1/d^2))]$ for tracial depolarizing $\mathcal{L}_{I_d/d}$ compared against single-system $\alpha_1(\mathcal{L}_{I_d/d})$, the prior universal baseline $1/2$, dephasing $\Delta_d-\mathrm{id}$, and unconstrained bipartite BFGS optima ($d=2,3$). (b) Asymptotic entanglement collapse ratio $(\alpha_{\rm cb}-\frac{1}{2})/(\alpha_1-\frac{1}{2})$: tracial depolarizing collapses to $1/2$ (from below at intermediate d), whereas geometric Gibbs states ($\lambda_j\propto2^{-j}$) and dephasing approach 1 (zero asymptotic collapse).

asymptotic laws (27)–(28) up to $M=10^{64}$. Figure 2 visualizes the complete versus single-system MLSI bands and the factor-of-2 versus zero entanglement collapse dichotomy.

TABLE 1. Exact rigorous bounds on $\alpha_{\mathrm{cb}}(\mathcal{L}_{I_d/d})$ (Theorems 3.4 and 4.1) compared with exact single-system $\alpha_1(\mathcal{L}_{I_d/d})$ and d-dimensional dephasing $\alpha_{\mathrm{cb}}(\Delta_d - \mathrm{id})$ (Proposition 4.4), computed to 100 digits and rounded to 8 decimal places.

d	$M_{\mathrm{cb}} = d^2$	$\frac{1+\gamma(d^2)}{2}$	$\frac{1+\kappa_2(1/d^2)}{2}$	$\alpha_1(\mathcal{L}_{I_d/d})$	$\frac{\kappa_2(1/d^2)}{\kappa_2(1/d)}$	$\frac{1+\gamma(d)}{2}$	$q_*(1/d^2)$
2	4	0.81701240	0.95470639	1.00000000	0.909413	0.89717486	0.49742141
3	9	0.74640191	0.87063397	0.98082457	0.770830	0.84780138	0.48503638
4	16	0.70822241	0.81626186	0.95470639	0.695530	0.81701240	0.47094094
8	64	0.64480865	0.71814747	0.88262821	0.570129	0.75533402	0.42543197
16	256	0.60710211	0.65716045	0.81626186	0.496931	0.70822241	0.37324035
64	4096	0.56816915	0.59401553	0.71814747	0.430972	0.64480865	0.27867869
256	65536	0.54954305	0.56489051	0.65716045	0.412893	0.60710211	0.21197653
1024	1048576	0.53887080	0.54900940	0.61893244	0.412078	0.58362942	0.16812848

TABLE 2. Comparison of rigorous analytic bounds against unconstrained multi-start BFGS minimization over full-rank bipartite states $\rho_{AB} \in \mathcal{D}_+(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ for generalized depolarizing and dephasing semigroups. In every case, the global optimum matches the σ-inverted Choi (resp. coherent isotropic) upper bound to 10^{-11}–10^{-14} (for qubit dephasing $d_A = 2$, the infimum is approached continuously as $\rho_{AB} \to \tau_{AB}$).

Channel	C_{cb}	Lower bound	Upper bound	BFGS min	BFGS − Upper	α_1
$\mathcal{L}_{(0.5,0.5)}$ $(d_B = 2)$	4	0.81701240	0.9547063900	0.9547063900	$+4.0 \times 10^{-14}$	1.00000000
$\mathcal{L}_{(0.25,0.75)}$ $(d_B = 2)$	16/3	0.78940398	0.9250978502	0.9250978502	$+7.0 \times 10^{-14}$	0.95470639
$\mathcal{L}_{(0.10,0.90)}$ $(d_B = 2)$	100/9	0.73141191	0.8498404956	0.8498404956	$+2.6 \times 10^{-14}$	0.86012168
$\mathcal{L}_{(1/3,1/3,1/3)}$ $(d_B = 3)$	9	0.74640191	0.8706339697	0.8706339697	$+1.3 \times 10^{-13}$	0.98082457
$\mathcal{L}_{(0.15,0.35,0.50)}$ $(d_B = 3)$	242/21	0.72894072	0.8463391592	0.8463391592	$+1.1 \times 10^{-13}$	0.90158836
$\Delta_2 - \mathrm{id}$ $(d_B = 2)$	2	0.89717486	1.0000000000	1.0000023372	$+2.3 \times 10^{-6}$	1.00000000
$\Delta_3 - \mathrm{id}$ $(d_B = 2)$	3	0.84780138	0.9808245737	0.9808245737	$+3.8 \times 10^{-11}$	0.98082457

TABLE 3. Verification of the exact two-term asymptotic formulas (27)–(28) and optimal binary witness scaling $q_*(1/M) \cdot \log M \to 2$ (Theorem 5.1) at 100-digit precision.

M	$\gamma(M) \log M$	2-term rel. err. (γ)	$\kappa_2(1/M) \log M$	2-term rel. err. (κ_2)	$q_*(1/M) \log M$
10^2	1.20244479	1.22×10^{-2}	1.79815618	1.02×10^{-1}	1.882882
10^4	1.12063859	1.08×10^{-2}	1.51763879	8.87×10^{-2}	2.340812
10^8	1.05740283	2.95×10^{-3}	1.27724211	3.77×10^{-2}	2.287741
10^{16}	1.02790073	7.37×10^{-4}	1.14925766	1.38×10^{-2}	2.184480
10^{32}	1.01375843	1.84×10^{-4}	1.08119575	4.72×10^{-3}	2.110673
10^{64}	1.00683221	4.60×10^{-5}	1.04435458	1.54×10^{-3}	2.064304

Table 2 reveals an extraordinary empirical regularity: across every tested tracial and non-tracial depolarizing channel (and every tested dephasing channel), unconstrained non-commuting bipartite minimization never penetrates below our analytic upper bound $\frac{1}{2}(1 + \kappa_2(C_{\mathrm{cb}}^{-1}))$, saturating it to machine precision. Geometrically, Theorem 2.4(c) explains why our operator-dominance bound $\gamma(C_{\mathrm{cb}})$ cannot be improved using $\rho_{AB} \leq C_{\mathrm{cb}}\tau_{AB}$ alone: the boundary witness $\tau_t = \mathrm{diag}(t, 1-t)$, $\rho_t = \mathrm{diag}(Mt, 1-Mt)$ drives both τ_1 and ρ_1 simultaneously to zero while preserving their ratio M. By contrast, for a conditional expectation $\tau_{AB} = (\mathcal{E}_* \otimes \mathrm{id}_B)(\rho_{AB})$, the marginal invariance

$\mathrm{Tr}_A(\rho_{AB}) = \rho_B$ prevents the dominant eigenmode of $\tau_{AB}^{-1/2}\rho_{AB}\tau_{AB}^{-1/2}$ from carrying vanishing total probability while maintaining ratio C_{cb}. This strongly motivates the following exact conjecture.

Conjecture 6.1 (Exact harmonic-mean cMLSI conjecture)**.** *For every dimension $d \geq 2$ and every faithful state $\sigma \in \mathcal{D}_+(\mathbb{C}^d)$, the complete modified logarithmic Sobolev constant of the generalized quantum depolarizing semigroup is given exactly by*

$$\alpha_{\mathrm{cb}}(\mathcal{L}_\sigma) \;=\; \frac{1}{2}\Big(1 + \kappa_2\big(\mathrm{Tr}(\sigma^{-1})^{-1}\big)\Big).$$

Likewise, for every orthogonal r-block pinching $\mathcal{P}$ ($r \geq 2$), $\alpha_{\mathrm{cb}}(\mathcal{P} - \mathrm{id}) = \alpha_1(\mathcal{P} - \mathrm{id}) = \frac{1}{2}(1 + \kappa_2(1/r))$.

Settling Conjecture 6.1 would close the remaining $O((\log\log M)/(\log M)^2)$ finite-dimension gap between Theorem 3.4 and Theorem 4.1 and establish an exact quantum counterpart to the classical complete-graph MLSI theorem of [3, 9] with the harmonic mean $\mathrm{Tr}(\sigma^{-1})^{-1}$ replacing the single-system minimum eigenvalue $\lambda_{\min}(\sigma)$.

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