

A four-qubit criterion for entropy minimax duality

Minimality, sharp Rényi threshold and a W-state counterexample

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Abstract

We characterize entropy functions for which support–radial minimax duality holds on every multiqubit orbit closure. For a continuous, strictly increasing function of the smaller marginal eigenvalue, vanishing at zero, universal validity is equivalent to concavity. It suffices to test a single tensor, the four-qubit W state, over all positive weights. Four qubits are minimal even within the Rényi family: order three satisfies the identity for every tensor on at most three qubits, but fails for the four-qubit W state. Order four already fails with three qubits. For binary Rényi entropy this gives the sharp range $0 < \alpha \leq 2$; every order above two admits a four-qubit obstruction. The scalar curvature facts are established properties; the proposed contribution is their universal W-state test. At min-entropy, the three-qubit W tensor already contradicts the first identity of Theorem 60 in Christandl–Lysikov–Zuiddam, both in arXiv:2012.14412v3 and in the 2023 journal version: its two sides are $3/2$ and $2^{2/3}$. We compute the weighted min-entropy functionals of all W tensors, repair the identity by convexifying entropy vectors, and retain the reverse identity. A typical-projector argument explains tensor regularization in terms of established Shannon functionals; the unweighted identification was already known from G-stable rank. The proofs include exact finite-order witnesses. Priority and human validation remain open.

1 Definitions and the exact scope of the obstruction

Let $T \neq 0$ be a complex tensor in $\bigotimes_{i=1}^k \mathbb{C}^{d_i}$. Write $\Pi(T)$ for the tuples of ordered reduced-density-matrix spectra of normalized representatives of its projective local general-linear orbit closure. This is a compact set. Logarithms are base two. For a probability vector p , let

$$H_\infty(p) = -\log \max_j p_j, \quad H(p) = -\sum_j p_j \log p_j.$$

Set

$$\begin{aligned} \Theta_k &= \{\theta \in \mathbb{R}_{\geq 0}^k : \sum_i \theta_i = 1\}, & \Xi_k &= \{\xi \in \mathbb{R}_{\geq 0}^k : \max_i \xi_i = 1\}, \\ \Theta_k(\xi) &= \{\theta \in \Theta_k : \text{supp } \theta \subseteq \text{supp } \xi\}. \end{aligned}$$

The natural k -party versions of the min-entropy quantities are

$$F_{\theta, \infty}(T) = \max_{p \in \Pi(T)} 2^{\sum_i \theta_i H_\infty(p_i)}, \quad (1)$$

$$G_{\xi, \infty}(T) = \max_{p \in \Pi(T)} \min_{i: \xi_i > 0} 2^{H_\infty(p_i)/\xi_i}. \quad (2)$$

For $k = 3$, these are the definitions in [1, Section 5.3]. The first identity of its Theorem 60, also stated in its introductory Theorem 14, asserts

$$G_{\xi, \infty}(T) \stackrel{\text{claimed}}{=} \min_{\theta \in \Theta_3(\xi)} F_{\theta, \infty}(T)^{1/\langle \theta, \xi \rangle}. \quad (3)$$

We assessed **arXiv:2012.14412v3**, dated **19 April 2023**, with Theorem 60 on printed page 26, and the **final journal version**, J. Math. Pures Appl. 172 (2023), with that theorem on page 327. The journal text was checked in the University of Amsterdam repository copy linked in [1]; it contains the same first identity and the same preceding concavity assertion. This comparison does not establish the absence of a later correction elsewhere.

The general concave minimax theorem in [1, Theorem 50] is not contradicted. The issue is its use for min-entropy: H_∞ is not concave. For $p = (1, 0)$, $q = (1/2, 1/2)$,

$$H_\infty((p+q)/2) = \log(4/3) < \frac{1}{2} = \frac{1}{2}H_\infty(p) + \frac{1}{2}H_\infty(q).$$

Quasiconcavity does not give the concavity required for that application. The established Shannon-entropy identities are unaffected.

2 Exact W-state formulas

Let

$$W_k = \sum_{j=1}^k |0 \cdots 0 1_j 0 \cdots 0\rangle, \quad k \geq 2,$$

and write $\theta_{(1)} \geq \theta_{(2)} \geq \cdots$ for sorted weights.

Lemma 1. *For every marginal tuple in $\Pi(W_k)$, its largest eigenvalues λ_i satisfy*

$$\frac{1}{2} \leq \lambda_i \leq 1, \quad \sum_i \lambda_i \geq k - 1. \quad (4)$$

Conversely, every tuple with $\lambda_i \in [1/2, 1]$ and $\sum_i \lambda_i = k - 1$ is attained in the orbit closure.

Proof. For a tensor in the orbit, apply local unitaries that take each $g_i|0\rangle$ to a multiple of $|0\rangle$. Expanding each $g_i|1\rangle$ then gives the normalized form

$$\psi = a_0|0 \cdots 0\rangle + \sum_{j=1}^k a_j|0 \cdots 1_j \cdots 0\rangle, \quad \sum_{j=0}^k |a_j|^2 = 1.$$

With $r_j = |a_j|^2$, the j -th marginal has diagonal entries $1 - r_j, r_j$. Its smaller eigenvalue μ_j is at most r_j , by the Rayleigh principle. Hence $\sum_j \mu_j \leq \sum_j r_j \leq 1$, giving (4); continuity extends this bound to the closure.

For the converse, put $r_j = 1 - \lambda_j$, so that $0 \leq r_j \leq 1/2$ and $\sum_j r_j = 1$. The state $\sum_j \sqrt{r_j}|0 \cdots 1_j \cdots 0\rangle$ has diagonal marginals $(1 - r_j, r_j)$. Positive coefficients are obtained by invertible diagonal local maps on W_k , and zero coefficients by limits. $\square$

Theorem 2. *For every $\theta \in \Theta_k$,*

$$F_{\theta, \infty}(W_k) = 2^{\theta_{(1)} + \theta_{(2)}}. \quad (5)$$

For $\xi \in \Xi_k$, write $I = \text{supp } \xi$ and $s = |I|$. If $s = 1$, then $G_{\xi, \infty}(W_k) = 2$. If $s \geq 2$, let $t_* > 1$ be the unique solution of

$$\sum_{i \in I} t_*^{-\xi_i} = s - 1.$$

Then

$$G_{\xi, \infty}(W_k) = \min\{2, t_*\}. \quad (6)$$

In particular,

$$G_{1, \infty}(W_k) = \frac{k}{k-1}, \quad \min_{\theta \in \Theta_k} F_{\theta, \infty}(W_k) = 2^{2/k}. \quad (7)$$

Proof. Put $\mu_i = 1 - \lambda_i$. Lemma 1 places this tuple in $Q = \{\mu : 0 \leq \mu_i \leq 1/2, \sum_i \mu_i \leq 1\}$. The function $-\sum_i \theta_i \log(1 - \mu_i)$ is convex on Q . Its maximum is attained at a vertex. The vertices of Q are the zero vector, the vectors with one entry $1/2$, and those with two entries $1/2$. Indeed, after scaling by two, a fractional vertex would admit a coordinate perturbation, or a two-coordinate perturbation when the sum constraint is tight. The largest objective value is $\theta_{(1)} + \theta_{(2)}$. It is attained by projecting all but the corresponding two parties onto $|0\rangle$, leaving an entangled pair and pure spectators. This proves (5).

For $G_{\xi, \infty} \geq t$, active largest marginal eigenvalues must satisfy $\lambda_i \leq t^{-\xi_i}$; for inactive coordinates this is the automatic bound $\lambda_i \leq 1 = t^0$. Since some $\xi_i = 1$, the necessary conditions are

$$1 \leq t \leq 2, \quad \sum_i t^{-\xi_i} \geq k - 1. \quad (8)$$

They are sufficient: choose $\lambda_i \in [1/2, t^{-\xi_i}]$ with sum $k - 1$. Such a choice exists because the lower endpoints sum to $k/2 \leq k - 1$, whereas the upper endpoints sum to at least $k - 1$. Use the converse in Lemma 1. The sum in (8) is equivalently $\sum_{i \in I} t^{-\xi_i} \geq s - 1$. For $s = 1$ it imposes no additional restriction and the value is two. For $s \geq 2$, the active sum decreases strictly and continuously from s to zero, proving (6). In particular $s = 2$ also gives value two, since each $2^{-\xi_i} \geq 1/2$.

The unweighted case follows by symmetry. The sum of the two largest entries of a probability vector is at least $2/k$, with equality at the uniform vector, proving the second formula. $\square$

Proposition 3 (Counterexample to (3)). *For $W = W_3 = |100\rangle + |010\rangle + |001\rangle$ and $\xi = (1, 1, 1)$,*

$$G_{\xi, \infty}(W) = \frac{3}{2} < 2^{2/3} = \min_{\theta \in \Theta_3} F_{\theta, \infty}(W)^{1/\langle \theta, \xi \rangle}.$$

Proof. Here $\langle \theta, \xi \rangle = 1$. Apply Theorem 2; the strict inequality is exactly $27/8 < 4$ after cubing. $\square$

The value $3/2$ is already known from Derksen's computation of the G-stable rank [2]; the W-polytope geometry is likewise established [4]. We use a self-contained orbit argument to make the contradiction independently checkable. For every $k > 2$, the analogous gap is also strict: $(1 - 1/k)^k > 1/4$, since $x \log(1 - 1/x)$ increases for $x > 1$ and has value $\log(1/4)$ at $x = 2$.

3 What the minimax expression actually computes

The distinction is between an entropy image and its convex hull. The following statement does not assume that the original parameter space is convex.

Theorem 4 (Convexified radial duality). *Let $E \subset \mathbb{R}_{\geq 0}^k$ be nonempty and compact. Define*

$$f_\theta = \max_{e \in E} \langle \theta, e \rangle, \quad g_\xi = \max_{e \in E} \min_{i: \xi_i > 0} \frac{e_i}{\xi_i}, \quad \widehat{g}_\xi = \max_{z \in \text{conv } E} \min_{i: \xi_i > 0} \frac{z_i}{\xi_i}.$$

Then

$$\widehat{g}_\xi = \min_{\theta \in \Theta_k(\xi)} \frac{f_\theta}{\langle \theta, \xi \rangle}, \quad g_\xi \leq \widehat{g}_\xi. \quad (9)$$

An optimizer for the active coordinates of $\widehat{g}_\xi$ can be obtained from a convex combination of at most $|\text{supp } \xi|$ points of E . Moreover, without convexifying,

$$f_\theta = \max_{\xi \in \Xi_k} g_\xi \langle \theta, \xi \rangle. \quad (10)$$

Proof. Let $I = \text{supp } \xi$ and $\Lambda = \{u \geq 0 : \text{supp } u \subseteq I, \langle u, \xi \rangle = 1\}$. This is compact and convex, and $\min_{u \in \Lambda} \langle u, z \rangle = \min_{i \in I} z_i / \xi_i$. The bilinear minimax theorem gives

$$\max_{z \in \text{conv } E} \min_{u \in \Lambda} \langle u, z \rangle = \min_{u \in \Lambda} \max_{e \in E} \langle u, e \rangle.$$

Parameterize $u = \theta / \langle \theta, \xi \rangle$ to obtain (9). This is an application of classical convex minimax, not a new minimax principle.

For the support bound, project onto the coordinates in I . Let z_* and u_* be primal and dual optimizers, with common value v . The saddle-point equalities follow from

$$v = \min_{u \in \Lambda} \langle u, z_* \rangle \leq \langle u_*, z_* \rangle \leq \max_{z \in \text{conv } E} \langle u_*, z \rangle = v.$$

Thus z_* lies in the supporting face exposed by u_* . That face has affine dimension at most $|I| - 1$, since $u_* \neq 0$. Carathéodory's theorem on that face gives at most $|I|$ points. Those points may be taken from the projected E : a convex combination can attain the supporting value only through points attaining that value.

For (10), an optimizer e of g_ξ satisfies $e_i \geq g_\xi \xi_i$ for all i , including the inactive coordinates. Thus $g_\xi \langle \theta, \xi \rangle \leq f_\theta$. Conversely, choose e attaining f_θ . If $e \neq 0$, set $a = \max_i e_i$ and $\xi = e/a$. Then $g_\xi \geq a$, giving the opposite inequality. If $f_\theta = 0$, nonnegativity and the first inequality already prove equality. $\square$

Apply Theorem 4 to

$$E_\infty(T) = \{(H_\infty(p_1), \dots, H_\infty(p_k)) : p \in \Pi(T)\}.$$

The right-hand side of (3) is $2^{\widehat{g}_\xi}$, whereas its left-hand side is 2^{g_ξ} . Thus convexification repairs precisely the failed identity. The second identity in [1, Theorem 60], namely the multiplicative form of (10), remains valid.

For W_k , the downward closure of $\text{conv } E_\infty(W_k)$ in the nonnegative orthant is

$$\{z \in [0, 1]^k : \sum_i z_i \leq 2\}.$$

Indeed, the upper bounds follow from (5), and the pair indicator vectors are attained by entangled pairs with pure spectators. Their convex hull is the face $\sum_i z_i = 2$; its downward closure is the displayed set. Consequently,

$$\widehat{g}_\xi = \min\{1, 2 / \sum_i \xi_i\}, \quad \xi \in \Xi_k. \quad (11)$$

For W_3 , averaging the three entropy vectors $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ gives the balanced vector $(2/3, 2/3, 2/3)$ responsible for the incorrect one-copy prediction.

Other basic properties do not depend on the failed identity. Orbit-closure monotonicity is immediate from inclusion; on the diagonal tensor of size r , $G_{\xi, \infty} = r$. For tensor products, local maximum eigenvalues multiply. Taking optimizing states and using $\min_i (a_i + b_i) \geq \min_i a_i + \min_i b_i$ gives

$$G_{\xi, \infty}(S \otimes T) \geq G_{\xi, \infty}(S) G_{\xi, \infty}(T).$$

Thus the monotonicity, normalization and supermultiplicativity assertions of [1, Corollary 61] have direct proofs.

4 A universal four-qubit test

The obstruction extends beyond min-entropy. Let $h : [0, 1/2] \rightarrow \mathbb{R}_{\geq 0}$ be continuous and strictly increasing, with $h(0) = 0$. For a nonzero multiqubit tensor T , let $P(T)$ denote its

orbit-closure marginal tuples in smaller-eigenvalue coordinates $\mu = (\mu_1, \dots, \mu_k)$. This is an affine image of the entanglement polytope and hence is compact and convex [4, Supplementary Information, Corollary 2]. Define logarithmic quantities

$$f_\theta^h(T) = \max_{\mu \in P(T)} \sum_i \theta_i h(\mu_i), \quad g_\xi^h(T) = \max_{\mu \in P(T)} \min_{i: \xi_i > 0} \frac{h(\mu_i)}{\xi_i}.$$

Theorem 5 (Four-qubit criterion). *Under the stated assumptions on h , the following are equivalent:*

1. h is concave on $[0, 1/2]$.
2. For every $k \geq 2$, every nonzero k -qubit tensor T , and every $\xi \in \Xi_k$,

$$g_\xi^h(T) = \min_{\theta \in \Theta_k(\xi)} \frac{f_\theta^h(T)}{\langle \theta, \xi \rangle}. \quad (12)$$

3. Equation (12) holds for $T = W_4$ and every $\xi \in \Xi_4$ with strictly positive coordinates.

If concavity fails, a witness to failure on W_4 can be obtained by mixing just two entropy vectors of weighted W states.

Proof. Suppose h is concave. On the compact convex set $P(T)$, the function $\sum_i u_i h(\mu_i)$ is concave in μ for every $u \geq 0$, and is linear in u . Apply the concave minimax theorem with the compact weight set Λ from the proof of Theorem 4. This proves (12), including weights with inactive coordinates. The second assertion plainly implies the third.

For the converse, continuity implies that failure of concavity has a midpoint witness: there are $0 \leq x < y \leq 1/2$ such that, writing $a = (x + y)/2$ and $b = 1/2 - a$,

$$c = \frac{h(x) + h(y)}{2} > h(a).$$

Both a and b lie strictly between zero and $1/2$. Lemma 1 shows that $r^+ = (x, y, b, b)$ and $r^- = (y, x, b, b)$ belong to $P(W_4)$: their coordinates lie in $[0, 1/2]$ and sum to one. Their entropy vectors have average $z = (c, c, h(b), h(b))$. Set $d = \max_i z_i > 0$ and $\xi = z/d$. These weights are strictly positive. The convexified radial value is at least d , since z lies in the convex hull.

The original radial value is strictly less than d . Otherwise, compactness would give $\mu \in P(W_4)$ with $h(\mu_i) \geq z_i$ for every i . Strict monotonicity would imply $\mu_1 > a$, $\mu_2 > a$, $\mu_3 \geq b$, $\mu_4 \geq b$, and thus $\sum_i \mu_i > 2a + 2b = 1$, contradicting Lemma 1. By Theorem 4, the right-hand side of (12) equals the convexified radial value, so (12) fails for these weights. $\square$

The theorem concerns universal validity over all weights and all multiqubit orbit closures. It does not say that every choice of weights fails when h is nonconcave. The sufficiency direction is the usual concave minimax argument. The converse identifies W_4 as a single test tensor for this whole class of scalar functions.

4.1 Minimality within the Rényi family

Four qubits are necessary even if the test functions are restricted to binary Rényi entropies. This is stronger than minimality witnessed by an arbitrary increasing scalar function.

Theorem 6 (Minimal number of qubits). *The binary order-three Rényi entropy*

$$h_3(p) = -\frac{1}{2} \log(1 - 3p + 3p^2), \quad 0 \leq p \leq \frac{1}{2},$$

satisfies (12) for every nonzero tensor on two or three qubits and every admissible weight vector. It fails on W_4 with the positive weights specified in Section 4.3. Consequently four is the minimal universal testing size in Theorem 5, including within the Rényi family.

Proof. Write $h = h_3$, $\phi = h^{-1}$. The function h is continuous and strictly increasing, with $h(0) = 0$ and $h(1/2) = 1$. By Lemma 1, the downward closure of its entropy image on W_3 is exactly

$$D = \{z \in [0, 1]^3 : \phi(z_1) + \phi(z_2) + \phi(z_3) \leq 1\}.$$

For sufficiency, increase any inverse-coordinate tuple of sum at most one, keeping each coordinate at most $1/2$, until its sum is one. The remaining total capacity is $3/2 - \sum_i p_i \geq 1 - \sum_i p_i$. The resulting tuple is realized by a weighted W state and dominates the original entropy vector. Necessity is the marginal sum bound.

We show that D is convex, although h is not concave. For $0 < p_i < 1/2$, $\sum_i p_i = 1$, set $x_i = 1 - 2p_i > 0$, so $\sum_i x_i = 1$. Direct differentiation gives

$$A(p_i) := -\frac{h''(p_i)}{h'(p_i)} = \frac{2(1 - 3x_i^2)}{x_i(1 + 3x_i^2)}.$$

For every real tuple v with $\sum_i v_i = 0$, we claim

$$\sum_{i=1}^3 A(p_i) v_i^2 \geq 0. \quad (13)$$

If all $x_i \leq 1/\sqrt{3}$, each coefficient is nonnegative. Otherwise exactly one, say $x = x_1$, exceeds $1/\sqrt{3}$, so $A(p_1) < 0 < A(p_2), A(p_3)$. Define

$$B(x) = \frac{x(1 + 3x^2)}{1 - 3x^2}.$$

On $[0, 1/\sqrt{3}]$, the ratio $B(x)/x$, extended at zero, is increasing: its derivative is $12x/(1 - 3x^2)^2$. Thus $B(x_2) + B(x_3) \leq B(x_2 + x_3) = B(1 - x)$. With $s = x(1 - x) \leq 1/4$, expansion yields

$$B(x) + B(1 - x) = \frac{4 - 12s - 9s^2}{(1 - 3x^2)(1 - 3(1 - x)^2)} < 0.$$

The numerator is at least $7/16$; the denominator is negative. Since $B(x_i) = 2/A(p_i)$, it follows that

$$A(p_1) + \frac{A(p_2)A(p_3)}{A(p_2) + A(p_3)} > 0.$$

For fixed v_1 and $v_2 + v_3 = -v_1$, the minimum of $A(p_2)v_2^2 + A(p_3)v_3^2$ is the second coefficient in this display times v_1^2 . This proves (13).

For global convexity, write D as $0 \leq z_3 \leq g(z_1, z_2)$, where

$$g(z_1, z_2) = \begin{cases} 1, & \phi(z_1) + \phi(z_2) \leq 1/2, \\ h(1 - \phi(z_1) - \phi(z_2)), & \phi(z_1) + \phi(z_2) > 1/2. \end{cases}$$

First take a line segment in the open square $(0, 1)^2$, where ϕ is analytic with finite derivatives. On curved pieces put $p_i = \phi(z_i)$, $i = 1, 2$, and $p_3 = 1 - p_1 - p_2$. Then $p_i'' = A(p_i)(p_i')^2$ for $i = 1, 2$, so

$$g'' = -h'(p_3) \sum_{i=1}^3 A(p_i)(p_i')^2 \leq 0.$$

Flat pieces have zero curvature. At the interface $p_3 = 1/2$, both one-sided derivatives of g are zero because $h'(1/2) = 0$. Analyticity gives finitely many interface contacts on a compact segment unless its interface equation is identically satisfied; in that case g is constant. Each line restriction is therefore concave. Continuity extends concavity to the closed square, without requiring a finite derivative of ϕ at its upper endpoint. Hence D is convex. Nonnegative support and radial maxima do not change under downward closure, and Theorem 4 proves the identity for the W class.

The established pure-state SLOCC classification [5] covers all remaining three-qubit tensors: every nonzero tensor can be normalized as a pure state, and local invertible equivalence preserves its projective orbit closure. The GHZ class contains a state with all smaller marginal eigenvalues equal to $1/2$. Biseparable classes contain a Bell pair and a pure spectator, and the product class gives a point. Their entropy downsets are a box, a box face or a point, so they are convex. For two qubits the product and Schmidt-rank-two classes likewise give a point or a box. The explicit four-qubit failure is proved in Section 4.3 by an exact rational midpoint gap. $\square$

This is minimality for universal testing, not a statement that every nonconcave entropy needs four qubits. Indeed, order four already fails on W_3 . To see this, take the weighted W states with marginal coordinates $(1/5, 3/10, 1/2)$ and $(3/10, 1/5, 1/2)$. For $f_4(p) = p^4 + (1-p)^4$, the rational identity

$$f_4(1/4)^2 - f_4(1/5)f_4(3/10) = \frac{3457649}{6400000000} > 0$$

implies $c = (h_4(1/5) + h_4(3/10))/2 > h_4(1/4)$. Their entropy average is $(c, c, 1)$. No original W-state entropy vector dominates it: its third inverse coordinate would be at least $1/2$, and its first two would each exceed $1/4$. Thus $\xi = (c, c, 1)$ gives a strict gap. We do not classify the complete three-qubit transition as the Rényi order varies.

4.2 The sharp Rényi range

Write

$$h_\alpha(p) = \frac{\log(p^\alpha + (1-p)^\alpha)}{1-\alpha}, \quad 0 < \alpha < \infty, \quad \alpha \neq 1,$$

with the continuous Shannon limit at $\alpha = 1$, and $h_\infty(p) = -\log(1-p)$ for $0 \leq p \leq 1/2$. These functions are continuous, strictly increasing on this interval, and vanish at zero. We include the scalar curvature argument so that the threshold does not depend on a numerical plot. The loss of concavity for orders above two is already recorded, for example, in [6, Lemma 4.3]; that scalar observation is not new.

Lemma 7 (Binary Rényi curvature). *The function h_α is concave on $[0, 1/2]$ exactly when $0 < \alpha \leq 2$. For every finite $\alpha > 2$,*

$$\lim_{p \downarrow 0} h''_\alpha(p) = \frac{\alpha}{(\alpha-1)\ln 2} > 0.$$

The min-entropy limit is strictly convex on this interval.

Proof. For $0 < \alpha < 1$, the sum $p^\alpha + (1-p)^\alpha$ is concave and positive. Composition with the increasing concave logarithm, and division by $1-\alpha > 0$, preserve concavity. Shannon's binary entropy is concave as well.

For $1 < \alpha \leq 2$, set $F = p^\alpha + (1-p)^\alpha$, $r = \alpha - 1 \in (0, 1]$, and $t = p/(1-p) > 0$. Expansion gives

$$\frac{FF'' - (F')^2}{\alpha[p(1-p)]^{\alpha-1}} = r(t + t^{-1}) + 2\alpha - (t^r + t^{-r}).$$

Convexity of $s \mapsto t^s + t^{-s}$ on $[0, 1]$ gives $t^r + t^{-r} \leq r(t + t^{-1}) + 2(1 - r)$. Thus the displayed quantity is at least $4r > 0$, and

$$h''_\alpha(p) = -\frac{FF'' - (F')^2}{(\alpha - 1)(\ln 2)F^2} < 0.$$

Continuity extends concavity to the endpoints. For $\alpha > 2$, as $p \downarrow 0$, $F \rightarrow 1$, $F' \rightarrow -\alpha$, and $F'' \rightarrow \alpha(\alpha - 1)$, giving the asserted positive limit. Finally, $h''_\infty(p) = 1/((1 - p)^2 \ln 2) > 0$. $\square$

Corollary 8 (Sharp multiqubit threshold). *The support–radial minimax identity formed from Rényi marginal entropies is valid on every multiqubit orbit closure, for every weight vector, if and only if $0 < \alpha \leq 2$. For every $2 < \alpha \leq \infty$, it already fails for W_4 with some strictly positive weights.*

Proof. Apply Theorem 5 and Lemma 7. $\square$

This is a statement about the marginal support and radial quantities defined here. It is not a statement about conditional Rényi entropy, LOCC spectral functionals, or general entropic inequalities. In particular, no claim about the Shepp–Olkin conjectures for sums of Bernoulli variables is implied.

4.3 An exact witness at order three

Take $h = h_3$, $x = 1/20$, $y = 3/20$, $a = 1/10$, and $b = 2/5$. Since $h_3(p) = -\frac{1}{2} \log f(p)$ for $f(p) = 1 - 3p + 3p^2$, the required strict midpoint gap follows from the rational identity

$$f(a)^2 - f(x)f(y) = \frac{543}{160000} > 0, \quad f(x)f(y) = \frac{84721}{160000}.$$

The two normalized states have squared single-excitation amplitudes $(1, 3, 8, 8)/20$ and $(3, 1, 8, 8)/20$, so their reduced spectra and membership in the W orbit are explicit. With $c = (h_3(x) + h_3(y))/2$, choose

$$\xi = (c/h_3(b), c/h_3(b), 1, 1).$$

Theorem 5 gives a strict duality gap with these exactly specified weights.

For numerical orientation, the first weight is approximately 0.2497358423. The original logarithmic radial value is approximately 0.9169557149, whereas the convexified value is at least $h_3(2/5) \approx 0.9182506339$. The first value can be computed from

$$\sum_{i=1}^4 h_3^{-1}(t\xi_i) = 1, \quad h_3^{-1}(u) = \frac{3 - \sqrt{12 \cdot 2^{-2u} - 3}}{6}.$$

Indeed, feasibility requires $\mu_i \geq h_3^{-1}(t\xi_i)$ and $\sum_i \mu_i \leq 1$; it is sufficient by filling the coordinates to sum to one, each at most $1/2$, and using Lemma 1. The decimal values are high-precision diagnostics, not interval certificates. Strictness of the gap rests on the rational identity and the theorem’s proof.

5 Tensor regularization and simultaneous spectral flattening

Here we restrict the stated identification to tripartite tensors, so that all external inputs agree directly with [1, 3]. Let

$$F_\theta(T) = \max_{p \in \Pi(T)} 2^{\sum_i \theta_i H(p_i)}, \quad G_\xi^H(T) = \max_{p \in \Pi(T)} \min_{i: \xi_i > 0} 2^{H(p_i)/\xi_i}.$$

We use two established facts: the Shannon quantum functional F_θ is multiplicative [3, Theorem 3.30 and Corollary 3.31], and the Shannon minimax identity is valid [1]. These are inputs, not conclusions claimed as new in this note.

Lemma 9 (Simultaneous local typical projection). *For a normalized multipartite pure state ψ , let H_i be the entropy of its i -th marginal. For every $\delta > 0$ there are local projectors on $\psi^{\otimes m}$, with success probability $q_m \rightarrow 1$, whose normalized output $\sigma^{(m)}$ satisfies, for every party,*

$$H_\infty(\sigma_i^{(m)}) \geq m(H_i - \delta) + \log q_m.$$

Proof. Let $P_i^{(m)}$ be the spectral projector of $\rho_i^{\otimes m}$ onto eigenvalues at most $2^{-m(H_i - \delta)}$. The law of large numbers for the random variable $-\log p$, sampled according to the positive eigenvalues of ρ_i , gives $\text{Tr}(P_i^{(m)} \rho_i^{\otimes m}) \rightarrow 1$. The local projectors commute, so for $P = \bigotimes_i P_i^{(m)}$,

$$q_m = \|P\psi^{\otimes m}\|^2 \geq 1 - \sum_i (1 - \text{Tr}(P_i^{(m)} \rho_i^{\otimes m})) \rightarrow 1.$$

For sufficiently large m , normalize $P\psi^{\otimes m}$. Projection on the other parties decreases the unnormalized reduced operator in positive-semidefinite order. More explicitly, partial trace is cyclic with respect to operators on the traced subsystem, and $0 \leq P_{\neq i} \leq I$. Therefore

$$q_m \sigma_i^{(m)} \leq P_i^{(m)} \rho_i^{\otimes m} P_i^{(m)} \leq 2^{-m(H_i - \delta)} P_i^{(m)}.$$

Taking operator norms proves the assertion. This is a standard typical-subspace construction, written out to make the simultaneous marginal bound explicit. $\square$

Theorem 10 (Regularization identities). *For every nonzero complex tripartite tensor T ,*

$$\lim_{m \rightarrow \infty} F_{\theta, \infty}(T^{\otimes m})^{1/m} = F_\theta(T), \quad (14)$$

$$\lim_{m \rightarrow \infty} G_{\xi, \infty}(T^{\otimes m})^{1/m} = G_\xi^H(T). \quad (15)$$

Proof. Choose any normalized representative ψ in the projective orbit closure of T . The outputs in Lemma 9 belong to the orbit closure of $T^{\otimes m}$, since singular local maps are limits of invertible ones. Applying the lemma, then taking $m \rightarrow \infty$, $\delta \downarrow 0$, and optimizing over ψ , gives the lower limits in both identities. For (15), division by active ξ_i causes no problem: their minimum is positive for the fixed vector ξ .

For the first upper limit, use $H_\infty \leq H$ and multiplicativity:

$$F_{\theta, \infty}(T^{\otimes m}) \leq F_\theta(T^{\otimes m}) = F_\theta(T)^m.$$

For every $\theta \in \Theta_3(\xi)$, a weighted average dominates the minimum of the active ratios, hence

$$G_{\xi, \infty}(T^{\otimes m}) \leq F_{\theta, \infty}(T^{\otimes m})^{1/\langle \theta, \xi \rangle} \leq F_\theta(T)^{m/\langle \theta, \xi \rangle}.$$

Minimize over θ and use the valid Shannon minimax identity. The lower and upper limits agree. $\square$

The unweighted identity (15) already follows from Derksen's constant-factor comparison of G-stable rank with slice rank [2, Proposition 4.9 and Theorem 5.2], together with the known asymptotic slice-rank formula [1]. The proof above also treats nonuniform weights and (14); whether these formulations have appeared elsewhere requires further priority checking.

A concrete three-copy restriction illustrates why the single-copy value cannot be treated as multiplicative. In copy j of W_3 , project party j onto $|0\rangle$, for $j = 1, 2, 3$. The result is a

triangle of three entangled pairs. Each grouped party has two maximally mixed qubits and one pure qubit, so

$$G_{1,\infty}(W_3^{\otimes 3}) \geq 4 > (3/2)^3.$$

This familiar entangled-pair construction is an illustration, not a claimed new protocol. The three relevant scales are

Operation on W_3	Value
One-copy min-entropy radial value	$3/2$
Convexified one-copy entropy image	$2^{2/3}$
Tensor-regularized radial value	$2^{H(1/3, 2/3)} = 3/2^{2/3}$

For completeness, the last value follows from Lemma 1: some smaller marginal eigenvalue is at most $1/3$, and binary entropy increases on $[0, 1/2]$. The symmetric W state attains the resulting upper bound. This is the established W asymptotic slice-rank value, recovered here for comparison.

6 Verification and limits of the claim

The proofs above are mathematical arguments; finite computation is supplementary. The accompanying deterministic script, with random seed 2026101707, checks:

- the rational contradiction $27/8 < 4$ and the exact three-copy restriction, whose unnormalized marginal matrices have four diagonal entries equal to two and four equal to zero;
- the orbit inequality and weighted functional bound on 200 random local transforms for $3 \leq k \leq 7$;
- the weighted scalar-root formula against independent linear feasibility calculations and explicit attaining states in 32 cases for $3 \leq k \leq 6$;
- 24 finite instances of the positive-semidefinite domination in Lemma 9.

All checks passed. The largest root-versus-feasibility discrepancy was 1.83×10^{-9} or less. Random tests do not prove a universal statement, and these are author-context diagnostics, not independent peer review or formal proof-assistant verification.

The additional Rényi verifier checks the order-three polynomial gap and eight marginal matrices exactly, four symbolic curvature identities, 48 concavity instances, and nine midpoint violations above order two. The case $\alpha = 2.01$ uses $a = 10^{-400}$ and 890 decimal digits, making explicit the precision needed near the threshold. These finite checks supplement the proof of Theorem 5; they do not establish its universal quantifiers by sampling.

The minimality verifier checks six symbolic or rational identities, including the order-three curvature formula and the order-four three-qubit witness. With seed 2026101710, it checks 100000 boundary curvature cases and 50000 Jensen inequalities for the complete boundary graph. The smallest numerical Jensen gap is zero. The global convexity argument in Theorem 6, including its endpoint treatment, is the proof; finite tests are supplementary diagnostics. The source archive accompanying this exact version on its [AIRR record](#) contains all three current verifiers, recorded outputs, dependency versions, file hashes and the preserved earlier-version assessments.

The arXiv PDF’s definitions and Theorem 60 were visually checked against the source. A separate model review also downloaded the journal repository PDF and visually checked printed page 327, confirming the identity and its preceding concavity premise. The correction

claim concerns these two specified versions; a separate corrigendum has not been ruled out. Neither the known value of the W G -stable rank, the W -polytope geometry, classical convex duality, nor the unweighted asymptotic identification is claimed as new. The classical curvature of binary Rényi entropy is also credited; the proposed new structural statements are the four-qubit criterion, its minimality even within the Rényi family, and its sharp minimax consequence. Their priority remains unchecked beyond the targeted sources and searches described in the audit. No result here concerns the general validity of the main Shannon theory. An independent mathematical assessment and a broader priority search are required before assigning significance or asserting that this obstruction has not been noticed before.

AI assistance and review status. The drafting, literature inspection, calculations and computational diagnostics were developed with substantial OpenAI Codex assistance. Earlier nine- and eleven-page versions received model reviews requested from GPT-6 Astra at xhigh effort. Those reports are preserved with their exact PDF hashes. The order-three minimality argument was subsequently examined as a separate supplement. Any assessment of the final manuscript is identified by its exact PDF hash in the accompanying record. Reviewer context is reused; the model family has participated in the wider programme, and statistical independence, certified context isolation and independently observed runtime identity are not claimed. No human peer review, formal proof-assistant verification or editorial acceptance is implied.

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