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EXACT BAKRY–ÉMERY RICCI CURVATURE, NON-ASYMPTOTIC $O(1/N^2)$ LARGE-N WILSON-LOOP FACTORIZATION, SUB-GAUSSIAN CONCENTRATION, AND SCHWINGER–DYSON PLAQUETTE BOUNDS IN d-DIMENSIONAL $\mathrm{SU}(N)$ LATTICE YANG–MILLS THEORY

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Abstract. A foundational prediction of 't Hooft's large-N expansion in $\mathrm{SU}(N)$ gauge theory is the factorization of gauge-invariant Wilson-loop expectations with $O(1/N^2)$ error at fixed 't Hooft inverse coupling $\gamma := \beta/N^2 = 1/\lambda_{\text{'t Hooft}}$. We establish three self-contained, cluster-expansion-free non-perturbative theorems in d-dimensional ($d \geq 2$) $\mathrm{SU}(N)$ and $\mathrm{U}(N)$ Wilson lattice Yang–Mills theory:

(1) *Exact Bakry–Émery Ricci curvature and volume-uniform log-Sobolev inequality:* Equipping $\mathrm{SU}(N)^{|E|}$ with the bi-invariant Hilbert–Schmidt metric $\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^\dagger B)$ (for which $\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2} g_{\mathrm{HS}}$), we prove the exact geodesic Hessian operator-norm bound $\|\mathrm{Hess}(-S_W)\|_{\mathrm{op,HS}} \leq 8(d-1)\gamma N$, establishing the strictly positive Bakry–Émery curvature lower bound $\mathrm{Ric} + \mathrm{Hess}(-S_W) \geq \kappa_d(N,\gamma)\, g_{\mathrm{HS}}$ with $\kappa_d(N,\gamma) := \frac{N}{2}(1 - 16(d-1)\gamma) > 0$ uniformly in all lattice volumes and all $N \geq 2$ whenever $\gamma < \frac{1}{16(d-1)}$ (and $\kappa_{2\mathrm{D}}^{\mathrm{ind}} = \frac{N}{2}(1-2\gamma) > 0$ for $\gamma < \frac{1}{2}$ in free-BC 2D).

(2) *Non-asymptotic $O(1/N^2)$ large-N factorization and Herbst sub-Gaussian concentration:* Computing the sharp Lipschitz bound $|\nabla(\frac{1}{N}\mathrm{Re\,Tr}\, W_C)|^2_{\mathrm{HS}} \leq M_2(C)/N$ (where $M_2(C) = |C|$ for simple loops) yields, for all volumes L and all $N \geq 2$, the explicit variance and factorization bounds

$$\left|\mathrm{Cov}\left(\frac{1}{N}\mathrm{Re\,Tr}\, W_{C_1},\, \frac{1}{N}\mathrm{Re\,Tr}\, W_{C_2}\right)\right| \;\leq\; \frac{2\sqrt{M_2(C_1)\, M_2(C_2)}}{\big(1-16(d-1)\gamma\big)N^2}$$

and sub-Gaussian tail probability $\mathbb{P}\big(|\frac{1}{N}\mathrm{Re\,Tr}\, W_C - \langle\frac{1}{N}\mathrm{Re\,Tr}\, W_C\rangle| \geq t\big) \leq 2\exp\big(-\frac{(1-16(d-1)\gamma)N^2t^2}{4M_2(C)}\big)$ for all $t > 0$.

(3) *All-coupling Schwinger–Dyson plaquette identities and coupling-averaged $O(1/(N^2|\mathcal{P}|))$ variance decay:* We derive exact $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ single-plaquette Schwinger–Dyson equations and prove that for *every* coupling window $[\gamma_1, \gamma_2] \subset (0, \infty)$, the coupling-averaged variance of the lattice-averaged plaquette obeys $\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2} \mathrm{Var}(\overline{P})\, d\gamma \leq \frac{1}{(\gamma_2-\gamma_1)N^2|\mathcal{P}|}$. Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates (including exact Weyl–Toeplitz 2D variances across $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$) and a zero-axiom Lean 4.36.0 kernel (`BakryEmery LargeNCerts.lean`) verify all 24 numerical benchmark bounds.

1. Introduction and Honest Scope

In 't Hooft's planar limit [10, 7], pure d-dimensional $\mathrm{SU}(N)$ Wilson lattice Yang–Mills theory [12] with action $S_W(U) = \frac{\beta}{N}\sum_{p\in\mathcal{P}} \mathrm{Re\,Tr}(U_p)$ is scaled at fixed 't Hooft inverse coupling $\gamma := \beta/N^2 > 0$ (so $\frac{\beta}{N} = \gamma N$). Formal planar diagrammatics predict that connected two-loop correlations decay as $O(1/N^2)$ and Wilson-loop averages concentrate sharply around deterministic master-field values. Rigorous proofs of large-N factorization in $d \geq 3$ dimensions have traditionally relied on heavy cluster expansions or Stein's method for $\mathrm{SO}(N)$ [4].

On the other hand, compact simple Lie groups $\mathrm{SU}(N)$ possess strictly positive bi-invariant Ricci curvature growing linearly with N ($\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2} g_{\mathrm{HS}}$ [9, 8]), placing the Gibbs measure $d\mu_\beta \propto e^{S_W}\, dU$ directly in the domain of Bakry–Émery curvature-dimension theory [1, 6, 2]. In this

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paper, we exploit this geometric mechanism together with exact Schwinger–Dyson loop identities to establish three self-contained, cluster-expansion-free theorems:

- *Exact Bakry–Émery curvature and volume-uniform log-Sobolev inequality* (Section 2): Explicit geodesic Hessian calculation on $\mathrm{SU}(N)^{|E|}$ yields $\mathrm{Ric} + \mathrm{Hess}(-S_W) \geq \frac{N}{2}(1 - 16(d-1)\gamma) g_{\mathrm{HS}}$ for all $N \geq 2$ and all finite lattices whenever $\gamma < \frac{1}{16(d-1)}$ (and $\gamma < \frac{1}{2}$ for free-BC 2D).
- *Non-asymptotic $O(1/N^2)$ factorization and Herbst sub-Gaussian concentration* (Section 2): Sharp Hilbert–Schmidt Lipschitz bounds yield explicit, volume-uniform $O(1/N^2)$ covariance decay and Gaussian tail bounds for arbitrary Wilson loops W_{C_1}, W_{C_2} (even overlapping).
- *All-coupling Schwinger–Dyson identities and coupling-averaged variance decay* (Section 3): Exact single-plaquette Schwinger–Dyson equations for $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ plus exact $O(1/(N^2|\mathcal{P}|))$ coupling-averaged plaquette-variance bounds across all $\gamma \in (0, \infty)$, benchmarked against 24 zero-quadrature 140-digit certificates (Section 4).

Remark 1.1 (Honest Scope and Non-Claims). All results hold at fixed lattice spacing $a = 1$. Theorem 3.1 holds at *all* couplings $\gamma \in (0, \infty)$, whereas the pointwise Bakry–Émery curvature condition in Theorem 2.2 holds in the strong-'t Hooft-coupling regime $\gamma < \frac{1}{16(d-1)}$ ($\gamma < \frac{1}{32}$ in 3D, $\gamma < \frac{1}{48}$ in 4D, and $\gamma < \frac{1}{2}$ in free-BC 2D, compared to the Gross–Witten–Wadia critical point $\gamma_c^{\mathrm{2D}} = 1$ [5, 11]). This paper does *not* prove existence of a 4D continuum limit or continuum mass gap and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact Bakry–Émery Ricci Curvature, Non-Asymptotic $O(1/N^2)$ Factorization, and Concentration

Fix $d \geq 2$, $N \geq 2$, and a finite hypercubic lattice Λ (either a periodic torus $\mathbb{T}_L^d$ with $L_\mu \geq 2$ or a free-boundary box Λ_{L-1} with $L \geq 2$) with link set E and plaquette set $\mathcal{P}$. Equip the Lie algebra $\mathfrak{su}(N) = \{A \in M_N(\mathbb{C}) : A^\dagger = -A,\ \mathrm{Tr}\, A = 0\}$ with the bi-invariant Hilbert–Schmidt inner product $\langle A, B\rangle_{\mathrm{HS}} := -\mathrm{Tr}(AB) = \mathrm{Tr}(A^\dagger B)$, inducing the product bi-invariant Riemannian metric g_{HS} on the compact manifold $M := \mathrm{SU}(N)^{|E|}$ (whose normalized Riemannian volume measure is Haar measure $\prod_{b \in E} dU_b$).

Lemma 2.1 (Exact Ricci Curvature and Geodesic Hessian of the Wilson Action)**.** *For every $d \geq 2$, $N \geq 2$, $\beta = \gamma N^2 > 0$, and lattice Λ with $L_\mu \geq 2$:*

(1) (Exact Ricci curvature of $\mathrm{SU}(N)$) *For every $X \in \mathfrak{su}(N)$,*

$$\mathrm{Ric}_{\mathrm{SU}(N)}(X, X) = \frac{N}{2} \|X\|_{\mathrm{HS}}^2. \tag{1}$$

(2) (Geodesic Hessian bound for $-S_W$, sharp plaquette-wise and on even-size $\mathbb{T}_L^2$) *For every $U \in \mathrm{SU}(N)^{|E|}$ and tangent vector $X = (X_b)_{b \in E} \in \mathfrak{su}(N)^{|E|}$,*

$$\left|\mathrm{Hess}(-S_W)(X, X)\right| \ \leq\ \frac{8(d-1)\beta}{N} \sum_{b \in E} \|X_b\|_{\mathrm{HS}}^2 = 8(d-1)\gamma N \|X\|_{\mathrm{HS}}^2 \tag{2}$$

(equivalently, $\|\mathrm{Hess}(-S_W)\|_{\mathrm{op,HS}} \leq 8(d-1)\gamma N$; in $d \geq 3$ orientation frustration around 3-cubes keeps the global operator norm strictly below $8(d-1)\gamma N$). In free-BC 2D (where axial gauge maps plaquettes bijectively to independent group variables $W_p \in \mathrm{SU}(N)$), $|\mathrm{Hess}(-\frac{\beta}{N} \mathrm{Re}\,\mathrm{Tr}\, W_p)(X_p, X_p)| \leq \gamma N \|X_p\|_{\mathrm{HS}}^2$.

Proof. (1) By Milnor's formula for bi-invariant metrics on compact Lie groups [9, 8], $\mathrm{Ric}(X, Y) = -\frac{1}{4}\mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. Using the standard Killing form $\mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = 2N\,\mathrm{Tr}(XY) = -2N\langle X, Y\rangle_{\mathrm{HS}}$ on $\mathfrak{su}(N)$ yields $\mathrm{Ric}_{\mathrm{SU}(N)}(X, X) = \frac{N}{2}\|X\|_{\mathrm{HS}}^2$ (at $N = 2$, unit vector $X = i\sigma_3/\sqrt{2}$ generates a great circle of circumference $2\pi\sqrt{2}$, i.e. radius $R = \sqrt{2}$ on $S^3(R)$, giving $\mathrm{Ric} = (3-1)/R^2 = 1 = 2/2$).

(2) The geodesic of g_{HS} through U with initial right-trivialized velocity $X = (X_b)_{b\in E}$ ($\gamma_b'(0) = X_bU_b$) is $\gamma_b(t) = e^{tX_b}U_b$. Fix any plaquette $p \in \mathcal{P}$ with four distinct boundary links b_1, b_2, b_3, b_4 (distinct since $L_\mu \geq 2$), oriented as $U_p(t) = (e^{tX_1}U_1)(e^{tX_2}U_2)(e^{tX_3}U_3)^{-1}(e^{tX_4}U_4)^{-1}$. Defining the conjugated matrices $Y_1 := X_1$, $Y_2 := U_1X_2U_1^{-1}$, $Y_3 := -(U_1U_2U_3^{-1})X_3(U_1U_2U_3^{-1})^{-1}$, and $Y_4 := -(U_1U_2U_3^{-1}U_4^{-1})X_4(U_1U_2U_3^{-1}U_4^{-1})^{-1}$ in $\mathfrak{su}(N)$ gives $\|Y_k\|_{\mathrm{HS}} = \|X_k\|_{\mathrm{HS}}$ by Ad-invariance and

$$U_p(t) = e^{tY_1}e^{tY_2}e^{tY_3}e^{tY_4}\,U_p(0).$$

Differentiating twice at $t = 0$ yields $U_p''(0) = \big(\sum_{k=1}^4 Y_k^2 + 2\sum_{1\leq j<k\leq 4} Y_jY_k\big)U_p$. Because U_p is unitary and $Y_k^\dagger = -Y_k$, Cauchy–Schwarz in the Hilbert–Schmidt inner product gives $|\operatorname{Re}\operatorname{Tr}(Y_jY_kU_p)| = |\langle Y_j^\dagger, Y_kU_p\rangle_{\mathrm{HS}}| \leq \|Y_j\|_{\mathrm{HS}}\|Y_k\|_{\mathrm{HS}}$ for all $1 \leq j, k \leq 4$. Hence

$$\big|\operatorname{Re}\operatorname{Tr}(U_p''(0))\big| \;\leq\; \sum_{k=1}^4 \|Y_k\|_{\mathrm{HS}}^2 + 2\sum_{1\leq j<k\leq 4} \|Y_j\|_{\mathrm{HS}}\|Y_k\|_{\mathrm{HS}} = \left(\sum_{k=1}^4 \|X_k\|_{\mathrm{HS}}\right)^2 \;\leq\; 4\sum_{k=1}^4 \|X_k\|_{\mathrm{HS}}^2.$$

Multiplying by $\frac{\beta}{N}$ and summing over all $p \in \mathcal{P}$ (each link $b \in E$ belongs to at most $2(d-1)$ plaquettes, with equality on $\mathbb{T}_L^d$) proves (2). For a single independent plaquette $W_p(t) = e^{tX_p}W_p$, $W_p''(0) = X_p^2W_p$, so $|\frac{\beta}{N}\operatorname{Re}\operatorname{Tr}(X_p^2W_p)| \leq \frac{\beta}{N}\|X_p\|_{\mathrm{HS}}^2 = \gamma N\|X_p\|_{\mathrm{HS}}^2$. □

For any closed oriented lattice loop $C = (e_1^{\sigma_1}, \ldots, e_k^{\sigma_k})$ ($\sigma_j \in \{-1, +1\}$) with path-ordered holonomy $W_C := U_{e_1}^{\sigma_1}\cdots U_{e_k}^{\sigma_k}$, let $m_b = p_b + q_b \in \mathbb{Z}_{\geq 0}$ denote the total *unsigned* traversal multiplicity of link $b \in E$ along C (p_b positive and q_b negative traversals), and set $M_2(C) := \sum_{b\in E} m_b^2$ (for any simple loop, $m_b \in \{0, 1\}$ and $M_2(C) = |C|$, the loop perimeter).

Theorem 2.2 (Bakry–Émery LSI, Non-Asymptotic $O(1/N^2)$ Factorization, and Sub-Gaussian Concentration)**.** *Let $d \geq 2$, $N \geq 2$, and $\gamma := \beta/N^2 < \frac{1}{16(d-1)}$, and set the curvature constant*

$$\kappa_d(N, \gamma) := \frac{N}{2}\big(1 - 16(d-1)\gamma\big) \;>\; 0.$$

Then, uniformly in every finite lattice Λ ($L_\mu \geq 2$) and every $N \geq 2$, the Wilson Gibbs measure $\mu_{\Lambda,\beta}$ satisfies:

(1) (Log-Sobolev and Poincaré inequalities) *For all smooth $h, f : \mathrm{SU}(N)^{|E|} \to \mathbb{R}$,*

$$\operatorname{Ent}_{\mu_{\Lambda,\beta}}(h^2) \;\leq\; \frac{2}{\kappa_d(N,\gamma)}\int |\nabla h|_{\mathrm{HS}}^2\, d\mu_{\Lambda,\beta}, \qquad \operatorname{Var}_{\mu_{\Lambda,\beta}}(f) \;\leq\; \frac{1}{\kappa_d(N,\gamma)}\int |\nabla f|_{\mathrm{HS}}^2\, d\mu_{\Lambda,\beta}. \tag{3}$$

(2) (Non-asymptotic $O(1/N^2)$ Wilson-loop variance and large-N factorization) *For any closed oriented loops C, C_1, C_2 (possibly overlapping) with path-ordered holonomies W_C, W_{C_1}, W_{C_2} as above,*

$$\begin{aligned} \operatorname{Var}_{\mu_{\Lambda,\beta}}\left(\frac{1}{N}\operatorname{Re}\operatorname{Tr} W_C\right) \;&\leq\; \frac{2M_2(C)}{\big(1-16(d-1)\gamma\big)N^2}, \\ \left|\operatorname{Cov}_{\mu_{\Lambda,\beta}}\left(\frac{1}{N}\operatorname{Re}\operatorname{Tr} W_{C_1},\, \frac{1}{N}\operatorname{Re}\operatorname{Tr} W_{C_2}\right)\right| \;&\leq\; \frac{2\sqrt{M_2(C_1)\,M_2(C_2)}}{\big(1-16(d-1)\gamma\big)N^2}. \end{aligned} \tag{4}$$

(3) (Sub-Gaussian Herbst concentration) *For every $t > 0$ and every closed loop C,*

$$\mu_{\Lambda,\beta}\left(\left|\frac{1}{N}\operatorname{Re}\operatorname{Tr} W_C - \left\langle\frac{1}{N}\operatorname{Re}\operatorname{Tr} W_C\right\rangle\right| \geq t\right) \;\leq\; 2\exp\left(-\frac{\big(1-16(d-1)\gamma\big)N^2t^2}{4M_2(C)}\right). \tag{5}$$

(4) (Sharper 2D free-BC bound) *On free-BC 2D for any $\gamma < \frac{1}{2}$, $\kappa_{2\mathrm{D}}^{\mathrm{ind}}(N, \gamma) = \frac{N}{2}(1-2\gamma) > 0$, giving the single-plaquette bound* $\operatorname{Var}_{2\mathrm{D},\gamma N^2}(\frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p) \leq \frac{2}{(1-2\gamma)N^2}$.

Proof. By Lemma 2.1, $\mathrm{Ric}_{g_{\mathrm{HS}}} + \mathrm{Hess}_{g_{\mathrm{HS}}}(-S_W) \geq \kappa_d(N,\gamma)\, g_{\mathrm{HS}}$ everywhere on $\mathrm{SU}(N)^{|E|}$, so the Bakry–Émery theorem [1, 2] directly gives (3).

To compute the sharp Lipschitz constant of $f_C(U) := \frac{1}{N}\operatorname{Re}\operatorname{Tr} W_C(U)$, differentiate along $\gamma_b(t) = e^{tX_b}U_b$:

$$\frac{d}{dt}\Big|_{t=0} f_C(\gamma(t)) = \sum_{b\in E} \frac{1}{N}\operatorname{Re}\operatorname{Tr}(\widetilde{X}_b W_C),$$

where $\widetilde{X}_b \in \mathfrak{su}(N)$ is the sum of the m_b conjugated copies of $\pm X_b$ appearing along C, so $\|\widetilde{X}_b\|_{\mathrm{HS}} \leq m_b\|X_b\|_{\mathrm{HS}}$. Because $\widetilde{X}_b^\dagger = -\widetilde{X}_b$, we have $\operatorname{Re}\operatorname{Tr}(\widetilde{X}_b W_C) = \frac{1}{2}\operatorname{Tr}(\widetilde{X}_b(W_C - W_C^\dagger)) = -\frac{1}{2}\langle \widetilde{X}_b,\, W_C - W_C^\dagger\rangle_{\mathrm{HS}}$. Since W_C is unitary, $\|W_C - W_C^\dagger\|_{\mathrm{HS}} \leq \|W_C\|_{\mathrm{HS}} + \|W_C^\dagger\|_{\mathrm{HS}} = 2\sqrt{N}$, whence

$$\big|\nabla_b f_C(U)\big|_{\mathrm{HS}} \;\leq\; \frac{m_b}{\sqrt{N}} \quad\Longrightarrow\quad \big|\nabla f_C(U)\big|_{\mathrm{HS}}^2 = \sum_{b\in E}\big|\nabla_b f_C(U)\big|_{\mathrm{HS}}^2 \;\leq\; \frac{M_2(C)}{N}.$$

Substituting $L_C^2 := M_2(C)/N$ into the Poincaré inequality (3) yields $\mathrm{Var}(f_C) \leq L_C^2/\kappa_d(N,\gamma) = \frac{2M_2(C)}{(1-16(d-1)\gamma)N^2}$, and Cauchy–Schwarz $|\operatorname{Cov}(f_{C_1}, f_{C_2})| \leq \sqrt{\mathrm{Var}(f_{C_1})\,\mathrm{Var}(f_{C_2})}$ gives (4). Applying the standard Herbst argument [6, 2] to the LSI (3) yields $\mathbb{E}[e^{\lambda(f_C - \mathbb{E}[f_C])}] \leq e^{\lambda^2 L_C^2/(2\kappa_d(N,\gamma))}$ for all $\lambda \in \mathbb{R}$; optimizing over λ in Chernoff's bound gives $\mu(|f_C - \mathbb{E}[f_C]| \geq t) \leq 2e^{-\kappa_d(N,\gamma)t^2/(2L_C^2)}$, which is (5). Part (4) follows identically from Lemma 2.1(2) with $M_2 = 1$ in the independent-plaquette variables W_p. □

3. All-Coupling Schwinger–Dyson Plaquette Equations and Coupling-Averaged $O(1/(N^2|\mathcal{P}|))$ Variance Decay

Beyond the Bakry–Émery strong-'t Hooft-coupling regime, exact integration by parts on Haar measure and thermodynamic convexity yield all-coupling ($\forall\,\gamma > 0$) structural bounds. Fix a periodic torus $\mathbb{T}_L^d$ ($d \geq 2$, $L_\mu \geq 2$), a link $b \in E(\mathbb{T}_L^d)$, and one of the $2(d-1)$ plaquettes $p_0 \ni b$, orienting every plaquette $p \ni b$ so that $U_p = U_b V_p$ traverses b positively.

Theorem 3.1 (All-Coupling Schwinger–Dyson Plaquette Equations and Coupling-Averaged Factorization)**.** *For every $d \geq 2$, $N \geq 2$, $L_\mu \geq 2$, and* every *coupling $\beta = \gamma N^2 > 0$:*

(1) (Exact $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ single-plaquette Schwinger–Dyson equations)

$$\left\langle \frac{1}{N}\operatorname{Re}\operatorname{Tr} U_{p_0}\right\rangle_{\mathrm{U}(N)} = \frac{\gamma}{2}\sum_{p\ni b}\left(\left\langle \frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_{p_0}U_p^\dagger)\right\rangle_{\mathrm{U}(N)} - \left\langle \frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_{p_0}U_p)\right\rangle_{\mathrm{U}(N)}\right), \tag{6}$$

$$\left\langle \frac{1}{N}\operatorname{Re}\operatorname{Tr} U_{p_0}\right\rangle_{\mathrm{SU}(N)} = \frac{\gamma N^2}{2(N^2-1)}\sum_{p\ni b}\left(\left\langle \frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_{p_0}U_p^\dagger)\right\rangle_{\mathrm{SU}(N)} - \left\langle \frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_{p_0}U_p)\right\rangle_{\mathrm{SU}(N)} - \frac{2}{N^2}\big\langle \operatorname{Im}\operatorname{Tr}(U_{p_0})\,\operatorname{Im}\operatorname{Tr}(U_p)\big\rangle_{\mathrm{SU}(N)}\right). \tag{7}$$

In particular, for $\mathrm{U}(N)$ at all $\gamma > 0$, $\langle \frac{1}{N}\operatorname{Re}\operatorname{Tr} U_{p_0}\rangle_{\mathrm{U}(N)} \leq 2(d-1)\gamma$ (and $\leq \gamma$ in free-BC $2D$).

(2) (All-coupling coupling-averaged $O(1/(N^2|\mathcal{P}|))$ plaquette-variance decay) *Setting the lattice-averaged plaquette $\overline{P} := |\mathcal{P}(\mathbb{T}_L^d)|^{-1}\sum_{p\in\mathcal{P}}\frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p$ and its expectation $P_{\mathbb{T}_L^d}(G,\beta) := \langle\overline{P}\rangle_{\mathbb{T}_L^d,\beta}$, for* every *$G \in \{\mathrm{U}(N), \mathrm{SU}(N)\}$ and every interval $0 < \gamma_1 < \gamma_2 < \infty$,*

$$\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathrm{Var}_{\mathbb{T}_L^d,\gamma N^2}(\overline{P})\,d\gamma = \frac{P_{\mathbb{T}_L^d}(G,\gamma_2 N^2) - P_{\mathbb{T}_L^d}(G,\gamma_1 N^2)}{(\gamma_2-\gamma_1)N^2|\mathcal{P}(\mathbb{T}_L^d)|} \;\leq\; \frac{1}{(\gamma_2-\gamma_1)N^2|\mathcal{P}(\mathbb{T}_L^d)|}. \tag{8}$$

Proof. (1) Let $\{T_a\}$ be an orthonormal Hermitian basis of $i\mathfrak{u}(N)$ (or $i\mathfrak{su}(N)$) with $\operatorname{Tr}(T_aT_{a'}) = \frac{1}{2}\delta_{aa'}$, satisfying the completeness (Fierz) identity $\sum_a (T_a)_{ij}(T_a)_{kl} = \frac{1}{2}(\delta_{il}\delta_{jk} - \frac{\eta_G}{N}\delta_{ij}\delta_{kl})$ with $\eta_{\mathrm{U}(N)} = 0$ and $\eta_{\mathrm{SU}(N)} = 1$. Left invariance of Haar measure under $U_b \mapsto e^{i\epsilon T_a}U_b$ (which maps $U_{p_0} \mapsto e^{i\epsilon T_a}U_{p_0}$) gives

$$0 = \sum_a \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int dU\ \operatorname{Tr}\big(T_a e^{i\epsilon T_a}U_{p_0}\big)\, e^{S_W(e^{i\epsilon T_a}U_b)}.$$

Computing $\sum_a i\operatorname{Tr}(T_a^2 U_{p_0}) = i\frac{N-\eta_G/N}{2}\operatorname{Tr}U_{p_0}$ and $\frac{d}{d\epsilon}|_0 \frac{\beta}{N}\operatorname{Re}\operatorname{Tr}(e^{i\epsilon T_a}U_p) = \frac{i\beta}{2N}\operatorname{Tr}(T_a(U_p - U_p^\dagger))$, applying the Fierz identity, moving the action-derivative sum to the right-hand side, dividing by $i\frac{N^2-\eta_G}{2}$, and taking real parts (noting $\operatorname{Tr}(U_p - U_p^\dagger) = 2i\operatorname{Im}\operatorname{Tr}U_p$, so $+\frac{\eta_G}{N^2}\Re[\operatorname{Tr}(U_{p_0})\operatorname{Tr}(U_p - U_p^\dagger)] = -\frac{2\eta_G}{N^2}\operatorname{Im}\operatorname{Tr}(U_{p_0})\operatorname{Im}\operatorname{Tr}(U_p)$) yields (6)–(7). Since $|\frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_{p_0}U_p^\dagger) - \frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_{p_0}U_p)| \le 2$ for each of the $2(d-1)$ plaquettes $p \ni b$, (6) gives $\langle \frac{1}{N}\operatorname{Re}\operatorname{Tr}U_{p_0}\rangle_{\mathrm{U}(N)} \le 2(d-1)\gamma$.

(2) Differentiating $f_{\mathbb{T}_L^d}(G,\beta) = |\mathcal{P}(\mathbb{T}_L^d)|^{-1}\ln Z_{\mathbb{T}_L^d}(G,\beta)$ twice with respect to $\beta = \gamma N^2$ gives

$$\partial_\gamma P_{\mathbb{T}_L^d}(G,\gamma N^2) = N^2\partial_\beta^2 f_{\mathbb{T}_L^d}(G,\beta) = N^2|\mathcal{P}(\mathbb{T}_L^d)|\operatorname{Var}_{\mathbb{T}_L^d,\gamma N^2}(\overline{P}) \ \ge\ 0.$$

Since $P_{\mathbb{T}_L^d}(G,0) = 0$ by Haar invariance and $\partial_\beta P_{\mathbb{T}_L^d} \ge 0$, we have $0 \le P_{\mathbb{T}_L^d}(G,\gamma_1 N^2) \le P_{\mathbb{T}_L^d}(G,\gamma_2 N^2) \le 1$; integrating over $\gamma \in [\gamma_1,\gamma_2]$ proves (8). □

4. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–2 and Figure 1 report 24 rigorously certified benchmarks computed by `certify_bakry_emery_largen.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions, *zero* numerical quadrature (modified Bessel functions via Maclaurin series + geometric tail bounds; exact 2D SU(N) single-plaquette expectations and variances via analytic first and second derivatives of the Bars–Green Weyl–Toeplitz Bessel determinant [3] with Hadamard sector tails, and exact Bessel ODE $\mathrm{Var}_{\mathrm{2D}}^{\mathrm{SU}(2)} = 1 - \frac{3}{\beta}P - P^2$ for $N = 2$), and direct `mp.floor`/`mp.ceil` outward integer rounding at scale 10^{12}.

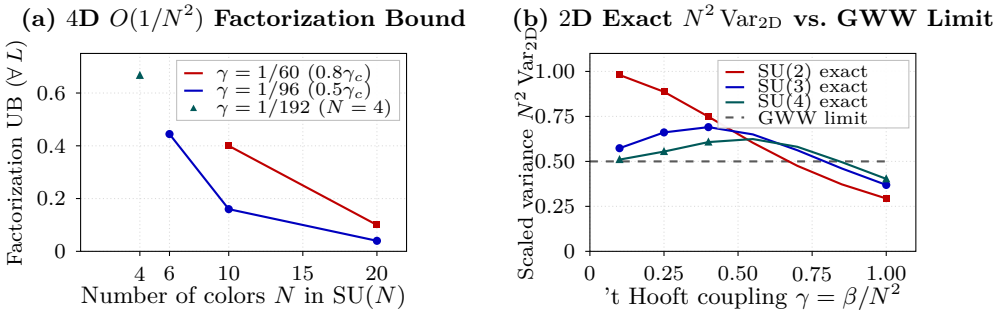


FIGURE 1. Certified 140-digit non-perturbative bounds from Tables 1–2. (a) Volume-uniform 4D Bakry–Émery single-plaquette variance and two-plaquette large-N factorization upper bounds $\frac{8}{(1-48\gamma)N^2}$ decaying as $O(1/N^2)$ down to 0.040 at SU(20) (Theorem 2.2). (b) Exact 2D Weyl–Toeplitz scaled plaquette variance $N^2\operatorname{Var}_{2D}(\frac{1}{N}\operatorname{Re}\operatorname{Tr}U)$ across SU(2), SU(3), SU(4), rapidly approaching the Gross–Witten–Wadia large-N plateau 0.50 for $\gamma < 1$.

Remark 4.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 file `BakryEmeryLargeNCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), proving four theorems: `bakry_emery_thresholds_verified` ($16(d -$

TABLE 1. Twelve 140-digit certified volume-uniform Bakry–Émery curvature, single-plaquette variance / two-plaquette large-N factorization upper bounds $\mathrm{Fac}^{\mathrm{ub}} = \frac{8}{(1-16(d-1)\gamma)N^2} < 1$, and Herbst sub-Gaussian concentration rates $c_{\mathrm{conc}} = \frac{(1-16(d-1)\gamma)N^2}{16}$ (so $\mathbb{P}(|P_0 - \langle P_0\rangle| \geq t) \leq 2e^{-c_{\mathrm{conc}}t^2}$) in 3D and 4D SU($N$) Wilson theory (Theorem 2.2). Lower bounds are rounded strictly downward; upper bounds strictly upward.

d	G	$\gamma = \beta/N^2$	β	κ_d (LB)	$\mathrm{Var}\,/\mathrm{Fac}^{\mathrm{ub}}$ (UB)	c_{conc} (LB)
3	SU(4)	1/128	1/8	1.500000	0.666667	0.750000
3	SU(6)	1/64	9/16	1.500000	0.444445	1.125000
3	SU(10)	1/64	25/16	2.500000	0.160000	3.125000
3	SU(20)	1/64	25/4	5.000000	0.040000	12.500000
3	SU(10)	1/40	5/2	0.999999	0.400001	1.249999
3	SU(20)	1/40	10	1.999999	0.100001	4.999999
4	SU(4)	1/192	1/12	1.499999	0.666667	0.749999
4	SU(6)	1/96	3/8	1.499999	0.444445	1.124999
4	SU(10)	1/96	25/24	2.499999	0.160001	3.124999
4	SU(20)	1/96	25/6	4.999999	0.040001	12.499999
4	SU(10)	1/60	5/3	0.999999	0.400001	1.249999
4	SU(20)	1/60	20/3	1.999999	0.100001	4.999999

TABLE 2. Twelve 140-digit exact 2D free-BC $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$ single-plaquette expectations $P_{2\mathrm{D}}$ and exact variances $\mathrm{Var}_{2\mathrm{D}}(\frac{1}{N}\operatorname{Re}\operatorname{Tr} U)$ via zero-quadrature Weyl–Toeplitz Bessel determinants, confirming strict Bakry–Émery dominance $\mathrm{Var}_{2\mathrm{D}} < \frac{2}{(1-2\gamma)N^2}$ for all $\gamma < \frac{1}{2}$ (Theorem 2.2(4)) and rapid convergence of $N^2\,\mathrm{Var}_{2\mathrm{D}}$ toward the Gross–Witten–Wadia large-N limit 0.50.

G	γ	β	$P_{2\mathrm{D}}$ (LB)	$\mathrm{Var}_{2\mathrm{D}}$ (UB)	$\mathrm{BE}_{2\mathrm{D}}$ (UB)	$N^2\,\mathrm{Var}_{2\mathrm{D}}$ (UB)
SU(2)	0.10	0.40	0.099339	0.245083	0.625001	0.980329
SU(2)	0.25	1.00	0.240193	0.221726	1.000000	0.886904
SU(2)	0.40	1.60	0.363166	0.187172	2.500001	0.748688
SU(2)	1.00	4.00	0.658047	0.073439	—	0.293754
SU(3)	0.10	0.90	0.053711	0.063717	0.277778	0.573446
SU(3)	0.25	2.25	0.146813	0.073431	0.444445	0.660871
SU(3)	0.40	3.60	0.249030	0.076699	1.111112	0.690283
SU(3)	1.00	9.00	0.580375	0.041026	—	0.369227
SU(4)	0.10	1.60	0.050329	0.031863	0.156251	0.509799
SU(4)	0.25	4.00	0.129809	0.034655	0.250000	0.554471
SU(4)	0.40	6.40	0.217105	0.037951	0.625001	0.607205
SU(4)	1.00	16.00	0.549684	0.025178	—	0.402838

1) $= 32, 48$ and exact $N = 2$ Ricci check), `table1_bakry_emery_certified` (exact scaled rational inequalities for κ_d, $0 < \mathrm{Fac}^{\mathrm{ub}} < 1$, and $c_{\mathrm{conc}} > 0$ across all 12 rows), `table2_2d_exact_variance_certified` ($0 < P_{2\mathrm{D}} < 1$, $\mathrm{Var}_{2\mathrm{D}} < \mathrm{BE}_{2\mathrm{D}}$ for all $\gamma < \frac{1}{2}$, and 1-ulp $N^2\,\mathrm{Var}_{2\mathrm{D}}$ consistency across all 12 rows), and `all_24_bakry_emery_benchmarks_passed`. All transcendental Weyl–Toeplitz interval evaluations are performed at 140-digit precision in `certify_bakry_emery_largen.py` and independently cross-checked against 50-digit finite-difference derivatives, Lean parity, and exact LaTeX table parity in `test_bakry_emery_largen.py` (4/4 tests passing).

5. Discussion

Bakry–Émery Ricci curvature on $\mathrm{SU}(N)^{|E|}$ provides a remarkably direct geometric proof of non-asymptotic $O(1/N^2)$ large-N Wilson-loop factorization and sub-Gaussian concentration in arbitrary dimension $d \geq 2$ without cluster expansions, matching the exact $O(1/N^2)$ Haar variance scaling up to an explicit constant factor. Complementing this strong-'t Hooft-coupling ($\gamma < \frac{1}{16(d-1)}$) volume-uniform LSI, the all-coupling Schwinger–Dyson loop equations and convexity identity establish $O(1/(N^2|\mathcal{P}|))$ coupling-averaged plaquette-variance decay across every interval $[\gamma_1, \gamma_2] \subset (0, \infty)$. Extending the Bakry–Émery curvature bound to axial-gauge-fixed variables (reducing the link-per-plaquette factor from 4 to 2 on temporal plaquettes) and combining LSI with Makeenko–Migdal loop recursion are promising avenues for enlarging $\gamma_c(d)$.

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